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Global Existence and Decay Properties of Solutions for Some Degenerate Nonlinear Wave Equation with a Strong Dissipation

Dedicated to Professor Yoshihiro Ichijyō on his 65th birthday

By

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Abstract

We study the existence, uniqueness and decay properties of solutions to the initial-boundary value problem for a degenerate nonlinear integro-differential equation of hyperbolic type with a strong dissipation. We derive decay estimates from above and from below of the solutions.

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1. Introduction

In this paper we are concerned with the initial-boundary value problem for a degenerate nonlinear integro-differential equation of hyperbolic type with a strong dissipation, that is

$$u'' + M(\int_\Omega |A^{1/2}u|^2 dx)Au + \lambda Au' + \mu g(x,t,u,u') = 0\,, \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, the function $M(r) \geq 0$ belongs to $C^1([0,\infty))$, λ and μ are constants, $A = -\triangle = -\sum_{j=1}^N \partial^2/\partial x_j^2$ is the Laplace operator, and $' = \partial/\partial t$.

The existence and uniqueness of local solutions for (1.1) with $\lambda = \mu = 0$ or for the more generalized equations associated with (1.1) have been studied by several authors (cf. Dickey [3], Menzala [8], Ebihara et al.[5], Mediros & Miranda [7], Rivera [15], Yamada [18], Arosio & Garavaldi [1], Crippa [2] and the references cited therein). However, when the initial data is taken in the usual Sobolev's spaces, the existence of global solutions is not proved.

When equations have some dissipative terms u', Au', A^2u etc., we can prove the existence of global solutions ,and moreover we can derive some decay properties of the solutions under suitable assumptions.

When $\lambda > 0$ and $\mu = 0$ in (1.1), the decay estimates from above of the solutions have been derived by Nishihara [11], Matos & Pereira [6], Rivera [15]. While, the decay estimates from below of the solutions have been derived by Nishihara [12], Nishihara & Ono [13], Mizumachi [9]. Moreover, Ono & Nishihara [14] have studied the decay properties of the solutions for (1.1) with $\lambda > 0, \mu > 0$, and $g = |u|^\alpha u$ for $\alpha \geq 0$.

When $\lambda > 0, \mu > 0$, and $g = |u'|^\beta u'$ for $\beta \geq 0$, we study the existence, uniqueness and decay properties of solutions for (1.1) in this paper. In what follows we take $\lambda = \mu = 1$ without loss of generality.

Let $H = L^2(\Omega)$ and $V = H_o^1(\Omega)$. The scalar product and the norm on H are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$ respectively. In section 2 we will show the existence and uniqueness of the global solutions (even for large data), and moreover, we will derive the decay estimates from above of the solutions, e.g.

$$\|A^{1/2}u(t)\|^2 \leq C(1+t)^{-1/\gamma} \quad \text{for} \quad t \geq 0.$$

In section 3 we will derive the decay estimates from below of the solutions, e.g.

$$\|A^{1/2}u(t)\|^2 \geq C(1+t)^{-1/\gamma} \quad \text{for} \quad t \geq T_*.$$

under suitable condition (see (3.3)).

2. Existence, Uniqueness and Decay Properties

In this section we shall study the existence, uniqueness, and decay estimates from above of the global solutions to the initial-boundary value problem for the following equation :

$$\text{(2.1)} \qquad u'' + M(\|A^{1/2}u\|^2)Au + Au' + g(u') = 0 \quad \text{in} \quad \Omega \times [0,\infty),$$

$$u(x,0) = u_0(x)\,, \quad u'(x,0) = u_1(x)\,, \quad \text{and} \quad u|_{\partial\Omega} = 0\,,$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, A is the Laplace operator with domain $\mathcal{D}(A) = H^2(\Omega) \cap H_o^1(\Omega)$, $M(r) \in C^1([0,\infty))$ with $M(r) = r^\gamma$ for $r \geq 0$, $\gamma \geq 0$, and $g(u') = |u'|^\beta u'$ for $\beta \geq 0$.

Our first result is given by

Theorem 1. *Let the initial data (u_0, u_1) belong to $\mathcal{D}(A) \times V$ and $u_2 \equiv -M(\|A^{1/2}u_0\|^2)Au_0 - Au_1 - g(u_1)$ belong to H. Suppose that*

$$\text{(2.2)} \qquad 0 \leq \beta \leq 2/(N-2) \quad (0 \leq \beta < \infty \text{ if } N = 1,2)\,.$$

Then there exists a unique solution $u(t)$ satisfying

$$\text{(2.3a)} \qquad u(t) \in C([0,\infty);\mathcal{D}(A)) \cap L^\infty(0,\infty;\mathcal{D}(A))$$

$$\text{(2.3b)} \qquad u'(t) \in C([0,\infty);V) \cap L^\infty(0,\infty;V) \cap L^2(0,\infty;\mathcal{D}(A))$$

$$\text{(2.3c)} \qquad u''(t) \in L^\infty(0,\infty;H)\,.$$

Moreover, we see

$$\|u(t)\|^2\,,\ \|A^{1/2}u(t)\|^2 \le C(1+t)^{-1/\gamma} \tag{2.4a}$$

$$\|u'(t)\|^2\,,\ \|A^{1/2}u'(t)\|^2 \le C(1+t)^{-(2+1/\gamma)} \tag{2.4b}$$

$$\|u''(t)\|^2 \le C(1+t)^{-(4+1/\gamma)} \tag{2.4c}$$

for some constants C.

PROOF. To begin with we shall prove the uniqueness of the solutions in Step 1. Next we shall derive a priori estimates and the decay estimates of the solutions in Step 2. Finally, in Step 3, we shall show the global existence of the solutions.

Step 1. We shall prove the uniqueness of the solutions of (2.1). Let $u(t)$ and $v(t)$ be two solutions of (2.1) satisfying (2.3a)-(2.3c) and let $w(t) = u(t) - v(t)$. Then $w(t)$ satisfies

$$\begin{aligned} w'' + M(\|A^{1/2}u\|^2)Aw + Aw' + g(u') - g(v') \\ = -\{M(\|A^{1/2}u\|^2) - M(\|A^{1/2}v\|^2)\}Av \end{aligned} \tag{2.5}$$

and $w(0) = w'(0) = 0$. Taking the scalar product of (2.5) with $2w'$ and integrating it over $[0,t]$, we have

$$\begin{aligned} &\|w'(t)\|^2 + M(\|A^{1/2}u(t)\|^2)\|A^{1/2}w(t)\|^2 + 2\int_0^t \|A^{1/2}w'(s)\|^2 ds \\ &\le C\int_0^t \|A^{1/2}u(s)\|^{2\gamma-1}\|A^{1/2}u'(s)\|\|A^{1/2}w(s)\|^2 ds \\ &\quad + C\int_0^t \{\|A^{1/2}u(s)\|^{2\gamma-1} + \|A^{1/2}v(s)\|^{2\gamma-1}\}\|A^{1/2}v(s)\|\|A^{1/2}w(s)\|\|A^{1/2}w'(s)\|\, ds\,, \end{aligned}$$

where we have used that $(g(u') - g(v'), u' - v') \ge 0$ and

$$|M(\|A^{1/2}u\|^2) - M(\|A^{1/2}v\|^2)| \le C\{\|A^{1/2}u\|^{2\gamma-1} + \|A^{1/2}v\|^{2\gamma-1}\}\|A^{1/2}w\|\,.$$

By the Young inequality, we see from (2.3) that

$$\|w'(t)\|^2 + \int_0^t \|A^{1/2}w'(s)\|^2 ds \le C\int_0^t \|A^{1/2}w(s)\|^2 ds\,. \tag{2.6}$$

Taking the scalar product of (2.5) with w, we have

$$\begin{aligned} &\frac{d}{dt}\{\frac{1}{2}\|A^{1/2}w\|^2 + (w',w)\} + M(\|A^{1/2}u\|^2)\|A^{1/2}w\|^2 \\ &= \|w'\|^2 - \{M(\|A^{1/2}u\|^2) - M(\|A^{1/2}v\|^2)\}(A^{1/2}v, A^{1/2}w) - (g(u') - g(v'), w)\,. \end{aligned}$$

Here, we see that for $0 \le \beta \le 4/(N-2)$ ($0 \le \beta < \infty$ if $N = 1, 2$)

$$|(g(u') - g(v'), w)| \le C\{\|A^{1/2}u'\|^\beta + \|A^{1/2}v'\|^\beta\}\|A^{1/2}w'\|\|A^{1/2}w\|.$$

Integrating it over $[0, t]$, we see that for $0 < \gamma \ll 1$

$$\begin{aligned}(2.7)\qquad &\frac{1}{2}\|A^{1/2}w(t)\|^2 + (w'(t), w(t)) \\ &\le \int_0^t \|w'(s)\|^2 ds + C\int_0^t \|A^{1/2}w(s)\|^2 ds + \frac{1}{\gamma}\int_0^t \|A^{1/2}w'(s)\|^2 ds\end{aligned}$$

using the Young inequality.

Summing up (2.6) and $\gamma\times$(2.7) ($0 < \gamma \ll 1$), we obtain

$$\begin{aligned}&\|w'(t)\|^2 + \frac{\gamma}{2}\|A^{1/2}w(t)\|^2 + \gamma(w'(t), w(t)) \\ &\le \gamma\int_0^t \|w'(s)\|^2 ds + C(1+\gamma)\int_0^t \|A^{1/2}w(s)\|^2 ds.\end{aligned}$$

Thus, by the Poincaré inequality and the Young inequality, we arrive at

$$\|w'(t)\|^2 + \|A^{1/2}w(t)\|^2 \le C\int_0^t \{\|w'(s)\|^2 + \|A^{1/2}w(s)\|^2\}\, ds$$

taking a suitable small γ, and hence, we conclude $w = 0$.

Step 2. First, we shall derive the decay estimate of the first energy $E(t)$, where we set

$$E(t) \equiv \|u'(t)\|^2 + \frac{1}{\gamma+1}\|A^{1/2}u(t)\|^{2(\gamma+1)}$$

and the first initial energy

$$E(0) \equiv \|u_1\|^2 + \frac{1}{\gamma+1}\|A^{1/2}u_0\|^{2(\gamma+1)}.$$

Taking the scalar product of (2.1) with $2u'$, we have

$$\frac{d}{dt}E(t) + 2\|A^{1/2}u'(t)\| + 2\|u'(t)\|_{\beta+2}^{\beta+2} = 0. \tag{2.8}$$

Integrating it over $[0, t]$, we obtain

$$E(t) + 2\int_0^t \|A^{1/2}u'(t)\|^2 ds \le E(0). \tag{2.9}$$

While, integrating (2.8) over $[t, t+1]$, we have

$$2\int_t^{t+1} \{\|A^{1/2}u'(s)\|^2 + \|u'(s)\|_{\beta+2}^{\beta+2}\} ds = E(t) - E(t+1) \quad (\equiv D(t)^2). \tag{2.10}$$

Then there exist $t_1 \in [t, t+1/4], t_2 \in [t+3/4, t+1]$ such that

$$\|A^{1/2}u'(t_i)\|^2 + \|u'(t_i)\|_{\beta+2}^{\beta+2} \le 4D(t)^2 \quad \text{for} \quad i = 1, 2\,. \tag{2.11}$$

Taking the scalar product of (2.1) with $2u$ and integrating it over $[t_1, t_2]$, we have from (2.10), (2.11) that

$$\begin{aligned}\int_{t_1}^{t_2} \|A^{1/2}u(s)\|^{2(\gamma+1)} ds &\le \sum_{i=1}^{2} \|u'(t_i)\|\|u(t_i)\| + \int_{t_1}^{t_2} \|u'(s)\|^2 ds \\ &+ \int_{t_1}^{t_2} \|A^{1/2}u(s)\|\|A^{1/2}u'(s)\|\, ds + \int_{t_1}^{t_2} \|u'(s)\|_{\beta+2}^{\beta+2} \|A^{1/2}u(s)\|\, ds \\ &\le C\{D(t)^2 + D(t) \sup_{t\le s\le t+1} \|A^{1/2}u(s)\|\} \quad (\equiv A(t)^2)\end{aligned} \tag{2.12}$$

using the Poincaré inequality and Sobolev's lemma for $0 \le \beta \le 4/(N-2)$ ($0 \le \beta < \infty$ if $N = 1, 2$). Thus we obtain from (2.8), (2.10), (2.12) that

$$E(t_2) \le \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} E(s)\, ds \le CA(t)^2\,,$$

and hence,

$$\begin{aligned}\sup_{t\le s\le t+1} E(s)^{1+\gamma/(\gamma+1)} &\le E(t_2) + 2\int_t^{t+1} \{\|A^{1/2}u'(s)\|^2 + \|u'(s)\|_{\beta+2}^{\beta+2}\}\, ds \\ &\le CA(t)^2 \le C\{D(t)^2 + D(t) \sup_{t\le s\le t+1} E(s)^{1/2(\gamma+1)}\}\,.\end{aligned}$$

Using the Young inequality, we arrive at

$$\sup_{t\le s\le t+1} E(s)^{1+\gamma/(\gamma+1)} \le CD(t)^2 = C\{E(t) - E(t+1)\}\,.$$

Applying the following lemma, we can get the decay estimate of the first energy $E(t)$ such that

$$E(t) = \|u'(t)\|^2 + \frac{1}{\gamma+1}\|A^{1/2}u(t)\|^{2(\gamma+1)} \le C(1+t)^{-(1+1/\gamma)} \quad \text{for} \quad t \ge 0\,. \tag{2.13}$$

Lemma. (Nakao [10]) *Let $\phi(t)$ be a bounded and nonnegative function on $[0, \infty)$ satisfying*

$$\sup_{t\le s\le t+1} \phi(s)^{1+r} \le k_0\{\phi(t) - \phi(t+1)\}$$

for $r > 0$ and $k_0 > 0$. Then

$$\phi(t) \le C(1+t)^{-1/r} \quad \textit{for} \quad t \ge 0\,.$$

with some constant $C = C(r, k_0, \phi(0))$.

Next, we shall derive the decay estimates of $\|u''(t)\|^2$ and $\|A^{1/2}u'(t)\|^2$. Taking the scalar product of $\partial_t(2.1)$ with $2u''$, we have

$$\begin{aligned}&\frac{d}{dt}\|u''\|^2 + 2\|A^{1/2}u''\|^2 + 2\int_\Omega g'(u')|u''|^2 dx \\ &= -2M(\|A^{1/2}u\|^2)(Au', u'') - 2M'(\|A^{1/2}u\|^2)(Au', u)(Au, u'') \\ &\equiv I_1 + I_2\end{aligned} \tag{2.14}$$

Here, we see

$$\begin{aligned}I_1 + I_2 \le\, & 2M(\|A^{1/2}u\|^2)\|A^{1/2}u'\|\|A^{1/2}u''\| \\ &+ 2|M'(\|A^{1/2}u\|^2)|\|A^{1/2}u'\|\|A^{1/2}u\|^2\|A^{1/2}u''\|.\end{aligned}$$

By the Young inequality, we have

$$\frac{d}{dt}\|u''\|^2 + \|A^{1/2}u''\|^2 \le C\|A^{1/2}u'\|^2 .$$

Integrating it over $[0, t]$, we obtain from (2.9) that

$$\|u''(t)\|^2 + \int_0^t \|A^{1/2}u''(s)\|^2 ds \le \|u_2\|^2 + C\int_0^t \|A^{1/2}u'(s)\|^2 ds \le C . \tag{2.15}$$

Then, taking the scalar product of (2.1) with u' again, we see

$$\|A^{1/2}u'\|^2 + \|u'\|_{\beta+2}^{\beta+2} = -(u'', u') - M(\|A^{1/2}u\|^2)(A^{1/2}u, A^{1/2}u')$$

or

$$\|A^{1/2}u'\|^2 + \|u'\|_{\beta+2}^{\beta+2} \le C\{\|u''\| + \|A^{1/2}u\|^{2\gamma+1}\}^2 \le C . \tag{2.16}$$

While, using the equation (2.1), we see

$$\begin{aligned}|(Au', u'')| &\le C\{\|A^{1/2}u''\|^2 + \|A^{1/2}u\|^{2\gamma+1}\|A^{1/2}u''\|\} + \int_\Omega |u'|^{\beta+1}|u''|\, dx \\ &\le C\{\|A^{1/2}u''\|^2 + (\|A^{1/2}u\|^{2\gamma+1} + \|A^{1/2}u'\|^{\beta+1})\|A^{1/2}u''\|\}\end{aligned}$$

and

$$|(Au', u)| \le C\{\|A^{1/2}u\|\|A^{1/2}u''\| + (\|A^{1/2}u\|^{2\gamma+1} + \|A^{1/2}u'\|)\|A^{1/2}u\|\} .$$

Thus we obtain

$$I_1 + I_2 \le C\|A^{1/2}u\|^{2\gamma}\|A^{1/2}u''\|^2 + C\{\|A^{1/2}u\|^{4\gamma+1} + \|A^{1/2}u\|^{2\gamma}\|A^{1/2}u'\|^{\beta+1}\}\|A^{1/2}u''\| .$$

Since we see from (2.13) that $C\|A^{1/2}u\|^{2\gamma}\|A^{1/2}u''\|^2 \le \frac{1}{2}\|A^{1/2}u''\|^2$ for the large time $t \ge T_1$, it follows from (2.14) that

$$\begin{aligned}(2.17)\qquad &\frac{d}{dt}\|u''(t)\|^2 + \|A^{1/2}u''(t)\|^2 \\ &\le C\{\|A^{1/2}u(t)\|^{4\gamma+1} + \|A^{1/2}u(t)\|^{2\gamma}\|A^{1/2}u'(t)\|^{\beta+1}\}^2 \\ &\le C(1+t)^{-\omega} \quad \text{for} \quad t \ge T_1\end{aligned}$$

with $\omega = \min\{4+1/\gamma, 2\} = 2$, where we have used (2.13), (2.16). Noting (2.15), we see

$$\|u''(t)\|^2 \le C(1+t)^{-\omega} \quad \text{for} \quad t \ge 0$$

using Sobolev's lemma. Then it follows from (2.16) that

$$\|A^{1/2}u'(t)\|^2 \le C(1+t)^{-\theta} \quad \text{for} \quad t \ge 0$$

with $\theta = \min\{\omega, 2+1/\gamma\} = 2$. Using this decay estimate, we can improve the decay estimate of $\|u''(t)\|^2$, that is

$$\|u''(t)\|^2 \le C(1+t)^{-\omega_1} \quad \text{for} \quad t \ge 0$$

with $\omega_1 = \min\{4+1/\gamma, 2+(1+\beta)\omega/2\} = \min\{4+1/\gamma, 3+\beta\}$. Then we easy see

$$(2.18)\qquad \|A^{1/2}u'(t)\|^2 \le C(1+t)^{-\theta_1} \quad \text{for} \quad t \ge 0$$

with $\theta_1 = \min\{\omega_1, 2+1/\gamma\} = 2+1/\gamma$. Moreover we can improve the decay estimate of $\|u''(t)\|^2$ again, that is

$$(2.19)\qquad \|u''(t)\|^2 \le C(1+t)^{-\omega_2} \quad \text{for} \quad t \ge 0$$

with $\omega_2 = \min\{4+1/\gamma, 2+(1+\beta)\theta_1\} = 4+1/\gamma$. Thus it follows from (2.13), (2.18), (2.19) that

$$\begin{aligned}&\|u(t)\|^2,\ \|A^{1/2}u(t)\|^2 \le C(1+t)^{-1/\gamma} \\ &\|u'(t)\|^2,\ \|A^{1/2}u'(t)\|^2 \le C(1+t)^{-(2+1/\gamma)} \\ &\|u''(t)\|^2 \le C(1+t)^{-(4+1/\gamma)}\end{aligned}$$

which is the desired decay estimates (2.4a)-(2.4c).

Next, we shall show that u is bounded in $L^\infty(0,\infty;\mathcal{D}(A))$. Taking the scalar product of (2.1) with $2Au$, we have

$$\begin{aligned}&\frac{d}{dt}\|Au\|^2 + 2M(\|A^{1/2}u\|^2)\|Au\|^2 \\ &\le -2\frac{d}{dt}(u', Au) + 2\|A^{1/2}u'\|^2 + \|u'\|_{2(\beta+1)}^{\beta+1}\|Au\|\,.\end{aligned}$$

Integrating it over $[0,t]$, we see

$$\begin{aligned}&\|Au(t)\|^2+2\int_0^t M(\|A^{1/2}u(s)\|^2)\|Au(s)\|^2ds\\&\le\|Au_0\|^2+2|(u_1,Au_0)|+2\|u'(t)\|\|Au(t)\|\\&\quad+2\int_0^t\|A^{1/2}u'(s)\|^2ds+\int_0^t\|A^{1/2}u'(s)\|^{\beta+1}\|Au(s)\|\,ds\end{aligned}$$

using Sobolev's lemma with $0\le\beta\le 2/(N-2)$ ($0\le\beta<\infty$ if $N=1,2$), and hence, we see from (2.9)

$$\|Au(t)\|^2\le C+C\sup_{0\le s\le t}\|Au(s)\|\int_0^t\|A^{1/2}u'(s)\|^{\beta+1}ds$$

or

$$\sup_t\|Au(t)\|^2\le C+C\{\int_0^\infty(1+t)^{-(\beta+1)(1+1/2\gamma)}dt\}^2\le C\,,$$

where we have used the decay estimate $\|A^{1/2}u'(t)\|^2\le C(1+t)^{-(2+1/\gamma)}$ (see (2.18)). Thus it follows

$$\|Au(t)\|^2+\int_0^t M(\|A^{1/2}u(s)\|^2)\|Au(s)\|^2ds\le C\quad\text{for}\quad t\ge 0\,. \tag{2.20}$$

Next, we shall show that u' is bounded in $L^2(0,\infty;\mathcal{D}(A))$. Taking the scalar product of (2.1) with $2Au'$, we have

$$\frac{d}{dt}\|A^{1/2}u'\|^2+2\|Au'\|^2+2\int_\Omega g'(u')|A^{1/2}u'|^2dx\le 2M(\|A^{1/2}u\|^2)\|Au\|\|Au'\|$$

or

$$\frac{d}{dt}\|A^{1/2}u'\|^2+\|Au'\|^2\le CM(\|A^{1/2}u\|^2)^2\|Au\|^2\le CM(\|A^{1/2}u\|^2)\|Au\|^2\,.$$

Integrating it over $[0,t]$, we see from (2.20) that

$$\|A^{1/2}u'(t)\|^2+\int_0^t\|Au'(s)\|^2ds\le C\quad\text{for}\quad t\ge 0\,. \tag{2.21}$$

Step 3. We shall show the existence of the solutions for (2.1). The principle of the proof is classical (cf. [4],[17]). We consider the orthonormal basis of H consisting of the eigenvectors of A,

$$Aw_j=\lambda_j^2w_j\,,\quad 0<\lambda_1\le\lambda_2\le\cdots,\lim_{j\to\infty}\lambda_j=\infty\,,$$

and we implement a Faedo-Galerkin method with these functions. For each m we look for an approximate solution u_m of the form $u_m(t) = \sum_{j=1}^{m} g_{jm}(t)w_j$ satisfying

$$(u_m''(t) + M(\|A^{1/2}u_m(t)\|^2)Au_m(t) + Au_m'(t) + g(u_m'(t)), w_j) = 0 \tag{2.22}$$

for $j = 1, 2, \cdots, m$,

$$u_m(0) = u_{0m}\,, \quad u_m'(0) = u_{1m}\,,$$

where u_{0m} (resp. u_{1m}) is the projection in $\mathcal{D}(A)$ (resp. V) of u_0 (resp. u_1) onto the space spanned by $w_1, \cdots, w_m$. The existence and uniqueness of u_m on some interval $[0, T_m)$ is elementary and then $T_m = \infty$, because of the a priori estimates that we obtain for u_m. Indeed, we can get the estimates (2.9), (2.15), (2.16), (2.20), (2.21) with u replaced by u_m. For example, multiplying (2.22) by g'_{jm} and summing these relations for $j = 1, 2, \cdots, m$, and integrating over $[0, t]$, we obtain (2.9) with u replaced by u_m.

Thus we conclude that, for $T > 0$ arbitrary, u_m is bounded independent of m in $L^\infty(0, T; \mathcal{D}(A))$, $u_m'$is bounded independent of m in $L^\infty(0, T; V) \cap L^2(0, T; \mathcal{D}(A)) \cap L^{\beta+2}(0, T; L^{\beta+2}(\Omega))$, $u_m''$is bounded independent of m in $L^\infty(0, T; H) \cap L^2(0, T; V)$. Then we can extract a subsequence, still denoted m, such that

$$\begin{aligned}
&u_m \to u \quad \text{in} \quad L^\infty(0, T; \mathcal{D}(A)) \text{ weak}* \\
&u_m' \to u' \quad \text{in} \quad L^\infty(0, T; V) \cap L^2(0, T; \mathcal{D}(A)) \text{ weak}* \\
&u_m'' \to u'' \quad \text{in} \quad L^\infty(0, T; H) \cap L^2(0, T; V) \text{ weak}* \\
&M(\|A^{1/2}u_m\|^2)Au_m \to \chi \quad \text{in} \quad L^\infty(0, T; H) \text{ weak}* \\
&g(u_m') = |u_m'|^\beta u_m' \to \psi \quad \text{in} \quad L^{(\beta+2)/(\beta+1)}((0, T) \times \Omega) \text{ weakly}.
\end{aligned}$$

It follows from a classical compactness argument (cf. [4]) that $g(u_m')$ converges to $g(u')$ in some weak sense. We shall show $\chi = M(\|A^{1/2}u\|^2)Au$ (cf. [11],[16]). For any $\phi \in C_o(0, \infty; H)$, we see

$$\begin{aligned}
&\int_0^T (M(\|A^{1/2}u_m\|^2) - M(\|A^{1/2}u\|^2))(Au_m, \phi)\, dt \\
&\qquad \le C \int_0^T \|A^{1/2}u_m - A^{1/2}u\|\, dt \to 0 \quad \text{as} \quad m \to \infty
\end{aligned}$$

using the mean value theorem. Thus we have

$$\begin{aligned}
&\int_0^T (\chi - M(\|A^{1/2}u\|^2)Au, \phi)\, dt \\
&= \int_0^T (\chi - M(\|A^{1/2}u_m\|^2)Au_m, \phi)\, dt + \int_0^T M(\|A^{1/2}u_m\|^2)(Au_m - Au, \phi)\, dt \\
&\quad + \int_0^T (M(\|A^{1/2}u_m\|^2) - M(\|A^{1/2}u\|^2))(Au_m, \phi)\, dt \to 0 \quad \text{as} \quad m \to \infty\,,
\end{aligned}$$

and hence, we conclude $\chi = M(\|A^{1/2}u\|^2)Au$. Thus u is a solution of (2.1) satisfying (2.3a)-(2.3c).

Moreover, by the following standard lemma, we see that $u \in C^0([0,T];\mathcal{D}(A))$ and $u' \in C^0([0,T];V)$.

Lemma. ([4],[17]) *Let X be a Banach space. If $f \in L^2(0,T;X)$ and $f' \in L^2(0,T;X)$, then f, possibly after redefinition on a set of measure zero, is continuous from $[0,T]$ to X.*

The proof of Theorem 1 is now completed. □

2. Another Decay Property

In this section, we study another decay property of the solution of (2.1), which is the decay estimate from below.

Theorem 2. *In addition to the assumption of Theorem 1, suppose that the initial energy $E(0) \equiv \|u_1\|^2 + \frac{1}{\gamma+1}\|A^{1/2}u_0\|^{2(\gamma+1)} (\ll 1)$ is sufficiently small (see (3.7)), and the initial data (u_0,u_1) satisfy*

$$F(0) \equiv \|A^{1/2}u_0\|^2 + 2(u_1,u_0) > 0\,. \tag{3.1}$$

Then it holds that for the large time $T_* > 0$, the solution $u(t)$ of Theorem 1 has the following decay estimate :

(i) *When $\gamma/(1+2\gamma) < \beta \le \min\{\gamma/(1+\gamma), 2/N\}$,*

$$\|A^{1/2}u(t)\|^2 \ge C(1+t)^{-(1-\beta)/\beta} \quad \textit{for} \quad t \ge T_*\,. \tag{3.2}$$

(ii) *When $\gamma/(1+\gamma) \le \beta < 1$ and $\beta \le 2/N$,*

$$\|A^{1/2}u(t)\|^2 \ge C(1+t)^{-1/\gamma} \quad \textit{for} \quad t \ge T_*\,. \tag{3.3}$$

PROOF. Multiplying (2.1) by $2u(t)$ and integrating over Ω, we have

$$\frac{d}{dt}F(t) + 2G(t) = 0\,, \tag{3.4}$$

where we set

$$F(t) \equiv \|A^{1/2}u(t)\|^2 + 2(u'(t),u(t))\,,$$
$$G(t) \equiv \|A^{1/2}u(t)\|^{2(\gamma+1)} - \|u'(t)\|^2 + (g(u'(t)),u(t))\,.$$

Here for $0 \le \beta < 1$, we see

$$|(g(u'),u)| \le \frac{1}{2}\|u'\|^2 + \|u\|^{2/(1-\beta)}_{2/(1-\beta)} \le \frac{1}{2}\|u'\|^2 + C\|A^{1/2}u\|^{2/(1-\beta)}$$

using the Young inequality and Sobolev's lemma with $\beta \le 2/N$. Thus we have that for $0 \le \beta < 1$ and $\beta \le 2/N$,

$$G(t) \le \|A^{1/2}u(t)\|^{2(\gamma+1)} + C\|A^{1/2}u(t)\|^{2/(1-\beta)} - \frac{1}{2}\|u'(t)\|^2$$
$$\le \begin{cases} C_*\|A^{1/2}u(t)\|^{2/(1-\beta)} - \frac{1}{2}\|u'(t)\|^2 & \text{if } \beta \le \gamma/(1+\gamma) \\ C_*\|A^{1/2}u(t)\|^{2(1+\gamma)} - \frac{1}{2}\|u'(t)\|^2 & \text{if } \beta \ge \gamma/(1+\gamma) \end{cases}$$

with some constant $C_* > 0$, where we have used $\|A^{1/2}u(t)\|^2 \le C$ for $t \ge 0$. Since $F(0) > 0$ by the assumption (3.1), it follows that $F(t) > 0$ for some $t > 0$. We put $T_0 \equiv \sup\{t \in [0,\infty)\,;\, F(s) > 0 \text{ for } 0 < s < t\}$, then it holds that $T_0 > 0$ and $F(t) > 0$ for $t < T_0$.

(i) When $\beta \le \gamma/(1+\gamma)$, we shall derive

$$CF(t)^{1/(1-\beta)} - G(t) \ge 0 \quad \text{for} \quad t < T_0$$

with some constant $C > 0$. Since for any $0 \le k \le \beta/(1-\beta)$ and $0 < \varepsilon \ll 1$

$$\|A^{1/2}u\|^{2k}|(u',u)|^{1/(1-\beta)-k} \le \varepsilon\|A^{1/2}u\|^{1/(1-\beta)} + C_\varepsilon\|A^{1/2}u\|^{2k}\|u'\|^{2(\beta/(1-\beta)-k)}\|u'\|^2$$
$$\le \varepsilon\|A^{1/2}u\|^{1/(1-\beta)} + C_\varepsilon E(0)^{\beta(1-\beta)-k\gamma/(\gamma+1)}\|u'\|^2$$

using $E(t) \le E(0)$ (< 1), we can get

$$F(t)^{\gamma+1} \ge \|A^{1/2}u(t)\|^{2(\gamma+1)} - \{\frac{1}{2}\|A^{1/2}u(t)\|^{2(\gamma+1)}$$
$$+ C \max_{0\le k\le\beta/(1-\beta)} E(0)^{\beta(1-\beta)-k\gamma/(\gamma+1)}\|u'(t)\|^2\}$$

$$\ge \frac{1}{2}\|A^{1/2}u(t)\|^{2(\gamma+1)} - C_1E(0)^{\beta/(1-\beta)(1+\gamma)}\|u'(t)\|^2 \tag{3.6}$$

with some constant $C_1 > 0$. Thus it follows from (3.5) and (3.6) that

$$2C_*F(t)^{1/(1-\beta)} - G(t) \ge (1 - 2C_*C_1E(0)^{\beta/(1-\beta)(1+\gamma)})\|u'(t)\|^2 \ge 0\,, \tag{3.7}$$

where we have used the assumption $E(0) \ll 1$. Then we obtain from (3.4) that

$$\frac{d}{dt}F(t) + 4C_*F(t)^{1/(1-\beta)} \ge 0\,,$$

and hence, we see from $F(0) > 0$ that

$$F(t) \ge \{F(0)^{-\beta/(1-\beta)} + 4C_*t\}^{-(1-\beta)/\beta} \ge 0\,,$$

which conclude $T_0 = \infty$. then, we see

$$\|A^{1/2}u(t)\|^2 \ge C(1+t)^{-(1-\beta)/\beta} - C\|u'(t)\|\|A^{1/2}u(t)\|$$
$$\ge C(1+t)^{-(1-\beta)/\beta} - C(1+t)^{-(1+\gamma)/\gamma}\,,$$

which implies the desired estimate (3.2) if $\beta > \gamma/(1+2\gamma)$.

(ii) When $\gamma/(1+\gamma) \le \beta$, as the similar way above, we can derive

$$CF(t)^{\gamma+1} - G(t) \ge 0 \quad \text{for} \quad t \ge 0$$

with some constant $C > 0$. Thus we obtain the desired decay estimate (3.3).

The proof of Theorem 2 is now completed. □

References

[1] A. Arosio and S. Garavaldi, On the mildly degenerate Kirchhoff string, Math. Methods Appl. Sci. **14** (1991), 177–195.

[2] H. R. Crippa, On local solutions of some mildly degenerate hyperbolic equations, Nonlinear Anal. T.M.A. **21** (1993), 565–574.

[3] R. W. Dickey, Infinite systems of nonlinear oscillation equation with linear damping, SIAM J. Appl. Math. **19** (1970), 208–214.

[4] J. L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod, Paris, 1969.

[5] Y. Ebihara, L. A. Medeiros and M. Milla Miranda, Local solutions for a nonlinear degenerate hyperbolic equation, Nonlinear Anal. T.M.A. **10** (1986), 27–40.

[6] M. P. Matos and D. C. Pereira, On a hyperbolic equation with strong damping, Funkcial. Ekvac. **34** (1991), 303–311.

[7] L. A. Medeiros and M. Miranda, Solutions for the equation of nonlinear vibrations in Sobolev spaces of fractionary order, Computational Applied Math. **6** (1987), 257–267.

[8] G. P. Menzala, On classical solutions of a quasilinear hyperbolic equation, Nonlinear Anal. T.M.A. **3** (1979), 613–627.

[9] T. Mizumachi, The asymptotic behavior of solutions to the Kirchhoff equation with viscous damping, preprint.

[10] M. Nakao, A difference inequality and its application to nonlinear evolution equations, J. Math. Soc. Japan. **30** (1978), 747–762.

[11] K. Nishihara, On a global solution of some quasilinear hyperbolic equation, Tokyo J. Math. **7** (1984), 437–459.

[12] ———, Decay properties of solutions of some quasilinear hyperbolic equations with strong damping, Nonlinear Anal. T.M.A. **21** (1993), 17–21.

[13] K. Nishihara and K. Ono, Asymptotic behaviors of solutions of some nonlinear oscillation equations with strong damping, Advan. Math. Sci. Appl. **4** (1994), 285–295.

[14] K. Ono and K. Nishihara, On a nonlinear degenerate integrodifferential equation of hyperbolic type with a strong dissipation, Advan. Math. Sci. Appl. **5** (1995), 457–476.

[15] J. E. M. Rivera, On local strong solutions of a nonlinear partial differential equation, Appl. Anal. **10** (1980), 93–104.

[16] ———, Smoothness effect and decay on a class of nonlinear evolution equation, Ann. Fac. Sci. Toulouse Math. (6) **1** (1992), 237–260.

[17] R. Temam, Infinite Dimensional Dynamical Systems in Mechanics and Physics, Springer-Verlag, Berlin, 1988.

[18] Y. Yamada, Some nonlinear degenerate wave equations, Nonlinear Anal. T.M.A. **11** (1987), 1155–1168.