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On the Class Numbers of Real Quadratic Fields of Richaud–Degert Type

By

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Abstract

The purpose of this paper is to give an explicit proof of the infinity of real quadratic fields of Richaud–Degert type and construct the infinite sequences of real quadratic fields with large class numbers.

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Introduction

Let a be an odd square-free positive integer. Then the real quadratic field $K = \mathbf{Q}(\sqrt{D})$ is called of Richaud-Degert type, if D is a square-free positive integer of the form $a^2n^2 + ia$ $(i = \pm 1, \pm 2, \pm 4)$. Here n is a positive integer. In the following, we shall abbreviate Richaud-Degert type to R-D type. $S_{i,a}$ denotes the set of R-D type real quadratic fields $\{K = \mathbf{Q}(\sqrt{D}) \mid D = a^2n^2 + ia \ (i = \pm 1, \pm 2, \pm 4)\}$. In this paper, we shall give an explicit proof of the infinity of real quadratic fields of Richaud-Degert type and construct the infinite sequences of real quadratic fields with large class numbers. It is well known that R-D type real quadaratic fields K have the good property that have the folllowing explict fundamental units $\varepsilon > 1$ except for the case $D = 5$.

(1) In the case $D = a^2n^2 \pm a$,
$\varepsilon = n + \sqrt{n^2 \pm 1}$ when $a = 1$,
$\varepsilon = 2an^2 \pm 1 + 2n\sqrt{a^2n^2 \pm a}$ otherwise.

(2) In the case $D = a^2n^2 \pm 2a$,
$\varepsilon = an^2 \pm 1 + n\sqrt{a^2n^2 \pm 2a}$.

(3) In the case $D = a^2n^2 \pm 4a$,
$\varepsilon = (n + \sqrt{n^2 \pm 4})/2$ when $a = 1$,
$\varepsilon = (an^2 \pm 2 + n\sqrt{a^2n^2 \pm 4a})/2$ otherwise.

Hence, one sees that almost all $K = \mathbf{Q}(D) \in S_{i,a}$ has the fundamental unit $\varepsilon < 3D$, i.e., the regulator $R_K = \log \varepsilon < \log D + \log 3$.

The proof of main theorem

Let q be an even integer such that $(a, q) = 1$ and b be an integer. We denote $D_{i,a}(x, b)$ the number of $D \leq x$ such that $D = a^2n^2 + ia$ is square-free for some integer $n \equiv b$ $(mod\ q)$. Then, modifying the methods used in [5], we have the following fact:

Proposition 1. *Suppose that* $a^2n^2 + ia \equiv a^2b^2 + ia \not\equiv 0$ $(mod\ 4)$. *Then*
$D_{i,a}(x, b) = c_{i,a}x^{1/2} + O(x^{1/3}\log x)$,
where $c_{i,a} = \frac{1}{aq}\prod_{p\nmid aq}(1 - (1 + \left(\frac{-ia}{p}\right))p^{-2})$.

To prove this proposition, we first note that

$$D_{i,a}(x, b) = \sum_{n \leq (x-ia)^{1/2}/a,\, n \equiv b\ (mod\ q)} \sum_{r^2 | (a^2n^2+ia)} \mu(r)$$

$$= \sum_{r \leq x} \mu(r) \sum_{n \leq (x-ia)^{1/2}/a,\, r^2|(a^2n^2+ia),\, n \equiv b\ (mod\ q)} 1.$$

We note that from the assumption that $a^2n^2 + ia \equiv a^2b^2 + ia \not\equiv 0$ $(mod\ 4)$, one may take only odd r in the above sums.

Put $y = x^{1/3}$. We shall consider the cases $r \leq y$ and $r > y$ separately. Put $a^2n^2 + ia = r^2s$. Then we see

$$\sum_{y<r\leq x} \mu(r) \sum 1 \ll \sum_{s \leq xy^{-2}} \sum_{(n,r),\, r^2s = a^2n^2+ia \equiv a^2b^2+ia\ (mod\ q)} 1.$$

From the theory of Pellian equation, the number of integer solutions of the pair (u, v) such that $u^2 - sv^2 = -ia$ and $1 \leq u \leq U$, is $\ll \log U$. Thus the above inner sum is $\ll \log x$, and so the contribution of $r > y$ is $O(xy^{-2}\log x) = O(x^{1/3}\log x)$. Thus we have shown

$$D_{i,a}(x, b) = \sum_{r \leq y} \mu(r) \sum_{n \equiv b\ (mod\ q),\, n \leq (x-ia)^{1/2}/a,\, r^2|(a^2n^2+ia)} 1 + O(x^{1/3}\log x).$$

Let $c(m)$ be the number of solutions n $(mod\ m)$ of the congruence
$a^2n^2 + ia \equiv 0$ $(mod\ m)$. Then one can easily see the following lemma.

Lemma 1. *In the case* $(m, 2a) = 1$, *we have* $c(m) = \prod_{p|m} c(p)$, *where* p *runs all the odd prime factors of* m , *and* $c(p) = 1 + \left(\frac{-ia}{p}\right)$.

Using the fact that the divisor function $d(r) = \sum_{(u,v),uv=r\ (u>0)} 1$, one can easily see the following well known lemma:

Lemma 2 ([2]).

$$\sum_{r\leq y} d(r) = y\log y + (2\gamma - 1)y + O(\sqrt{y}),$$

$$\sum_{r>y} d(r)r^{-2} = y^{-1}\log y + O(y^{-1}),$$

where γ is the Euler constant.

Since

$$\sum_{n\leq(x-ia)^{1/2}/a,r^2|(a^2n^2+ia),n\equiv b\ (mod\ q)} 1 = c(r^2)(a^{-1}x^{1/2}r^{-2} + O(1)),$$

and $c(r^2) = c(r) \leq d(r)$ for odd r, we see

$$\sum_{r\leq y} c(r^2) \leq \sum_{r\leq y} d(r) \ll y\log y$$

and

$$\sum_{r>y} c(r^2)r^{-2} \leq \sum_{r>y} d(r)r^{-2} \ll y^{-1}\log y.$$

Therefore

$$D_{i,a}(x,b) = \frac{x^{1/2}}{aq}\sum_{r=1,(r,qa)=1}^{\infty} \mu(r)c(r)r^{-2} + O(y\log y) + O(x^{1/2}y^{-1}\log y) + O(x^{1/3}\log x).$$

Since $\mu(r)c(r)$ is a multiplicative function, we see

$$D_{i,a}(x,b) = \frac{x^{1/2}}{aq}\sum_{r=1,(r,aq)=1}^{\infty} \mu(r)c(r)r^{-2} + O(x^{1/3}\log x)$$

$$= c_{i,a}x^{1/2} + O(x^{1/3}\log x).$$

Here $c_{i,a} = \frac{1}{aq}\prod_{p\nmid aq}(1 - (1 + \left(\frac{-ia}{p}\right))p^{-2})$. Hence, we have complete the proof of the above proposition.

We note that the following elementary lemma holds for R-D type real quadartic fields.

Lemma 3.
1) *The case $D = a^2n^2 + a$.*
In the case $a \equiv 1\ (mod\ 4)$, $D \not\equiv 0\ (mod\ 4)$ for any n.
In the case $a \equiv 3\ (mod\ 4)$, $D \equiv 0\ (mod\ 4)$ for any odd n, and $D \not\equiv 0\ (mod\ 4)$ for any

even n.
2) *The case* $D = a^2n^2 - a$.
In the case $a \equiv 1 \pmod 4$, $D \equiv 0 \pmod 4$ *for any odd n, and* $D \not\equiv 0 \pmod 4$ *for any even n.*
In the case $a \equiv 3 \pmod 4$, $D \not\equiv 0 \pmod 4$ *for any n.*
3) *The case* $D = a^2n^2 \pm 2a$.
$D \not\equiv 0 \pmod 4$ *for any n.*
4) *The case* $D = a^2n^2 \pm 4a$.
$D \equiv 0 \pmod 4$ *for any even n, and* $D \not\equiv 0 \pmod 4$ *for any odd n.*

Since D is square-free, we may not consider the cases $D \equiv 0 \pmod 4$. Hence, taking $q = 2$, and $b = 0$ or 1 according to i, a in the above proposition, we have the following proposition:

Proposition 2. *Let* $D_{i,a}(x)$ *be the number of square-free* $D \le x$ *such that* $K = \mathbf{Q}(\sqrt{D}) \in S_{i,a}$. *Then*

$$D_{i,a}(x) = c'_{i,a}x^{1/2} + O(x^{1/3}\log x).$$

Here $c'_{i,a} = \frac{1}{a}\prod_{p\nmid 2a}(1 - (1 + \left(\frac{-ia}{p}\right))p^{-2})$ *for the cases* $i = 1$ $(a \equiv 1 \pmod 4)$, $i = -1$ $(a \equiv 3 \pmod 4)$, *and* $i = \pm 2$, *and* $c'_{i,a} = \frac{1}{2a}\prod_{p\nmid 2a}(1 - (1 + \left(\frac{-ia}{p}\right))p^{-2})$ *for the cases* $i = 1$ $(a \equiv 3 \pmod 4)$, $i = -1$ $(a \equiv 1 \pmod 4)$, *and* $i = \pm 4$.

Remark 1. In [5], H.C.Montgomery and P.J.Weinberger have shown the case $a = i = 1$, that is, $c'_{1,1} = \prod_{p\equiv 1 \ (mod\ 4)}(1 - 2p^{-2})$

Here we shall recall the following lemma (c.f. Lemma 2 of [5]).

Lemma 4. *Let* δ *be a positive number* < 1, q *be a natural number and* χ *be a primitive character modulas* q. *Then for any* $(\log q)^\delta \le y \le \log q$, *we have*

$$\log L(1, \chi) = \sum_{p \le y} \chi(p)p^{-1} + O_\delta(1)$$

unless χ *lies in an exceptional set* $E(\delta)$. *Here the set* $E(\delta)$ *contains* $\ll Q^\delta$ *primitive character* χ *with conductor* $q \le Q$.

In the following, we shall construct the infinite subset $S^0_{i,a} \subset S_{i,a}$ with the following property.

Theorem. *For any $K = \mathbf{Q}(\sqrt{D}) \in S^0_{i,a}$, there is a constant c_1 such that the class number $h(D)$ of K satisfies*

$$h(D) > c_1 D^{1/2} (\log D)^{-1} \log\log D$$

To prove this main theorem, we need the following elementary lemma.

Lemma 5. *For any natural number t, and any odd prime $p > 4t+1$, there exists a natural number n_1 such that $\left(\frac{n_1}{p}\right) = \left(\frac{n_1+t}{p}\right) = 1$.*

Proof. Let r_1 be the natural number $r_1 \equiv 2^{-1}$ $(mod\ p)$. Put $n_1 = r_1^4(4t-1)^2$. Then $n_1 + t \equiv r_1^4(16t^2 - 8t + 1) + 16r_1^4 t \equiv r_1^4(4t+1)^2$ $(mod\ p)$. Since $n_1 \not\equiv 0$ $(mod\ p)$ and $n_1 + t \not\equiv 0$ $(mod\ p)$, we have proved this lemma.

We now complete the proof of the above theorem. Let $y = \frac{1}{9}\log x$, and $q_x = \prod_{8a+1<p\leq y} p$. From Lemma 5 and the Chinese remainder theorem, there exists a natural number b_x, such that $\left(\frac{a^2 b_x^2 + ia}{p}\right) = 1$ for any $p \mid q_x$. Then, $q_x < x^{1/8}$, $\chi(p) = \left(\frac{D}{p}\right) = 1$ for any $D = a^2n^2 + ia, n \equiv b_x$ $(mod\ q_x)$ for any $p \mid q_x$. We note that for any $x' > x$, one can take $b_{x'}, q_{x'}$ which satisfy $q_x \mid q_{x'}$ and $b_{x'} \equiv b_x$ $(mod\ q_{x'})$. By Proposition 1, there are $\gg x^{1/2} q_x^{-1} \gg x^{1/8}$ such square-free $D \leq x$. From Lemma 4, with $\delta < 3/8$, we see that $L(1,\chi) > a_1 \log y > a_2 \log\log D$ for almost all of these D, where a_1 and a_2 are some constants. This completes the proof of Theorem.

In his paper [4], J.E.Littlewoods has shown the following fact:

Lemma 6. *Under the Generalized Riemann Hypothesis, any class number of the real quadratic field $K = \mathbf{Q}(\sqrt{D})$ should satisfy*

$h(D) < c_2 D^{1/2} (\log D)^{-1} \log\log D,$

where c_2 is a positive constant.

Combinig this lemma and Theorem, we have:

Corollary. *Under the GRH, for any sequence of R-D type real quadratic fields $S_{i,a}$, there exists infinitely many $K \in S_{i,a}$ which satisfy*

$$c_1 < \frac{h(D)}{D^{1/2}(\log D)^{-1} \log\log D} < c_2,$$

where c_1 and c_2 are some positive constants.

Remark 2. The infinity of R-D type real quadratic fields have also been proved in [5]. Finally, we note that the infinity of R-D type real quadratic fields with odd class numbers is reduced to the following problem.

Does the given quadratic irreducible polynomial $g(x)$ take infinitely many prime values when $x \in \mathbf{Z}$?

This is nothing but the famous unsolved conjecture of Hardy-Littlewoods proposed in their paper [1]. Concerning the almost primes represented by quadratic polynaomials, we refer to see H.Iwaniec's paper [3].

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