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Blowup Phenomena for Nonlinear Dissipative Wave Equations

By

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Abstract

We study the initial-boundary value problem for the nonlinear wave equations with nonlinear dissipative terms : $\Box u + |u'|^\beta u' = |u|^\alpha u$ with $u(0) = u_0$, $u'(0) = u_1$, and $u|_{\partial\Omega} = 0$. When the initial energy $E(u_0, u_1) < 0$ and the inner product $(u_0, u_1) > 0$, the solution blows up at some finite time T which is estimated from above. On the other hand, when $0 \leq E(u_0, u_1) \ll 1$ and $u_0 \in \mathcal{W}_*$, the solution exists globally in time and has the energy decay $E(u(t), u'(t)) \leq c(1+t)^{-2/\beta}$ for $t \geq 0$.

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1. Introduction

In this paper we mainly investigate on the blowup phenomena to the initial-boundary value problem for the following nonlinear wave equations with nonlinear dissipative terms :

$$(0.1)\qquad \begin{cases} u'' + Au + \delta_1 u' + \delta_2 |u'|^\beta u' + \delta_3 Au' = |u|^\alpha u & \text{in} \quad \Omega \times [0, +\infty) \\ u(x,0) = u_0(x)\,, \quad u'(x,0) = u_1(x)\,, \quad \text{and} \quad u(x,t)|_{\partial\Omega} = 0\,, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $' = \partial_t \equiv \partial/\partial t$, $A = -\Delta \equiv \sum_{j=1}^N \partial^2/\partial x_j^2$ is the Laplace operator with the domain $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$, $\delta_1 \geq 0, \delta_2 \geq 0, \delta_3 \geq 0, \beta > 0$, and $\alpha > 0$ are constants. Let H be the usual real separable Hilbert space $L^2(\Omega)$ with norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$. We denote $L^p(\Omega)$-norm by $\|\cdot\|_p$ $(\|\cdot\| = \|\cdot\|_2)$.

We define the energy associated with Eq.(0.1) by

$$E(u,u') \equiv \|u'\|^2 + J(u)\,, \tag{0.2}$$

where we put

$$J(u) \equiv \|A^{1/2}u\|^2 - \frac{2}{\alpha+2}\|u\|_{\alpha+2}^{\alpha+2}\,, \tag{0.3}$$

and following Nakao and Ono [17], we introduce the K-positive set and the K-negative set :

$$\mathcal{W}_* \equiv \{u \in \mathcal{D}(A) \ : \ K(u) > 0\} \cup \{0\} \tag{0.4}$$

and

$$\mathcal{V}_* \equiv \{u \in \mathcal{D}(A) \ : \ K(u) < 0\}\,, \tag{0.5}$$

respectively, where we set

$$K(u) \equiv \|A^{1/2}u\|^2 - \|u\|_{\alpha+2}^{\alpha+2} \tag{0.6}$$

(cf. see [9, 20, 26]).

In the non-dissipative case ($\delta_1 = \delta_2 = \delta_3 = 0$), many authors have already studied on blowup solutions for the problem (0.1), see for example the works of [1–5, 9, 24, 26]. In particular, when $\alpha \leq 4/(N-2)$ ($\alpha < +\infty$ if $N = 1, 2$), we observe that the solution of (0.1) with $\delta_1 = \delta_2 = \delta_3 = 0$ can not be extended globally in time under the assumptions which $u_0 \in \mathcal{V}_*$ and $E(u_0, u_1) < d$ (i.e. $E(u_0, u_1) \ll 1$), where d is the so-called potential well depth, and that the solution can be extended globally in time under the assumptions which $u_0 \in \mathcal{W}_*$ and $0 \leq E(u_0, u_1) < d$ (e.g. see [20]).

In the case of $\delta_2 = 0$ in (0.1), Levine [10–12] has given an upper estimate of the blowup time T under $E(u_0, u_1) < 0$ by using the so-called concavity methods. We note that $u \in \mathcal{V}_*$ if $E(u, u') < 0$. Recently, when $\alpha \leq 4/(N-2)$ ($\alpha < +\infty$ if $N = 1, 2$), in the case of $\delta_1 > 0$ and $\delta_2 = \delta_3 = 0$ in (0.1), Ohta [18] has proved that the solution can not be extended globally in time under the assumptions which $u_0 \in \mathcal{V}_*$ and $E(u_0, u_1) < d$. On the other hand, we shall prove that the problem (0.1) admits a unique global solution, and that the solution has some decay properties under the assumptions which $u_0 \in \mathcal{W}_*$ and $0 \leq E(u_0, u_1) \ll 1$ without the assumption $\alpha \leq 4/(N-2)$ in Section 5.

In the case of $\delta_2 > 0$ and $\delta_1 = \delta_3 = 0$ in (0.1), Georgiev and Todorova [6] have proved that the solution can not be extended globally in time under the assumptions which the initial energy is sufficiently negative ($E(u_0, u_1) \ll -1$) and $\beta < \alpha \leq 2/(N-2)$ without any estimates of the blowup time T. Our purpose of the present paper is mainly to relax the assumption connected with the initial energy to $E(u_0, u_1) < 0$, and to derive an upper estimate of the blowup time T, where we treat the case of $\delta_1 \geq 0, \delta_2 > 0$, and $\delta_3 \geq 0$ in Section 2. Moreover, we also give the arranged proof for blowup results in the case of $\delta_2 = 0$ in Sections 3 and 4.

On the other hand, in the case of $\delta_1 + \delta_2 + \delta_3 > 0$ in (0.1), we show that the problem (0.1) admits a unique global solution, and that the solution and its energy have some decay properties under the assumptions which $u_0 \in \mathcal{W}_*$ and $0 \le E(u_0, u_1) \ll 1$. In particular, when $\delta_2 > 0$ in (0.1), the energy $E(u(t), u'(t))$ has some decay rate polynomially. When $\delta_1 + \delta_3 > 0$ in (0.1), the energy $E(u(t), u'(t))$ has some decay rate exponentially (see Section 5). Nakao [16] has studied the existence and decay properties of a unique global solution for the problem (0.1) with $-|u|^\alpha u$ (monotone) instead of $|u|^\alpha u$ in the case of $\delta_2 > 0$ and $\delta_1 = \delta_3 = 0$, but his results can not apply our problem immediately. Our results of the global in time solvability can apply to the problem (0.1) with $|u|^\alpha u$ replaced by the nonlinear function $f(u)$ such that $|f(u)| \le k_1 |u|^{\alpha+1}$ and $|f'(u)| \le k_2 |u|^\alpha$ with positive constants k_1 and k_2.

We denote $[a]^+ = \max\{0, a\}$ where $1/[a]^+ = +\infty$ if $[a]^+ = 0$.

1. Preliminaries

We give the local in time existence theorem for the problem (0.1) applying the Banach contraction mapping theorem.

Theorem 1. (Local Existence) *Let the initial data* $\{u_0, u_1\}$ *belong to* $\mathcal{D}(A) \times \mathcal{D}(A^{1/2})$. *Suppose that*

$$\alpha \le 2/(N-4) \quad (\alpha < +\infty \ \text{if}\ N \le 4).$$

Then the problem (0.1) *admits a unique local solution* u *belonging to*

$$C_w^0([0,T); \mathcal{D}(A)) \cap C_w^1([0,T); \mathcal{D}(A^{1/2})) \cap C^0([0,T); \mathcal{D}(A^{1/2})) \cap C^1([0,T); H)$$

for some $T = T(\|Au_0\|, \|A^{1/2}u_1\|) > 0$, *and* u *satisfies*

$$u' \in L^{\beta+2}((0,T) \times \Omega) \quad \text{if} \quad \delta_2 > 0, \tag{1.1}$$

$$u' \in L^2(0,T; \mathcal{D}(A^{1/2})) \quad \text{if} \quad \delta_3 > 0. \tag{1.2}$$

Moreover, if $\delta_2 = 0$, *then*

$$u \in C^0([0,T); \mathcal{D}(A)) \cap C^1([0,T); \mathcal{D}(A^{1/2})) \cap C^2([0,T); H). \tag{1.3}$$

Furthermore, at least one of the following statements is valid :

(i) $T = +\infty$

(ii) $\|Au(t)\|^2 + \|A^{1/2}u'(t)\|^2 \to +\infty \quad$ *as* $\quad t \to T-$.

PROOF. For $T > 0$ and $R > 0$, we define the two-parameter space $X_{T,R}$ of the solutions by

$$\begin{aligned} X_{T,R} \equiv \{v(t) \in & C_w^0([0,T]; \mathcal{D}(A)) \cap C_w^1([0,T]; \mathcal{D}(A^{1/2})) \cap C^0([0,T); \mathcal{D}(A^{1/2})) \\ & \cap C^1([0,T); H) \ : \ \|A^{1/2}v'(t)\|^2 + \|Av(t)\|^2 \le R^2 \text{ on } [0,T]\}. \end{aligned}$$

It is easy to see that $X_{T,R}$ can be organized as a complete metric space with the distance :

$$d(u,v) \equiv \sup_{0\le t\le T} \{\|u'(t)-v'(t)\|^2 + \|A^{1/2}(u(t)-v(t))\|^2\} .$$

We define a nonlinear mapping $\mathcal{S}$ as follows. For $v \in X_{T,R}, u = \mathcal{S}v$ is the unique solution of the following equations :

$$\begin{cases} u'' + Au' + \delta_1 u' + \delta_2 |u'|^\beta u' + \delta_3 Au' = |v|^\alpha v & \text{in} \quad \Omega \times [0,T] \\ u(0) = u_0 , \quad u'(0) = u_1 , \quad \text{and} \quad u|_{\partial\Omega} = 0 . \end{cases}$$

Using the fact that $(|u_1'|^\beta u_1' - |u_2'|^\beta u_2', u_1' - u_2') \ge 0$, we can prove that there exist $T > 0$ and $R > 0$ such that $\mathcal{S}$ maps $X_{T,R}$ into itself ; $\mathcal{S}$ is a contraction mapping with respect to the metric $d(\cdot,\cdot)$ (e.g. see Theorem 3.1 in [13], Theorem 2.1 in [6]). By applying the Banach contraction mapping theorem, we obtain a unique solution u belonging to $X_{T,R}$ of (0.1). Moreover, noting that $(|u'|^\beta u', u') = \|u'\|_{\beta+2}^{\beta+2}$ and $(Au', u') = \|A^{1/2}u'\|^2$, we get (1.1) and (1.2), respectively (see [23]). When $\delta_2 = 0$, by the continuity argument for wave equations (e.g. see [13, 22, 25]), we see that the solution u belongs to (1.3). We omit the detail here. □

In what follows, we put $E(t) = E(u(t), u'(t))$, $f(u) = |u|^\alpha u$, and $g(u') = |u'|^\beta u'$ for simplicity.

Multiplying Eq.(0.1) by $2u'$ (or u) and integrating it over Ω, we have immediately the following differential equalities associated with Eq.(0.1).

Lemma 1.1. *Let u be a solution of* (0.1). *Then*

$$\partial_t E(t) + 2\{\delta_1\|u'(t)\|^2 + \delta_2\|u'(t)\|_{\beta+2}^{\beta+2} + \delta_3\|A^{1/2}u'(t)\|^2\} = 0 \tag{1.4}$$

where $E(t) = E(u(t), u'(t))$, and

$$K(u(t)) = \|u'(t)\|^2 - \partial_t(u(t), u'(t)) - (\delta_1 u'(t) + \delta_2 g(u'(t)) + \delta_3 Au'(t), u(t)) \tag{1.5}$$

where $K(u) = \|A^{1/2}u\|^2 - \|u\|_{\alpha+2}^{\alpha+2}$ and $g(u') = |u'|^\beta u'$.

We see from (1.4) that

$$E(t) + 2\int_0^t \{\delta_1\|u'(s)\|^2 + \delta_2\|u'(s)\|_{\beta+2}^{\beta+2} + \delta_3\|A^{1/2}u'(s)\|^2\} ds = E(0) \tag{1.6}$$

where $E(0) = E(u_0, u_1)$.

To pull out blowup properties of solutions, we apply the concavity methods (see Levine [10–12]). We define the nonnegative function P by

$$\begin{aligned} P(t) \equiv & \|u(t)\|^2 + \int_0^t (\delta_1\|u(s)\|^2 + \delta_3\|A^{1/2}u(s)\|^2)\, ds \\ & + (T_0 - t)(\delta_1\|u_0\|^2 + \delta_3\|A^{1/2}u_0\|^2) + r(t+\tau)^2 \end{aligned} \tag{1.7}$$

for a solution $u(t), t \in [0, T_0]$, where $T_0 > 0, r \geq 0$, and $\tau > 0$ are some constants which are specified later on, then we observe the following properties.

Proposition 1.2. *The function $P(t)$ satisfies*

$$P''(t) = 2\{r - E(t) + 2\|u'(t)\|^2 + \frac{\alpha}{\alpha+2}\|u(t)\|_{\alpha+2}^{\alpha+2} - \delta_2(g(u'(t)), u(t))\} \tag{1.8}$$

with $g(u') = |u'|^\beta u'$, and

$$P(t)P''(t) - (\alpha/4 + 1)P'(t)^2 \geq P(t)Q(t), \tag{1.9}$$

where

$$\begin{aligned} Q(t) = &- (\alpha + 2)(r + E(0)) \\ &+ \alpha\{\|A^{1/2}u(t)\|^2 + \int_0^t (\delta_1\|u'(s)\|^2 + \delta_3\|A^{1/2}u'(s)\|^2)\, ds\} \\ &+ 2\delta_2\{(\alpha + 2)\int_0^t \|u'(s)\|_{\beta+2}^{\beta+2} ds - (g(u'(t)), u(t))\}. \end{aligned} \tag{1.10}$$

PROOF. Differentiating (1.7) with respect to t, we have

$$\begin{aligned} P'(t) &= 2(u(t), u'(t)) + (\delta_1\|u(t)\|^2 + \delta_3\|A^{1/2}u(t)\|^2) \\ &\quad - (\delta_1\|u_0\|^2 + \delta_3\|A^{1/2}u_0\|^2) + 2r(t + \tau) \\ &= 2\{(u(t), u'(t)) + \int_0^t (\delta_1(u(s), u'(s)) + \delta_3(A^{1/2}u(s), A^{1/2}u'(s)))\, ds \\ &\quad + r(t + \tau)\} \end{aligned} \tag{1.11}$$

and

$$\begin{aligned} P''(t) &= 2\{(u(t), u''(t) + \delta_1 u'(t) + \delta_3 Au'(t)) + \|u'(t)\|^2 + r\} \\ &= 2\{r + \|u'(t)\|^2 - K(u(t)) - \delta_2(g(u'(t)), u(t))\} \\ &= 2\{r - E(t) + 2\|u'(t)\|^2 + \frac{\alpha}{\alpha+2}\|u(t)\|_{\alpha+2}^{\alpha+2} - \delta_2(g(u'(t)), u(t))\}, \end{aligned} \tag{1.12}$$

which implies the desired equality (1.8). Next, we set

$$\begin{aligned} R(t) \equiv &\{\|u(t)\|^2 + \int_0^t (\delta_1\|u(s)\|^2 + \delta_3\|A^{1/2}u(s)\|^2)\, ds + r(t + \tau)^2\} \times \\ &\times \{\|u'(s)\|^2 + \int_0^t (\delta_1\|u'(s)\|^2 + \delta_3\|A^{1/2}u'(s)\|^2)\, ds + r\} \\ &- \{(u(t), u'(t)) + \int_0^t (\delta_1(u(s), u'(s)) + \delta_3(A^{1/2}u(s), A^{1/2}u'(s)))\, ds \\ &+ r(t + \tau)\}^2, \end{aligned}$$

then we see $R(t) \geq 0$ and

$$R(t) =\{P(t) - (T_0 - t)(\delta_1\|u_0\|^2 + \delta_3\|A^{1/2}u_0\|^2)\}$$
$$\times \{\|u'(t)\|^2 + \int_0^t (\delta_1\|u'(s)\|^2 + \delta_3\|A^{1/2}u'(s)\|^2)\, ds + r\} - (1/4)P'(t)^2$$

or

(1.13)
$$P'(t) =4[\{P(t) - (T_0 - t)(\delta_1\|u_0\|^2 + \delta_3\|A^{1/2}u_0\|^2)\}$$
$$\times \{\|u'(t)\|^2 + \int_0^t (\delta_1\|u'(s)\|^2 + \delta_3\|A^{1/2}u'(s)\|^2)\, ds + r\} - R(t)]\,.$$

Thus it follows from (1.13) that

$$P(t)P''(t) - (\alpha/4 + 1)P'(t)^2$$
$$\geq P(t)[P''(t) - (\alpha + 4)\{\|u'(t)\|^2$$
$$+ \int_0^t (\delta_1\|u'(s)\|^2 + \delta_3\|A^{1/2}u'(s)\|^2)\, ds + r\}]\,.$$

To derive (1.9), we shall show that the above $[\cdots]$ is equal to $Q(t)$.

$$[\cdots] = 2\{r + \|u'(t)\|^2 - K(u(t)) - \delta_2(g(u'(t)), u(t))\}$$
$$- (\alpha + 4)\{r + \|u'(t)\|^2 + \int_0^t (\delta_1\|u'(s)\|^2 + \delta_3\|A^{1/2}u'(s)\|^2)\, ds\}$$
$$= -(\alpha + 2)(r + \|u'(t)\|^2) + 2\{-K(u(t)) - \delta_2(g(u'(t)), u(t))\}$$
$$- (\alpha + 4) \int_0^t (\delta_1\|u(s)\|^2 + \delta_3\|A^{1/2}u(s)\|^2)\, ds$$
$$= -(\alpha + 2)\{r + E(t) + 2 \int_0^t (\delta_1\|u'(s)\|^2 + \delta_2\|u'(s)\|_{\beta+2}^{\beta+2}$$
$$+ \delta_3\|A^{1/2}u'(s)\|^2)\, ds\}$$
$$+ \alpha\{\|A^{1/2}u(t)\|^2 + \int_0^t (\delta_1\|u'(s)\|^2 + \delta_3\|A^{1/2}u'(s)\|^2)\, ds\}$$
$$+ 2\delta_2\{(\alpha + 2) \int_0^t \|u'(s)\|_{\beta+2}^{\beta+2} ds - (g(u'(t)), u(t))\}\,,$$

and noting (1.6), it is equal to $Q(t)$. The proof of Proposition 1.2 is now completed. □

2. Blow Up I $(\delta_2 > 0)$

When $\delta_2 > 0$ $(\delta_1 \geq 0, \delta_3 \geq 0)$ in Eq.(0.1), we shall show that the solution $u(t)$ blows up at some finite time under the assumptions which $E(0) < 0$ and $(u_0, u_1) > 0$ and $\alpha > \beta$. We denote by $|\Omega|$ the measure of Ω, and we assume $|\Omega| \geq 1$ for simplicity.

Our main result is as follows.

Theorem 2. ($\delta_2 > 0$) *Let $\delta_2 > 0$ in (0.1). Suppose that $\alpha > \beta$ and*

$$E(0) < 0 \quad \text{and} \quad G(0) \equiv (-E(0))^{\omega} + 2\omega m_0^{-1}(u_0, u_1) > 0 .$$

Then there exists a T such that

$$0 < T \le m_0 m_1 \omega (1-\omega)^{-1} G(0)^{-(1-\omega)/\omega}$$

and the local solution $u(t)$ in the sense of Theorem 1 blows up at the finite time T, where ω, m_0, and m_1 are positive constants such that

$$\omega = 1 - (\frac{1}{\beta+2} - \frac{1}{\alpha+2}) \quad (1/2 < \omega < 1),$$

$$m_0 = (2\delta_2(1+2/\alpha)|\Omega|^{\frac{\alpha-\beta}{\alpha+2}}(-E(0))^{-(1-\omega)})^{1/(\beta+1)},$$

$$m_1 = 2\max\Big\{1, (1+2/\alpha)(2|\Omega|^{\frac{\alpha}{2(\alpha+2)}} m_0^{-1})^{\frac{2}{2\omega-1}}(-E(0))^{-(1-\frac{2}{(2\omega-1)(\alpha+2)})}\Big\}.$$

PROOF. We put $r = 0$ in (1.7) and we shall estimate $P''(t)$ given by (1.8) with $r = 0$. We have that for $g(u') = |u'|^{\beta}u'$

$$|\delta_2(g(u'), u)| \le \delta_2 \|u'\|_{\beta+2}^{\beta+1}\|u\|_{\beta+2} \le \delta_2 B_1 \|u'\|_{\beta+2}^{\beta+1}\|u\|_{\alpha+2}$$

$$= \delta_2 B_1 \|u'\|_{\beta+2}^{\beta+1}\|u\|_{\alpha+2}^{\frac{\alpha+2}{\beta+2}}\|u\|_{\alpha+2}^{-(\frac{\alpha+2}{\beta+2}-1)} \tag{2.1}$$

with $B_1 = |\Omega|^{\frac{\alpha-\beta}{(\alpha+2)(\beta+2)}}$. Since we see from (0.2), (0.3), and (1.6) that

$$\|u(t)\|_{\alpha+2} \ge (-E(t))^{1/(\alpha+2)} \ge (-E(0))^{1/(\alpha+2)} > 0 \tag{2.2}$$

and from the Young inequality that

$$\delta_2 B_1 \|u'\|_{\beta+2}^{\beta+1}\|u\|_{\alpha+2}^{\frac{\alpha+2}{\beta+2}} \le \frac{\beta+1}{\beta+2}(\varepsilon^{-1}\delta_2 B_1)^{\frac{\beta+2}{\beta+1}}\|u'\|_{\beta+2}^{\beta+2} + \frac{\varepsilon^{\beta+2}}{\beta+2}\|u\|_{\alpha+2}^{\alpha+2}$$

for any $\varepsilon > 0$, we observe from (2.1) that

$$\begin{aligned}\delta_2|(g(u'(t)), u(t))| \le\ & (\varepsilon^{-1}\delta_2 B_1)^{\frac{\beta+2}{\beta+1}}(-E(t))^{-(\frac{1}{\beta+2}-\frac{1}{\alpha+2})}\|u'(t)\|_{\beta+2}^{\beta+2} \\ & + \varepsilon^{\beta+2}(-E(0))^{-(\frac{1}{\beta+2}-\frac{1}{\alpha+2})}\|u(t)\|_{\alpha+2}^{\alpha+2}\end{aligned}$$

if $\alpha > \beta$, and we obtain from (1.8) with $r = 0$ that

$$\begin{aligned}P''(t) \ge\ & 2\{(-E(t)) + 2\|u'(t)\|^2 + \frac{\alpha}{\alpha+2}\|u(t)\|_{\alpha+2}^{\alpha+2} \\ & - (\varepsilon^{-1}\delta_2 B_1)^{\frac{\beta+2}{\beta+1}}(-E(t))^{-(\frac{1}{\beta+2}-\frac{1}{\alpha+2})}\|u'(t)\|_{\beta+2}^{\beta+2} \\ & - \varepsilon^{\beta+2}(-E(0))^{-(\frac{1}{\beta+2}-\frac{1}{\alpha+2})}\|u(t)\|_{\alpha+2}^{\alpha+2}\} \\ \ge\ & 2\{(-E(t)) + 2\|u'(t)\|^2 + \frac{\alpha}{2(\alpha+2)}\|u(t)\|_{\alpha+2}^{\alpha+2} \\ & - \delta_2 m_0(-E(t))^{-(\frac{1}{\beta+2}-\frac{1}{\alpha+2})}\|u'(t)\|_{\beta+2}^{\beta+2}\},\end{aligned} \tag{2.3}$$

where we put $\varepsilon^{\beta+2}=(\alpha/2)(\alpha+2)^{-1}(-E(0))^{(\frac{1}{\beta+2}-\frac{1}{\alpha+2})}$ (>0 if $E(0)<0$) and $m_0=(2\delta_2(1+2/\alpha)B_1^{\beta+2}(-E(0))^{-(\frac{1}{\beta+2}-\frac{1}{\alpha+2})})^{1/(\beta+1)}$.

We introduce the function $G(t)$ as

$$G(t)\equiv(-E(t))^{\omega}+\omega m_0^{-1}P'(t)$$

with $\omega=1-(\frac{1}{\beta+2}-\frac{1}{\alpha+2})$ $(1/2<\omega<1)$, then we observe the following.

Claim A. *If $E(0)<0$ and $\alpha>\beta$, then*

$$G'(t)\geq m_0^{-1}H(t)\,, \tag{2.4}$$

where

$$H(t)\equiv(-E(t))+2\|u'(t)\|^2+\frac{\alpha}{2(\alpha+2)}\|u(t)\|_{\alpha+2}^{\alpha+2}\quad(>0)\,. \tag{2.5}$$

Indeed, we see from (1.4) and (2.3) that

$$\begin{aligned}
G'(t)&=\omega(-E(t))^{-(1-\omega)}(-E'(t))+\omega m_0^{-1}P''(t)\\
&\geq 2\omega(-E(t))^{-(1-\omega)}(\delta_1\|u'(t)\|^2+\delta_2\|u'(t)\|_{\beta+2}^{\beta+2}+\delta_3\|A^{1/2}u'(t)\|^2)\\
&\quad+2\omega m_0^{-1}\{(-E(t))+2\|u'(t)\|^2+\frac{\alpha}{2(\alpha+2)}\|u(t)\|_{\alpha+2}^{\alpha+2}\\
&\quad-\delta_2 m_0(-E(t))^{-(1-\omega)}\|u'(t)\|_{\beta+2}^{\beta+2}\}\\
&\geq 2\omega m_0^{-1}\{(-E(t))+2\|u'(t)\|^2+\frac{\alpha}{2(\alpha+2)}\|u(t)\|_{\alpha+2}^{\alpha+2}\}\geq m_0^{-1}H(t)\,,
\end{aligned}$$

where we used the fact $1/2<\omega<1$, which implies (2.4).

Moreover, we observe the following.

Claim B. *If $E(0)<0$ and $\alpha>\beta$, then*

$$G(t)^{1/\omega}\leq m_1H(t) \tag{2.6}$$

with $m_1=2\max\{1\,,\,(1+2/\alpha)(2B_2m_0^{-1})^{\frac{2}{2\omega-1}}(-E(0))^{-(1-\frac{2}{(2\omega-1)(\alpha+2)})}\}$ *and* $B_2=|\Omega|^{\frac{\alpha}{2(\alpha+2)}}$.

Indeed, Since $|(u',u)|\leq B_2\|u'\|\|u\|_{\alpha+2}$, we have that

$$\begin{aligned}
G(t)^{1/\omega}&\leq 2\{(-E(t))+(m_0^{-1}|P'(t)|)^{1/\omega}\}\\
&\leq 2\{(-E(t))+(2B_2m_0^{-1}\|u'(t)\|\|u(t)\|_{\alpha+2})^{1/\omega}\}\\
&\leq 2\{(-E(t))+2\|u'(t)\|^2+(1/2)(2B_2m_0^{-1}\|u(t)\|_{\alpha+2})^{2/(2\omega-1)})\}\,,
\end{aligned}$$

where we used the Young inequality. Moreover, since $2/(2\omega - 1) < \alpha + 2$ and

$$(-E(0))^{-1/(\alpha+2)}\|u(t)\|_{\alpha+2} \geq 1 \quad \text{if} \quad E(0) < 0$$

(see (2.2)), we observe that

$$\begin{aligned} G(t)^{1/\omega} \leq 2\{&(-E(t)) + 2\|u'(t)\|^2 \\ &+ (1/2)(2B_2 m_0^{-1})^{\frac{2}{2\omega-1}}(-E(0))^{-(1-\frac{2}{(2\omega-1)(\alpha+2)})}\|u(t)\|_{\alpha+2}^{\alpha+2}\}, \end{aligned}$$

and hence, we obtain (2.6).

Therefore it follows from Claim A and Claim B that

$$\partial_t\{G(t)^{1-1/\omega}\} = -\frac{1-\omega}{\omega}G(t)^{-1/\omega}G'(t) \leq -\frac{1-\omega}{\omega}(m_0 m_1)^{-1},$$

and hence,

$$G(t) \geq \{G(0)^{-(1-\omega)/\omega} - \frac{1-\omega}{\omega}(m_0 m_1)^{-1}t\}^{-\omega/(1-\omega)}$$

for some $t > 0$ if $G(0) > 0$. Here, we put $T_0 \equiv m_0 m_1 \omega(1-\omega)^{-1}G(0)^{-(1-\omega)/\omega}$. Then there exists a T such that $0 < T \leq T_0$ and $\lim_{t\to T^-} G(t) = +\infty$.

Since it follows from (0.2) and (0.3) that $(-E(t)) + \|u'(t)\|^2 \leq 2(\alpha+2)^{-1}\|u(t)\|_{\alpha+2}^{\alpha+2}$, we have from (2.5) and (2.6) that $G(t)^{1/\omega} \leq \text{Const.}\|u(t)\|_{\alpha+2}^{\alpha+2}$. Thus, we see that $\lim_{t\to T^-}\|u(t)\|_{\alpha+2}^2 = \lim_{t\to T^-}\{\|Au(t)\|^2 + \|A^{1/2}u'(t)\|^2\} = +\infty$, and hence, the local solution $u(t)$ can not be continued to the finite time T. The proof of Theorem 2 is now completed. □

Remark 2.1. Since $G'(t) \geq m_0^{-1}H(t) \geq m_0^{-1}(-E(0)) > 0$, there exits a $t_0 > 0$ such that $G(t) > 0$ for $t \geq t_0$, and hence, we see that if $\alpha > \beta$ and $E(0) < 0$ (without $G(0) > 0$), the local solution blows up at some finte time.

3. Blow Up II ($\delta_2 = 0$)

When $\delta_2 = 0$ ($\delta_1 \geq 0, \delta_3 \geq 0$) in Eq.(0.1), we shall show that the solution blows up at some finite time under the assumptions which $E(0) < 0$, or $E(0) = 0$ and $(u_0, u_1) > 0$ (see [10–12]).

Our results are as follows.

Theorem 3. ($\delta_2 = 0$) *Let* $\delta_2 = 0$ *in* (0.1). *Suppose that*

$$E(0) < 0.$$

Then there exists a T *such that*

$$\begin{aligned} 0 < T \leq &\alpha^{-2}(-E(0))^{-1}[\{(2\delta_1\|u_0\|^2 + 2\delta_3\|A^{1/2}u_0\|^2 - \alpha(u_0, u_1))^2 \\ &+ \alpha^2(-E(0))\|u_0\|^2\}^{1/2} + 2\delta_1\|u_0\|^2 + 2\delta_3\|A^{1/2}u_0\|^2 - \alpha(u_0, u_1)] \end{aligned}$$

and the local solution $u(t)$ in the sense of Theorem 1 blows up at the finite time T.

Theorem 4. ($\delta_2 = 0$) *Let $\delta_2 = 0$ in* (0.1). *Suppose that*

$$E(0) = 0 \quad \textit{and} \quad (u_0, u_1) > 0 .$$

Then there exists a T such that

$$0 < T \leq 2\alpha^{-1}(u_0, u_1)^{-1}\|u_0\|^2$$

and the local solution $u(t)$ in the sense of Theorem 1 blows up at the finite time T.

Here, we denote the Sobolev-Poincaré constant by

$$c_{*,p} \equiv \sup\{\|v\|_p\|A^{1/2}v\|^{-1} \ : \ v \in \mathcal{D}(A^{1/2}) \, , \ v \neq 0\} \tag{3.1}$$

for $2 \leq p \leq 4/[N-2]^+$ ($2 \leq p < +\infty$ if $N = 2$).

Theorem 5. ($\delta_1 = \delta_2 = \delta_3 = 0$) *Let $\delta_1 = \delta_2 = \delta_3 = 0$ in* (0.1). *Suppose that*

$$E(0) \leq \alpha(\alpha+2)^{-1}c_{*,2}^{-2}\|u_0\|^2 \quad \textit{and} \quad (u_0, u_1) > 0 . \tag{3.2}$$

Then there exists a T such that

$$0 < T \leq 2\alpha^{-1}(u_0, u_1)^{-1}\|u_0\|^2 \tag{3.3}$$

and the local solution $u(t)$ in the sense of Theorem 1 blows up at the finite time T.

PROOF OF THEOREM 3 AND 4. We put $r = -E(0)$ (≥ 0) and $\delta_2 = 0$ in (1.7), then we see from (1.10) that

$$Q(t) = \alpha\{\|A^{1/2}u(t)\|^2 + \int_0^t (\delta_1\|u'(s)\|^2 + \delta_3\|A^{1/2}u'(s)\|^2)\,ds\} \geq 0$$

and from (1.9) that

$$(P(t)^{-\alpha/4})'' = -(\alpha/4)P(t)^{-(\alpha/4+2)}\{P(t)P''(t) - (\alpha/4+1)P'(t)^2\} \leq 0 ,$$

and hence,

$$P(t) \geq \{\frac{P(0)^{\alpha/4+1}}{4P(0) - \alpha P'(0)t}\}^{\alpha/4} \tag{3.4}$$

for some $t > 0$ if $P(0) > 0$.

Case I. When $E(0) < 0$, we choose $\tau > 0$ such that

$$P'(0) = 2\{(u_0, u_1) + (-E(0))\tau\} > 0 ,$$

and we take

$$T_0 = 4P(0)/(\alpha P'(0)) \quad (> 0).$$

Then we see that

$$T_0 = T(\tau) \equiv \frac{2\{\|u_0\|^2 + (-E(0))\tau^2\}}{\alpha\{(u_0, u_1) + (-E(0))\tau\} - 2(\delta_1\|u_0\|^2 + \delta_3\|A^{1/2}u_0\|^2)},$$

and we find that $T(\tau)$ takes a minimum at

$$\begin{aligned}\tau = \tau_0 \equiv & \alpha^{-2}(-E(0))^{-1}[\{(2\delta_1\|u_0\|^2 + 2\delta_3\|A^{1/2}u_0\|^2 - \alpha(u_0, u_1))^2 \\ & + \alpha^2(-E(0))\|u_0\|^2\}^{1/2} + 2\delta_1\|u_0\|^2 + 2\delta_3\|A^{1/2}u_0\|^2 - \alpha(u_0, u_1)].\end{aligned}$$

Here, we put

$$T_0 = \min_{\tau>0} T(\tau) = T(\tau_0).$$

Then, we see from (3.4) that there exists a T such that $0 < T \le T_0$ and

$$\text{(3.5)} \qquad \lim_{t \to T^-} \{\|u(t)\|^2 + \int_0^t (\delta_1\|u(s)\|^2 + \delta_3\|A^{1/2}u(s)\|^2)\, ds\} = +\infty,$$

that is, $\lim_{t\to T^-} \|A^{1/2}u(t)\| = +\infty$ if $\delta_3 > 0$ and $\lim_{t\to T^-} \|u(t)\| = +\infty$ if $\delta_3 = 0$, and hence, the local solution $u(t)$ can not be continued to the finite time T. The proof of Theorem 3 is now completed.

Case II. When $E(0) = 0$ and $(u_0, u_1) > 0$, we see that

$$P(0) > 0 \quad \text{and} \quad P'(0) > 0.$$

Putting $T_0 = 4P(0)/(\alpha P'(0))$ (> 0), we see from (3.4) that (3.5) holds for some $0 < T \le T_0$. The proof of Theorem 4 is now completed. □

PROOF OF THEOREM 5 We put $r = 0$ and $\delta_1 = \delta_2 = \delta_3 = 0$ in (1.7) i.e. $P(t) = \|u(t)\|^2$, then we see from (1.10) that

$$Q(t) = -(\alpha + 2)E(0) + \alpha\|A^{1/2}u(t)\|^2.$$

We assume that $(u_0, u_1) > 0$, then

$$P'(t) = 2(u(t), u'(t)) > 0$$

for near $t = 0$, that is, $P(t)$ is a increasing function and

$$0 < \|u_0\|^2 = P(0) \le P(t) = \|u(t)\|^2 \le c_{*,2}^2\|A^{1/2}u(t)\|^2$$

for near $t = 0$. Thus we obtain that

$$Q(t) \ge -(\alpha + 2)E(0) + \alpha c_{*,2}^{-2}\|u_0\|^2 \ge 0$$

if $E(0) \le \alpha(\alpha+2)^{-1}c_{*,2}^{-2}\|u_0\|^2$. Then it follows from (1.9) that

$$(P(t)^{-\alpha/4})'' = -(\alpha/4)P(t)^{-(\alpha/4+2)}\{P(t)P''(t) - (\alpha/4+1)P'(t)^2\} \le 0$$

and

$$(P(t)^{-\alpha/4})' = -(\alpha/4)P(t)^{-(\alpha/4+1)}P'(t) < 0$$

for near $t = 0$. Thus we have that

$$\partial_t\{(P(t)^{-\alpha/4})'\}^2 = 2(P(t)^{-\alpha/4})''(P(t)^{-\alpha/4})' \ge 0\,,$$

and hence,

$$\{(P(t)^{-\alpha/4})'\}^2 \ge \{(P(0)^{-\alpha/4})'\}^2 = \{-(\alpha/4)P(0)^{-(\alpha/4+1)}P'(0)\}^2 > 0$$

for near $t = 0$. Therefore, we conclude that $(P(t)^{-\alpha/4})'$ can not be change sign for $t \ge 0$, and we see that

$$P(t) > 0\,, \quad P'(t) > 0\,, \quad \text{and} \quad (P(t)^{-\alpha/4})'' \le 0$$

for $t \ge 0$. Putting $T_0 = 4P(0)/(\alpha P'(0))$ (> 0), we see from (3.4) that (3.5) with $\delta_1 = \delta_3 = 0$ holds for some $0 < T \le T_0$. The proof of Theorem 5 is now completed. □

4. Blow Up III ($\delta_2 = \delta_3 = 0$ & $\alpha \le 4/(N-2)$)

In this section, even if initial energy $E(0)$ is positive, we shall show that the solution for the problem (0.1) with $\delta_2 = \delta_3 = 0$ ($\delta_1 \ge 0$) can not be continued globally under the assumptions which $u_0 \in \mathcal{V}_*$ and $E(0) \ll 1$ and $\alpha \le 4/(N-2)$ ($\alpha < +\infty$ if $N = 1, 2$).

We observe the following useful results connected with the K-negative set $\mathcal{V}_*$.

Proposition 4.1. *Let u be a solution of Eq.(0.1). Suppose that*

$$\alpha \le 4/(N-2) \quad (\alpha < +\infty \text{ if } N = 1,2)\,,$$
$$u_0 \in \mathcal{V}_* \equiv \{u \in \mathcal{D}(A) \;:\; K(u) < 0\}\,, \tag{4.1}$$

and

$$E(0) < \alpha(\alpha+2)^{-1}c_{*,\alpha+2}^{-2(\alpha+2)/\alpha} \quad (\equiv D_*) \tag{4.2}$$

with a positive constant $c_{*,\alpha+2}$ given by (3.1). Then

$$K(u(t)) \equiv \|A^{1/2}u(t)\|^2 - \|u(t)\|_{\alpha+2}^{\alpha+2} < 0 \tag{4.3}$$

and

$$E(t) < D_* \le \alpha(\alpha+2)^{-1}\|A^{1/2}u(t)\|^2 \tag{4.4}$$

for $t \geq 0$ (*cf.* (3.2)).

PROOF. Since $E(t) \leq E(0)$ (see (1.6)), we get from (4.2) immediately that

$$E(t) < D_* . \tag{4.5}$$

Let

$$T \equiv \sup\{t \in [0, +\infty) \ : \ K(u(s)) < 0 \quad \text{for} \quad 0 \leq s < t\},$$

then we see $T > 0$ by (4.1) and $K(u(t)) < 0$ and $u(t) \neq 0$ for $0 \leq t < T$. If $T < +\infty$, then $K(u(T)) = 0$, and hence,

$$J(u(T)) = \frac{\alpha}{\alpha + 2} \|A^{1/2}u(T)\|^2 . \tag{4.6}$$

Now, when $K(u) < 0$ and $u \neq 0$, we see from (3.1) that

$$\|A^{1/2}u\|^2 < \|u\|_{\alpha+2}^{\alpha+2} \leq c_{*,\alpha+2}^{\alpha+2} \|A^{1/2}u\|^{\alpha+2}$$

for $\alpha \leq 4/(N-2)$ ($\alpha < +\infty$ if $N = 1, 2$), and hence,

$$\|A^{1/2}u\|^2 > c_{*,\alpha+2}^{-2(\alpha+2)/\alpha} \quad (> 0) . \tag{4.7}$$

Thus, we have from (4.7) and the continuity that

$$\|A^{1/2}u(T)\|^2 \geq c_{*,\alpha+2}^{-2(\alpha+2)/\alpha} . \tag{4.8}$$

Thus we get from (0.2), (4.6), and (4.8) that

$$E(T) \geq J(T) \geq \alpha(\alpha + 2)^{-1} \|A^{1/2}u(T)\|^2 \geq D_* ,$$

which contradicts (4.5), and hence, we see $T = +\infty$. Moreover, from (4.5) and (4.7) we obtain (4.4). □

When $\delta_1 = \delta_2 = \delta_3 = 0$ in (0.1) (non-dissipative case), we obtain the following result.

Theorem 6. ($\delta_1 = \delta_2 = \delta_3 = 0$) *Let* $\delta_1 = \delta_2 = \delta_3 = 0$ *in* (0.1). *Under the assumption of proposition 4.1, the local solution blows up at some finite time.*

Remark 4.2. If we assume that $(u_0, u_1) > 0$, then the conclusion of Theorem 3 holds true, that is, the local solution blows up at the finite time T given by (3.3).

PROOF. We put $r = 0$ and $\delta_1 = \delta_2 = \delta_3 = 0$ in (1.7) i.e. $P(t) = \|u(t)\|^2$, then we see from (1.12) that

$$P''(t) = 2\{\|u'(t)\|^2 - K(u(t))\} \tag{4.9}$$

$$\begin{aligned} &= (\alpha + 4)\|u'(t)\|^2 + \{\alpha\|A^{1/2}u(t)\|^2 - (\alpha + 2)E(t)\} \\ &\geq (\alpha + 4)\|u'(t)\|^2 , \end{aligned} \tag{4.10}$$

where we used (4.4) at the last inequality. Thus we have that

$$P''(t)P(t) - (\alpha/4+1)P'(t)^2 \\ \geq (\alpha+4)\{\|u'(t)\|^2\|u(t)\|^2 - (u(t),u'(t))^2\} \geq 0 \tag{4.11}$$

for $t \geq 0$.

On the other hand, we see from (4.10), (4.7), and (1.6) with $\delta_1 = \delta_2 = \delta_3 = 0$ that

$$P''(t) \geq \alpha\|A^{1/2}u(t)\|^2 - (\alpha+2)E(t) \\ \geq (\alpha+2)\{D_* - E(0)\} \equiv n_0 > 0\,,$$

where we used the assumption (4.2). Then we obtain that

$$P'(t) \geq P'(0) + n_0 t\,,$$

and hence, there exists t_0 such that

$$P'(t) = 2(u(t),u'(t)) > 0 \tag{4.12}$$

for $t \geq t_0$. Thus, from (4.11) and (4.12) we arrived at our conclusion by the argument as in Section 2. □

Theorem 7. ($\delta_1 > 0, \delta_2 = \delta_3 = 0$) *Let $\delta_1 > 0$ and $\delta_2 = \delta_3 = 0$ in* (0.1). *Under the assumption of Proposition 4.1, the local solution blows up at some finite time.*

Proof. Following Ohta [18], we shall prove the theorem. We put

$$\tilde{P}(t) \equiv \|u(t)\|^2\,,$$

then we see from (1.5) (cf. (4.9)) that

$$\tilde{P}''(t) + \delta_1\tilde{P}'(t) = 2(\|u'(t)\|^2 - K(u(t))) \\ = (\alpha+4)\|u'(t)\|^2 + \{\alpha\|A^{1/2}u(t)\|^2 - (\alpha+2)E(t)\} \\ \geq (\alpha+4)\|u'(t)\|^2 + (\alpha+2)\{D_* - E(t)\}\,, \tag{4.13}$$

where we used (4.7). Next, we put

$$H(t) \equiv \delta_1\tilde{P}'(t) - (\alpha/2+2)\{D_* - E(t)\}\,, \tag{4.14}$$

then we see from (1.4) with $\delta_2 = \delta_3 = 0$ and (4.13) that

$$H'(t) = \delta_1\tilde{P}''(t) + (\alpha/2+2)E'(t) \\ = \delta_1\tilde{P}''(t) - (\alpha+4)\delta_1\|u'(t)\|^2 \\ \geq -\delta_1^2\tilde{P}'(t) + \delta_1(\alpha+2)\{D_* - E(t)\} \\ \geq -\delta_1 H(t) + \delta_1(\alpha/2)\{D_* - E(0)\}\,,$$

where we used the fact $E(t) \leq E(0)$ (see (1.6)). Thus we get

$$H(t) \geq e^{-\delta_1 t}(H(0) - n_1) + n_1 ,$$

where $n_1 = (\alpha/2)\{D_* - E(0)\}$ (> 0 by (4.2)), and hence, there exists a t_1 such that

$$H(t) > 0 \qquad \text{for} \quad t \geq t_1 .$$

Therefore, it follows from (4.14) and (4.4) that

$$\delta_1 \tilde{P}'(t) > (\alpha/2 + 2)\{D_* - E(t)\} > 0 , \tag{4.15}$$

that is,

$$P(t) > 0 \quad \text{and} \quad P'(t) > 0 \quad \text{for} \quad t \geq t_1 .$$

On the other hand, we observe from (4.15) and (1.4) with $\delta_2 = \delta_3 = 0$ that

$$\begin{aligned}
&\partial_t\{(D_* - E(t))\tilde{P}(t)^{-(\alpha/4+1)}\} \\
&\quad = -E'(t)\tilde{P}(t)^{-(\alpha/4+1)} - (\alpha/4+1)(D_* - E(t))\tilde{P}'(t)\tilde{P}(t)^{-(\alpha/4+2)} \\
&\quad \geq -\{E'(t)\tilde{P}(t) + (\delta_1/2)\tilde{P}'(t)^2\}\tilde{P}(t)^{-(\alpha/4+2)} \\
&\quad = 2\delta_1\{\|u'(t)\|^2\|u(t)\|^2 - (u(t), u'(t))^2\}\tilde{P}(t)^{-(\alpha/4+2)} \geq 0 ,
\end{aligned}$$

and hence,

$$\{D_* - E(t)\} \geq n_2 \tilde{P}(t)^{\alpha/4+1} \tag{4.16}$$

for $t \geq t_1$, where $n_2 = \{D_* - E(t_1)\}P(t_1)^{-(\alpha/4+1)}$ (> 0 by (4.4)). Thus we have from (4.13) and (4.16) that

$$\tilde{P}''(t) + \delta_1\tilde{P}'(t) \geq n_2\tilde{P}(t)^{\alpha/4+1}$$

with $\tilde{P}(t) > 0$ and $\tilde{P}'(t) > 0$ for $t \geq t_1$, and hence, we conclude from Lemma 4.3 below that $\tilde{P}(t) = \|u(t)\|^2$ blows up at some finite time. The proof of Theorem 7 is now completed. □

Lemma 4.3. (see [14, 21]) *Let the function $P(t)$ satisfy*

$$P''(t) + \delta P'(t) \geq c_0 P(t)^{1+r} \tag{4.17}$$

for $t \geq 0$ with $\delta \geq 0, c_0 > 0, r > 0$, and $P(0) > 0$ and $P'(0) > 0$. Then $P(t)$ blows up at some finite time.

PROOF. We consider that the differential equation $Q'(t) = \varepsilon Q(t)^{1+r/2}$ for $Q(t) \in C^2([0, +\infty))$ and $0 < \varepsilon \ll 1$ with $Q(0) = P(0)$ (> 0). Then we see that $Q(t) = \{Q(0)^{-r/2} - (r/2)\varepsilon t\}^{-2/r}$ for some $t > 0$, and that $Q(t)$ blows up at some finite time T_0. Since

$$\varepsilon Q(0)^{1+r/2} (= Q'(0)) < P'(0)$$

for some small $\varepsilon > 0$, we have that

$$Q''(t) = \varepsilon(1 + r/2)Q(t)^{r/2}Q'(t) = \varepsilon^2(1 + r/2)Q(t)^{1+r},$$

and hence, from $Q(t) \geq Q(0)$,

$$\begin{aligned} Q''(t) + \delta Q'(t) &= \varepsilon^2(1 + r/2)Q(t)^{1+r} + \varepsilon\delta Q(t)^{1+r/2} \\ &\leq \{\varepsilon^2(1 + r/2) + \varepsilon\delta Q(0)^{-r/2}\}Q(t)^{1+r} \leq c_0 Q(t)^{1+r} \end{aligned} \tag{4.18}$$

for small $\varepsilon > 0$. Since $Q'(0) < P'(0)$, we see that $Q'(t) < P'(t)$ for near $t = 0$. Let

$$T \equiv \sup\{t \in [0, +\infty) \,:\, Q'(s) < P'(s) \text{ for } 0 \leq s < t\},$$

then we see $T > 0$ and $Q'(t) < P'(t)$ for $0 \leq t < T$ and $Q(t) < P(t)$ for $0 < t < T$. If $T < T_0$, then we observe that

$$Q'(T) = P'(T), \quad Q''(T) \geq P''(T), \quad \text{and} \quad Q(T) < P(T).$$

On the other hand, it follows from (4.17) and (4.18) that

$$(Q''(T) - P''(T)) + \delta(Q'(T) - P'(T)) \leq c_0(Q(T)^{1+r} - P(T)^{1+r}),$$

which is a contradiction, and hence, we see that $T \geq T_0$ and

$$Q(t) \leq P(t) \quad \text{for} \quad 0 \leq t \leq T_0.$$

Thus, $P(t)$ blows up at some finite time. □

5. Global Existence and Decay

In this section we shall study on the global in time existence and energy decay properties of the solution for Eq.(0.1) with $\delta_1 + \delta_2 + \delta_3 > 0$ under the assumptions that $0 \leq E(0) \equiv E(u_0, u_1) \ll 1$ and

$$u_0 \in \mathcal{W}_* \equiv \{u \in \mathcal{D}(A) \,:\, K(u) > 0\} \cup \{0\}.$$

We observe the following useful results connected with the K-positive set $\mathcal{W}_*$.

Proposition 5.1. (i) *If* $\alpha < 4/[N-4]^+$, *then*

$$\mathcal{W}_* \textit{ is a neighborhood of } 0 \textit{ in } \mathcal{D}(A^{1/2}) = H_0^1(\Omega) \textit{ and an open set.} \tag{5.1}$$

(ii) *If* $u \in \overline{\mathcal{W}_*}$, *then*

$$d_*^{-1}\|A^{1/2}u\|^2 \leq J(u) \quad (\leq E(u, u')) \tag{5.2}$$

where $d_* = (1 + 2\alpha^{-1})$ (≥ 1).

PROOF. We see from Lemma 5.2 below that

$$\|u\|_{\alpha+2}^{\alpha+2} \le c_*^{\alpha+2}\|A^{1/2}u\|^{\alpha-(\alpha+2)\theta_1}\|Au\|^{(\alpha+2)\theta_1}\|A^{1/2}u\|^2 , \tag{5.3}$$

where $\theta_1 = [(N-2)\alpha - 4]^+/(2(\alpha+2))$ and $\alpha - (\alpha+2)\theta_1 > 0$ if $\alpha < 4/[N-4]^+$, and hence, $K(u) > 0$ if $\mathcal{D}(A^{1/2})$-norm of u is sufficiently small and $u \neq 0$, which implies (5.1). From the definitions of $\mathcal{W}_*$ and $J(u)$, (5.2) follows immediately. □

We use well-known lemma without the proof.

Lemma 5.2. (Gagliardo-Nirenberg) *Let* $1 \le r < p \le +\infty$ *and* $p \ge 2$. *Then, the inequality*

$$\|v\|_p \le c_*\|A^{m/2}v\|^{\theta}\|v\|_r^{1-\theta} \quad \textit{for} \quad v \in \mathcal{D}(A^{m/2}) \cap L^r(\Omega)$$

holds with some constant c_* *and*

$$\theta = (\frac{1}{r} - \frac{1}{p})(\frac{1}{r} + \frac{m}{N} - \frac{1}{2})^{-1}$$

provided that $0 < \theta \le 1$ $(0 < \theta < 1$ *if* $m - N/2$ *is a nonnegative integer*).

(Sobolev-Poincaré) *Let* $1 \le p \le 2N/[N-2m]^+$ $(1 \le p < +\infty$ *if* $N = 2m)$. *Then, the inequality*

$$\|v\|_p \le c_*\|A^{m/2}v\| \quad \textit{for} \quad v \in \mathcal{D}(A^{m/2})$$

holds with some constant c_*.

Moreover, we use the inequality $\|u\| \le c_*\|u\|_p$ for $u \in L^p(\Omega), p \ge 2$, with some constant c_*. In what follows, we assume $c_* \ge 1$ for simplicity.

To state our results we define the second energy associated with Eq.(0.1) by

$$E_2(u, u') \equiv \|A^{1/2}u'\|^2 + \|Au\|^2 .$$

Then, multiplying Eq.(0.1) by $2Au'$ and integrating it over Ω, we have

$$\begin{aligned} \partial_t E_2(t) + 2\{\delta_1\|A^{1/2}u'(t)\|^2 &+ \delta_2(\beta+1)\int_\Omega |u'(t)|^\beta |A^{1/2}u'(t)|^2 dx \\ &+ \delta_3\|Au(t)\|^2\} = 2(f(u(t)), Au(t)) , \end{aligned} \tag{5.4}$$

where we put $E_2(t) \equiv E_2(u(t), u'(t))$ $(E_2(0) \equiv E_2(u_0, u_1))$ for simplicity.

In what follows, we denote by $c_j, j = 1, 2, \cdots$, constants independent of the initial data and depending only on $\alpha, \beta, N, c_*, \delta_1, \delta_2$, and δ_3.

Our results are as follows :

Theorem 8. $(\delta_2 > 0)$ *Let* $\delta_2 > 0$ *and* $\delta_1 = \delta_3 = 0$ *in* (0.1), *and let the initial data* $\{u_0, u_1\}$ *belong to* $\mathcal{W}_*$ $(\subset \mathcal{D}(A)) \times \mathcal{D}(A^{1/2})$. *Suppose that*

$$\alpha < 2/[N-4]^+ , \qquad \beta \le 4/(N-2) \quad (\beta < +\infty \textit{ if } N = 1, 2) ,$$
$$\beta < \alpha - [(N/2-1)\alpha - 1]^+ ,$$

and that the initial energy $E(0)$ *is small* ($0 \le E(0) \ll 1$ *but* $E_2(0) \ge 1$) *such that*
(i) *when* $\alpha \le 4/(N-2)$ ($\alpha < +\infty$ *if* $N \le 2$),

$$(0 \le) \quad c_1 E(0)^{\alpha/2} < 1 \quad \textit{and} \quad \omega_1 c_3 E(0)^{\omega_2} E_2(0)^{\omega_1} < 1\,, \tag{5.5}$$

(ii) *when* $4/(N-2) < \alpha < 2/[N-4]^+$ ($N \ge 3$),

$$(0 \le) \quad \{\omega_1 c_3 E(0)^{\omega_2} + c_4 E(0)^{\omega_3}\} E_2(0)^{\omega_1} < 1\,, \tag{5.6}$$

where $\omega_1 = [(N-2)\alpha]^+/4$ (≥ 0), $\omega_2 = (\alpha - \beta)/2 - \omega_1$ (> 0), *and* $\omega_3 = \omega_1(4-(N-4)\alpha)/((N-2)\alpha - 4)$ (> 0). *Then, the problem* (0.1) *admits a unique global solution* $u \in \mathcal{W}_*$ *satisfying*

$$\|u'(t)\|^2 + \|A^{1/2}u(t)\|^2 \le d_* E(t) \le c(1+t)^{-2/\beta} \tag{5.7}$$

for $t \ge 0$ *with a constant* c.

Theorem 9. ($\delta_1 + \delta_3 > 0$) *Let* $\delta_1 + \delta_3 > 0$ *and* $\delta_2 \ge 0$ *in* (0.1), *and let the initial data* $\{u_0, u_1\}$ *belong to* $\mathcal{W}_* \times \mathcal{D}(A^{1/2})$. *Suppose that*

$$\alpha < 2/[N-4]^+ \quad \textit{and} \quad \beta \le 4/(N-2) \quad (\beta < +\infty \ \textit{if}\ N = 1,2)\,,$$

and that the initial energy $E(0)$ *is small* ($0 \le E(0) \ll 1$ *but* $E_2(0) \ge 1$) *such that*
(i) *when* $\alpha \le 4/(N-2)$ ($\alpha < +\infty$ *if* $N = 1,2$),

$$(0 \le) \quad c_1 E(0)^{\alpha/2} < 1 \quad \textit{and} \quad \omega_1 c_5 E(0)^{\tilde{\omega}_2} E_2(0)^{\omega_1} < 1\,, \tag{5.8}$$

(ii) *when* $4/(N-2) < \alpha < 2/[N-4]^+$ ($N \ge 3$),

$$(0 \le) \quad \{\omega_1 c_5 E(0)^{\tilde{\omega}_2} + c_4 E(0)^{\omega_3}\} E_2(0)^{\omega_1} < 1\,. \tag{5.9}$$

where $\omega_1 = [(N-2)\alpha]^+/4$ (≥ 0), $\tilde{\omega}_2 = \alpha/2 - \omega_1$ (> 0), *and* $\omega_3 = \omega_1(4-(N-4)\alpha)/((N-2)\alpha - 4)$ (> 0). *Then, the problem* (0.1) *admits a unique global solution* $u \in \mathcal{W}_*$ *satisfying*

$$\|u'(t)\|^2 + \|A^{1/2}u(t)\|^2 \le d_* E(t) \le ce^{-kt} \tag{5.10}$$

for $t \ge 0$ *with constants* c *and* $k > 0$.

Remark 5.3. When we consider the problem (0.1) with $|u|^\alpha u$ replaced by the nonlinear function $f(u)$ such that

$$|f(u)| \le k_1 |u|^{\alpha+1} \quad \text{and} \quad |f'(u)| \le k_1 |u|^\alpha$$

with positive constants k_1 and k_2, we can get the similar results as Theorem 8 and Theorem 9. Then we need to redefine (0.3) and (0.6) by

$$J(u) \equiv \|A^{1/2}u\|^2 - 2\int_\Omega F(u)\,dx$$

with $F(u) = \int_0^u f(\eta)\,d\eta$ and

$$K(u) \equiv \|A^{1/2}u\|^2 - k_1\|u\|_{\alpha+2}^{\alpha+2},$$

respectively.

First, we shall prepare for those proof. We put

$$T_1 \equiv \sup\{t \in [0,+\infty) \ : \ u(s) \in \mathcal{W}_* \text{ for } 0 \le s < t\},$$

then we see $T_1 > 0$ and $u(t) \in \mathcal{W}_*$ for $0 \le t < T_1$ because $u_0 \in \mathcal{W}_*$ being an open set (see (5.1)). If $T_1 < +\infty$, then $u(T_1) \in \partial\mathcal{W}_*$, that is,

$$K(u(T_1)) = 0 \quad \text{and} \quad u(T_1) \neq 0. \tag{5.11}$$

We see from (1.6), (5.2), and (5.3) that

$$\|u(t)\|_{\alpha+2}^{\alpha+2} \le (1/2)B(t)\|A^{1/2}u(t)\|^2 \tag{5.12}$$

for $0 \le t \le T_1$ where

$$B(t) \equiv c_1 E(0)^{(\alpha-(\alpha+2)\theta_1)/2}\|Au(t)\|^{(\alpha+2)\theta_1} \tag{5.13}$$

with $c_1 = 2c_*^{\alpha+2}d_*^{(\alpha-(\alpha+2)\theta_1)/2}$.

We put

$$T_2 \equiv \sup\{t \in [0,+\infty) \ : \ B(s) < 1 \text{ for } 0 \le s < t\},$$

then we see $T_2 > 0$ and $B(t) < 1$ for $0 \le t < T_2$ because $B(0) < 1$ by (5.5), (5.6), (5.8), or (5.9). If $T_2 < T_1$ $(< +\infty)$, then

$$B(T_2) = 1, \tag{5.14}$$

and

$$K(u(t)) \ge \|A^{1/2}u(t)\|^2 - (1/2)B(t)\|A^{1/2}u(t)\|^2 \ge (1/2)\|A^{1/2}u(t)\|^2 \tag{5.15}$$

for $0 \le t \le T_2$.

PROOF OF THEOREM 8. Following Nakao [16], we shall derive the decay property of the energy $E(t) \equiv E(u(t), u'(t))$ associated with Eq.(0.1) with $\delta_2 > 0$ and $\delta_1 = \delta_3 = 0$. In what follows, we put $\delta_2 = 1$ without loss of generality.

For a moment, we assume that $T_2 > 1$. Integrating (1.4) with $\delta_2 = 1$ and $\delta_1 = \delta_3 = 0$ over $[t, t+1], 0 < t < T_2 - 1$, we have

$$2\int_t^{t+1} \|u'(s)\|_{\beta+2}^{\beta+2} ds = E(t) - E(t+1) \quad (\equiv 2D(t)^{\beta+2}) \tag{5.16}$$

and

$$\int_t^{t+1} \|u'(s)\|^2 ds \le c_*^2 \int_t^{t+1} \|u'(s)\|_{\beta+2}^2 ds \le c_*^2 D(t)^2 . \tag{5.17}$$

Then there exist $t_1 \in [t, t+1/4]$ and $t_2 \in [t+3/4, t+1]$ such that

$$\|u'(t_i)\| \le 2c_* D(t) \quad i = 1, 2 . \tag{5.18}$$

Since $|(g(u'), u)| \le \|u'\|_{\beta+2}^{\beta+1} \|u\|_{\beta+2}$, we see from (1.5) and (5.15) that

$$\begin{aligned}
&(1/2) \int_{t_1}^{t_2} \|A^{1/2} u(s)\|^2 ds \le \int_{t_1}^{t_2} K(u(s))\, ds \\
&\le \int_t^{t+1} \|u'(s)\|^2 ds + \sum_{i=1}^{2} \|u'(t_i)\| \|u(t_i)\| + \int_t^{t+1} \|u'(s)\|_{\beta+2}^{\beta+1} \|u(s)\|_{\beta+2} ds \\
&\le \int_t^{t+1} \|u'(s)\|^2 ds + c_* \{ \sum_{i=1}^{2} \|u'(t_i)\| \\
&\quad + \int_t^{t+1} \|u'(s)\|_{\beta+2}^{\beta+1} ds \} \sup_{t \le s \le t+1} \|A^{1/2} u(s)\| ,
\end{aligned} \tag{5.19}$$

where we used the fact that $\|u\|_{\beta+2} \le c_* \|A^{1/2} u\|$ for $\beta \le 4/(N-2)$. Integrating (1.4) over $[t, t_2]$, we have from (5.19) that

$$\begin{aligned}
E(t) &= E(t_2) + 2 \int_t^{t_2} \|u'(s)\|_{\beta+2}^{\beta+2} ds \\
&\le 2 \int_{t_1}^{t_2} E(s)\, ds + 2 \int_t^{t+1} \|u'(s)\|_{\beta+2}^{\beta+2} ds \\
&\le 2 \int_t^{t+1} \{ \|u'(s)\|^2 + \|u'(s)\|_{\beta+2}^{\beta+2} \} ds + 2 \int_{t_1}^{t_2} \|A^{1/2} u(s)\|^2 ds \\
&\le 2 \int_t^{t+1} \{ 3\|u'(s)\|^2 + \|u'(s)\|_{\beta+2}^{\beta+2} \} ds \\
&\quad + 4c_* \{ \sum_{i=1}^{2} \|u'(t_i)\| + (\int_t^{t+1} \|u'(s)\|_{\beta+2}^{\beta+2} ds)^{\frac{\beta+1}{\beta+2}} \} \sup_{t \le s \le t+1} \|A^{1/2} u(s)\| ,
\end{aligned}$$

and from (5.16), (5.17), and (5.18) that

$$E(t) \le 2\{ 3c_*^2 D(t)^2 + D(t)^{\beta+2} \} + 4c_* \{ 4c_* D(t) + D(t)^{\beta+1} \} (d_* E(t))^{1/2} .$$

Since $2D(t)^{\beta+2} \le E(t) \le E(0) \le 1$, we see

$$E(t) \le 2^8 c_*^4 d_* D(t)^2 + (1/2) E(t) ,$$

and hence,

$$E(t)^{1+\beta/2} \le (2^9 c_*^4 d_*)^{(\beta+2)/2} D(t)^{\beta+2}$$
$$\le 2^{-1}(2^9 c_*^4 d_*)^{(\beta+2)/2}\{E(t) - E(t+1)\}.$$

Thus, noting the fact $E(t) \le E(0)$ and applying Lemma 5.4 below, we obtain the following energy decay estimate :

$$E(t) \le \{E(0)^{-\beta/2} + d_0^{-1}[t-1]^+\}^{-2/\beta} \tag{5.20}$$

for $0 \le t \le T_2$ with $d_0 = \beta^{-1}(2^9 c_*^4 d_*)^{(\beta+2)/2}$ (≥ 1).

Next, using the energy decay (5.20), we shall estimate the second energy $E_2(t) \equiv E_2(u(t), u'(t))$. It follows from (5.4) and Lemma 5.2 that

$$\partial_t E_2(t) \le 2(f(u(t)), Au'(t)) \le 2c_*(\alpha+1)\|u(t)\|_{N\alpha}^{\alpha}\|Au(t)\|\|A^{1/2}u'(t)\|$$
$$\le 2c_*^{\alpha+1}(\alpha+1)\|A^{1/2}u(t)\|^{\alpha(1-\theta_2)}\|Au(t)\|^{\alpha\theta_2+1}\|A^{1/2}u'(t)\|$$
$$\le c_2 E(t)^{\alpha(1-\theta_2)/2} E_2(t)^{\omega_1+1}, \tag{5.21}$$

where $c_2 = 2c_*^{\alpha+1}(\alpha+1)d_*^{\alpha(1-\theta_2)/2}$, $\theta_2 = [(N-2)\alpha - 2]^+/(2\alpha)$, and $\omega_1 = \alpha\theta_2/2$. We observe from (5.20) that if $\alpha(1-\theta_2) > \beta$,

$$\int_0^t c_2 E(s)^{\alpha(1-\theta_2)/2} ds = \int_0^1 + \int_1^t \le c_3 E(0)^{\omega_2} \tag{5.22}$$

with $c_3 = c_2 d_0(\alpha(1-\theta_2))/(\alpha(1-\theta_2)-\beta)$ and $\omega_2 = (\alpha(1-\theta_2)-\beta)/2$.

When $\alpha \le 2/(N-2)$ (i.e. $\omega_1 = 0$), we have from (5.21) and (5.22) that

$$E_2(t) \le E_2(0)\exp\{\int_0^t c_2 E(s)^{\alpha(1-\theta_2)/2} ds\}$$
$$\le E_2(0)\exp\{c_3 E(0)^{\omega_2}\} \quad (< +\infty). \tag{5.23}$$

On the other hand, when $\alpha > [N-2]^+$ (i.e. $\omega_1 > 0$), we have

$$E_2(t) \le \{E_2(0)^{-\omega_1} - \omega_1 \int_0^t c_2 E(s)^{\alpha(1-\theta_2)/2} ds\}^{-1/\omega_1}$$
$$\le \{E_2(0)^{-\omega_1} - \omega_1 c_3 E(0)^{\omega_2}\}^{-1/\omega_1} \quad (< +\infty) \tag{5.24}$$

if $\omega_1 c_3 E(0)^{\omega_2} E_2(0)^{\omega_1} < 1$.

When $\alpha \le 4/(N-2)$ (i.e. $\theta_1 = 0$), we have from (5.5) and (5.13) that

$$B(t) = c_1 E(0)^{\alpha/2} < 1. \tag{5.25}$$

On the other hand, when $\alpha > 4/[N-2]^+$ (i.e. $\theta_1 > 0$), we have from (5.13) and (5.24) that

$$B(t) \le c_1 E(0)^{1-(N-4)\alpha/4} E_2(t)^{(N-2)\alpha/4-1}$$
$$\le c_1 E(0)^{1-(N-4)\alpha/4}\{E_2(0)^{-\omega_1} - \omega_1 c_3 E(0)^{\omega_2}\}^{-\frac{(N-2)\alpha-4}{4\omega_1}} < 1 \tag{5.26}$$

if we assume (5.6), that is,

$$\{\omega_1 c_3 E(0)^{\omega_2} + c_4 E(0)^{\omega_3}\} E_2(0)^{\omega_1} < 1$$

with $c_4 = c_1^{4\omega_1/((N-2)\alpha-4)}$ and $\omega_3 = \omega_1(4-(N-4)\alpha)/((N-2)\alpha-4)$. Thus we conclude that (5.25) and (5.26) contradict (5.14), and hence, we see that $T_2 \geq T_1$. Moreover, we observe from (5.11) and (5.15) that

$$0 = K(u(T_1)) \geq (1/2)\|A^{1/2}u(T_1)\|^2 > 0\,,$$

which is a contradiction, and hence, we see that $T_1 = +\infty$, that is, (5.20), (5.23), and (5.24) hold true for all $t \geq 0$. The proof of Theorem 8 is now completed. □

We used the following useful lemma in the proof of Theorem 8. (We omit the proof here, see [15, 17].)

Lemma 5.4. (Nakao [15]) *Let ϕ be a bounded and nonnegative function on $[0, +\infty)$ satisfying*

$$\sup_{t\leq s\leq t+1} \phi(s)^{1+r} \leq k\{\phi(t) - \phi(t+1)\}$$

for $t > 0$ and $k > 0$. Then

$$\phi(t) \leq \{\phi(0)^{-r} + rk^{-1}[t-1]^+\}^{-1/r} \quad \textit{for} \quad t \geq 0\,.$$

PROOF OF THEOREM 9. From (1.5), we have

$$\begin{aligned}&\partial_t\{2(u(t), u'(t)) + \delta_1\|u(t)\|^2 + \delta_3\|A^{1/2}u(t)\|^2\}\\ &\quad = 2\|u'(t)\|^2 - 2K(u(t)) - 2(g(u'(t)), u(t))\,,\end{aligned}$$

and hence, from this and (1.4), we have

$$\begin{aligned}\partial_t E^*(t) = &-2\{(\delta_1 - \varepsilon)\|u'(t)\|^2 + \delta_2\|u'(t)\|_{\beta+2}^{\beta+2} + \delta_3\|A^{1/2}u'(t)\|^2\}\\ &- 2\varepsilon K(u(t)) - 2\varepsilon\delta_2(g(u'(t)), u(t))\end{aligned} \tag{5.27}$$

for $< \varepsilon < 1$, where we set

$$E^*(t) \equiv E(t) + \varepsilon\{2(u(t), u'(t)) + \delta_1\|u(t)\|^2 + \delta_3\|A^{1/2}u(t)\|^2\}\,. \tag{5.28}$$

Then we see that for

$$(2d_*)^{-1}(\|u'(t)\|^2 + \|A^{1/2}u(t)\|^2) \leq E^*(t) \leq 2(\|u'(t)\|^2 + \|A^{1/2}u(t)\|^2)\,. \tag{5.29}$$

if $\varepsilon \leq (2d_*(c_* + c_*^2\delta_1 + \delta_3))^{-1}$. Indeed, since

$$d_*^{-1}(\|u'(t)\|^2 + \|A^{1/2}u(t)\|^2) \leq E(t) \leq \|u'(t)\|^2 + \|A^{1/2}u(t)\|^2 \tag{5.30}$$

by (5.2) and

$$\begin{aligned}
&|2(u,u') + \delta_1\|u\|^2 + \delta_3\|A^{1/2}u\|^2| \\
&\quad\le 2c_*\|A^{1/2}u\|\|u'\| + c_*^2\delta_1\|A^{1/2}u\|^2 + \delta_3\|A^{1/2}u\|^2 \\
&\quad\le (c_* + c_*^2\delta_1 + \delta_3)(\|u'\|^2 + \|A^{1/2}u\|^2),
\end{aligned}$$

we see (5.29) immediately.

To proceed the estimation of (5.27), we observe from (1.6) and (5.2) that

$$\begin{aligned}
&|\delta_2(g(u'),u)| \le \delta_2\|u\|_{\beta+2}\|u'\|_{\beta+2}^{\beta+1} \\
&\le \delta_2 c_*\|A^{1/2}u\|\|u'\|_{\beta+2}^{\beta+1}, \quad \beta\le 4/(N-2) \\
&= \delta_2 c_*\|A^{1/2}u\|^{\frac{\beta}{\beta+2}}\|A^{1/2}u\|^{\frac{2}{\beta+2}}\|u'\|_{\beta+2}^{\beta+1} \\
&\le \delta_2 c_*(d_*E(0))^{\frac{\beta}{2(\beta+2)}}\|u'\|_{\beta+2}^{\beta+1}\|A^{1/2}u\|^{\frac{2}{\beta+2}} \\
&\le \frac{\beta+1}{\beta+2}(\delta_2 c_*(d_*E(0))^{\frac{\beta}{2(\beta+2)}})^{\frac{\beta+2}{\beta+1}}\|u'\|_{\beta+2}^{\beta+2} + \frac{1}{\beta+2}\|A^{1/2}u\|^2 \\
&\le (\delta_2 c_* d_*)^{\frac{\beta+2}{\beta+1}}\|u'\|_{\beta+2}^{\beta+2} + (1/2)\|A^{1/2}u\|^2,
\end{aligned}$$

and hence,

$$\begin{aligned}
\partial_t E^*(t) &\le -2(\delta_1 + c_*^{-2}\delta_3 - \varepsilon)\|u'(t)\|^2 - \varepsilon\|A^{1/2}u(t)\|^2 \\
&\quad - 2(\delta_2 - \varepsilon(\delta_2 c_* d_*)^{\frac{\beta+2}{\beta+1}})\|u'(t)\|_{\beta+2}^{\beta+2} \\
&\le -2\varepsilon(\|u'(t)\|^2 + \|A^{1/2}u(t)\|^2), \qquad (5.31)
\end{aligned}$$

where we used (5.15) and we put

$$\varepsilon = \min\{(\delta_1 + c_*^{-2}\delta_3)/2,\ \delta_2(\delta_2 c_* d_*)^{-\frac{\beta+2}{\beta+1}},\ (2d_*(c_* + c_*^2\delta_1 + \delta_3))^{-1}\}$$

(We note that $\varepsilon > 0$ by $\delta_1 + \delta_3 > 0$). Thus we obtain from (5.29), (5.30), and (5.31) that

$$E^*(t) \le E^*(0)e^{-\varepsilon t}$$

or

$$E(t) \le \|u'(t)\|^2 + \|A^{1/2}u(t)\|^2 \le (2d_*)^2E(0)e^{-\varepsilon t} \qquad (5.32)$$

for $0 \le t \le T_2$.

Next, using the decay (5.32), we shall estimate the second energy $E_2(t)$. It follows from (5.4) and (5.21) that

$$\partial_t E_2(t) \le c_2 E(t)^{\alpha(1-\theta_2)/2}E_2(t)^{\omega_1+1}. \qquad (5.33)$$

We observe from (5.32) that if $\alpha(1-\theta_2) > \beta$,

$$\int_0^t c_2 E(s)^{\alpha(1-\theta_2)/2} ds \le c_5 E(0)^{\tilde{\omega}_2} \tag{5.34}$$

with $c_5 = c_2(2d_*)^{2\tilde{\omega}_2}/\tilde{\omega}_2$ and $\tilde{\omega}_2 = \alpha(1-\theta_2)/2$ (>0).

When $\alpha \le 2/(N-2)$ (i.e. $\omega_1 = 0$), we have from (5.33) and (5.34) that

$$E_2(t) \le E_2(0) \exp\{c_5 E(0)^{\alpha/2}\} \quad (< +\infty)\,. \tag{5.35}$$

On the other hand, when $\alpha > 2/[N-2]^+$, we have that

$$\begin{aligned} E_2(t) &\le \{E_2(0)^{-\omega_1} - \omega_1 \int_0^t c_2 E(s)^{\alpha(1-\theta_2)/2} ds\}^{-1/\omega_1} \\ &\le \{E_2(0)^{-\omega_1} - \omega_1 c_5 E(0)^{\tilde{\omega}_2}\}^{-1/\omega_1} \quad (< +\infty) \end{aligned} \tag{5.36}$$

if $\omega_1 c_5 E(0)^{\tilde{\omega}_2} E_2(0)^{\omega_1} < 1$.

When $\alpha \le 4/(N-2)$ (i.e. $\theta_1 = 0$), we have from (5.8) and (5.13) that

$$B(t) = c_1 E(0)^{\alpha/2} < 1\,. \tag{5.37}$$

On the other hand, when $\alpha > 4/[N-2]^+$ (i.e. $\theta_1 > 0$), we have (5.13) and (5.36) that

$$\begin{aligned} B(t) &\le c_1 E(0)^{1-(N-4)\alpha/4} E_2(t)^{(N-2)\alpha/4-1} \\ &\le c_1 E(0)^{1-(N-4)\alpha/4} \{E_2(0)^{-\omega_1} - \omega_1 c_5 E(0)^{\tilde{\omega}_2}\}^{-\frac{(N-2)\alpha-4}{4\omega_1}} < 1 \end{aligned} \tag{5.38}$$

if we assume (5.9), that is,

$$\{\omega_1 c_5 E(0)^{\tilde{\omega}_2} + c_4 E(0)^{\omega_3}\} E_2(0)^{\omega_1} < 1\,.$$

Thus we conclude that (5.37) and (5.38) contradict (5.14), and hence, we see $T_2 \ge T_1$. Moreover, we observe from (5.11) and (5.15) that

$$0 = K(u(T_1)) \ge (1/2)\|A^{1/2}u(T_1)\|^2 > 0\,,$$

which is a contradiction, and hence, we see $T_1 = +\infty$, that is, (5.32), (5.35), and (5.36) hold true for all $t \ge 0$. The proof of Theorem 9 is now completed. □

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