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Miura Transformation and S-Matrix

Dedicated to Professor Yoshihiro Ichijyō on his 65th birthday

By

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Abstract

The Miura transformations

$$u_{\pm}(x) = v(x)^2 \pm v'(x) - m^2, \quad m > 0$$

are explicitly represented in terms of the scattering data of the 1-dimensional Schrödinger operators

$$H(u_{\pm}) = -\partial^2 + u_{\pm}(x)$$

with the rapidly decreasing potentials $u_{\pm}(x)$ and the 1-dimensional Dirac operator

$$D(v) = \sqrt{-1}\begin{pmatrix} \partial & -v(x) \\ v(x) & -\partial \end{pmatrix}$$

with the step type potential $v(x)$ such that $\lim_{x\to\pm\infty} v(x) = \pm m$.

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1. Introduction

The purpose of the present paper is to obtain the explicit representation of the nonlinear transformations $v(x) \mapsto u_{\pm}(x)$ defined by

$$(1) \qquad u_{\pm}(x) = v(x)^2 \pm v'(x) - m^2$$

in terms of the scattering data of the 1-dimensional Schrödinger operators

$$H(u_\pm) = -\partial^2 + u_\pm(x), \quad x \in \boldsymbol{R}$$

and the 1-dimensional Dirac operator

$$D(v) = \sqrt{-1}\begin{pmatrix} \partial & -v(x) \\ v(x) & -\partial \end{pmatrix}, \quad x \in \boldsymbol{R},$$

where $\partial =' = d/dx$. We assume that $u(x) \in \boldsymbol{L}_1^1(\boldsymbol{R})$ and $v(x) \in \boldsymbol{D}_1^1(\boldsymbol{R}; m)$ for some positive real number m, where

$$\boldsymbol{L}_l^1(\boldsymbol{R}) = \{\, u(x) \mid \text{real valued measurable and } \int_{-\infty}^{\infty} (1+|x|^l)|u(x)|dx < \infty\}$$

and

$$\boldsymbol{D}_l^1(\boldsymbol{R}; m) = \{v(x) \quad | \quad \text{real valued measurable, } \lim_{x\to\pm\infty} v(x) = \pm m \\ \text{and } \int_{-\infty}^{\infty} (1+|x|^l)(|v(x)^2 - m^2| + |v'(x)|)dx < \infty\}$$

for $l \in \boldsymbol{N}$. The nonlinear transformations (1), which were first introduced by R. M. Miura [9], are called the Miura transformations and convert such a step type solution as above of the mKdV equation into rapidly decreasing solutions of the KdV equation, that is, if $v = v(x,t) \in C^4(\boldsymbol{R}) \cap \boldsymbol{D}_1^1(\boldsymbol{R}; m)$ $(\forall t \in \boldsymbol{R})$ solves the mKdV equation

$$v_t - 6v^2 v_x + v_{xxx} = 0, \quad (x,t) \in \boldsymbol{R}^2$$

then

$$q_\pm = q_\pm(x,t) = v(x + 6m^2 t, t)^2 \pm v_x(x + 6m^2 t, t) - m^2$$

solve the KdV equation

$$q_{\pm t} - 6q_\pm q_{\pm x} + q_{\pm xxx} = 0, \quad (x,t) \in \boldsymbol{R}^2,$$

where the subscripts x, t denote the partial differentiations. Refer [4] for more recent development related to this fact.

One easily verifies

$$(2) \qquad {}^t T D(v)^2 T - m^2 E = \begin{pmatrix} H(u_-) & 0 \\ 0 & H(u_+) \end{pmatrix},$$

where E is the 2×2 unit matrix and

$$T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

See also [11] for this identity. We represent the Miura transformation in terms of scattering data on the basis of the operator identity (2).

The contents of this paper are as follows. In the sections 2 and 3, the scattering data of $H(u)$ and $D(v)$ are explained respectively. In the section 4, the Miura transformations (1) are represented in terms of these scattering data.

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2. Scattering data of $H(u)$

In this section, we summarize the definition of the scattering data of the 1-dimensional Schrödinger operator $H(u)$. Refer [1], [2], [3], [7] and [8] for detail.

Suppose $u(x) \in \boldsymbol{L}_1^1(\boldsymbol{R})$ and consider the eigenvalue problem

$$H(u)f(x) = -f''(x) + u(x)f(x) = \zeta^2 f(x) \tag{3}$$

on the real line, where $\zeta = \sigma + \sqrt{-1}\eta$. If $\Im\zeta \geq 0$ then there exist the unique solutions $f_\pm(x, \zeta; u)$ of (3) bound by the conditions

$$f_\pm(x, \zeta; u) = \exp(\pm\sqrt{-1}\zeta x) + o(1), \quad x \to \pm\infty \tag{4}$$

respectively. We call the solutions $f_\pm(x, \zeta; u)$ the Schrödinger-Jost solutions. The solutions $f_\pm(x, \zeta; u)$ are analytic in ζ, $\Im\zeta > 0$. If $\zeta = \sigma$ is real then

$$W_H[f_+(x, \sigma; u), \overline{f_+(x, \sigma; u)}] = -2\sqrt{-1}\sigma$$

follows, where $W_H[f, g] = fg' - f'g$ is the Wronskian and $\overline{z}$ is the complex conjugate of $z \in \boldsymbol{C}$. Hence, if $\sigma \in \boldsymbol{R} \setminus \{0\}$ then $f_+(x, \sigma; u)$ and $\overline{f_+(x, \sigma; u)}$ are linearly independent. Therefore, one can uniquely express $f_-(x, \sigma; u)$, $\sigma \in \boldsymbol{R} \setminus \{0\}$ as

$$f_-(x, \sigma; u) = a(\sigma; u)\overline{f_+(x, \sigma; u)} + b(\sigma; u)f_+(x, \sigma; u).$$

One has immediately

$$a(\sigma; u) = \frac{1}{2\sqrt{-1}\sigma} W_H[f_+(x, \sigma; u), f_-(x, \sigma; u)], \tag{5}$$

$$b(\sigma; u) = \frac{1}{2\sqrt{-1}\sigma} W_H[f_-(x, \sigma; u), \overline{f_+(x, \sigma; u)}]$$

and

$$|a(\sigma; u)|^2 = 1 + |b(\sigma; u)|^2.$$

By (5), the coefficient $a(\sigma; u)$ can be extended to the analytic function $a(\zeta; u)$ in $\Im\zeta > 0$. Hence, there exist finite number of zeros of $a(\zeta; u)$ in $\Im\zeta > 0$, which are purely imaginary and simple. Suppose $\sqrt{-1}\eta_1, \cdots, \sqrt{-1}\eta_n$ ($\eta_1 > \cdots > \eta_n > 0$) be the zeros of $a(\zeta; u)$. Then, $f_\pm(x, \sqrt{-1}\eta_j; u)$ are real valued and linearly independent. Hence, there exists the real number d_j such that

$$f_+(x, \sqrt{-1}\eta_j; u) = d_j f_-(x, \sqrt{-1}\eta_j; u).$$

Since, by (4), $f_\pm(x,\sqrt{-1}\eta_j;u)$ behave as $\exp(\mp\eta_j x)$ as $x\to\pm\infty$ respectively, the solutions $f_\pm(x,\sqrt{-1}\eta_j;u)$ belong to $L^2(\boldsymbol{R})$, i.e., $-\eta_j^2$ are the discrete eigenvalues of the selfadjoint operator $H(u)$. Put

$$r_\pm(\sigma;u)=\pm\frac{b(\pm\sigma;u)}{a(\sigma;u)},\quad \sigma\in\boldsymbol{R}\setminus\{0\}$$

and

$$t_\pm(\zeta;u)=\frac{1}{a(\zeta;u)},\quad \Im\zeta\geq 0.$$

We have

$$|r_\pm(\sigma;u)|<1,\quad \sigma\neq 0,$$
$$r_\pm(-\sigma;u)=\overline{r_\pm(\sigma;u)}$$

and

$$r_\pm(\sigma;u)=O(\frac{1}{\sigma}),\quad |\sigma|\to\infty.$$

The coefficients $r_\pm(\sigma;u)$ and $t_\pm(\sigma;u)$ are called the reflection coefficients and the transmission coefficients respectively. The 2×2 matrix

$$S(\sigma;u)=\begin{pmatrix} t_+(\sigma;u) & r_-(\sigma;u)\\ r_+(\sigma;u) & t_-(\sigma;u)\end{pmatrix}$$

is the S-matrix. By the GLM method (see e.g., [2], [3] and [7]), the operator $H(u)$ can be reconstructed from the reflection coefficient $r_+(\sigma;u)$, the eigenvalues $-\eta_n^2>\cdots>-\eta_1^2$ and the norming constants

$$\gamma_j=\frac{1}{\displaystyle\int_{-\infty}^{\infty}f_+(x,\sqrt{-1}\eta_j;u)^2dx}=\frac{d_j}{\sqrt{-1}a'(\sqrt{-1}\eta_j;u)},\quad j=1,2,\cdots,n, \tag{6}$$

where $a'(\sqrt{-1}\eta_j;u)=\frac{d}{d\zeta}a(\sqrt{-1}\eta_j;u)$. We call the collection

$$\Sigma_H(u)=\{r_+(\sigma;u);-\eta_1^2<\cdots<-\eta_n^2;\gamma_1,\cdots,\gamma_n\}$$

the scattering data.

3. Scattering data of $\boldsymbol{D}(\boldsymbol{v})$

In this section we summarize the definition of the scattering data of the 1-dimensional Dirac operator $D(v)$ from [10]. See also [5] and [6] for more recent development.

Suppose $v(x)\in\boldsymbol{D}_1^1(\boldsymbol{R};m)$ for some positive real number m. Consider the eigenvalue problem

$$D(v)\phi(x)=\sqrt{-1}\begin{pmatrix}\phi_1'(x)-v(x)\phi_2(x)\\ v(x)\phi_1(x)-\phi_2'(x)\end{pmatrix}=\lambda\phi(x) \tag{7}$$

on the real line, where $\phi(x)={}^t(\phi_1(x),\phi_2(x))$ and $\lambda\in\boldsymbol{C}$.

Let $\zeta = \zeta(\lambda)$ be the two-valued algebraic function defined by

$$\zeta^2 = \lambda^2 - m^2$$

and $\mathcal{R}_+$ be the upper leaf of the 2-sheeted Riemann surface associated with the algebraic function $\zeta(\lambda)$ such that $\Im\zeta(\lambda) > 0$ for $\lambda \in \mathcal{R}_+$. Put

$$I_m = \boldsymbol{R} \setminus [-m, m] = (-\infty, -m) \cup (m, \infty)$$

and

$$\sigma(\xi) = (\mathrm{sgn}\ \xi)\sqrt{\xi^2 - m^2}, \quad \xi \in I_m,$$

where

$$\mathrm{sgn}\ \xi = \frac{\xi}{|\xi|}.$$

If $\lambda \in \mathcal{R}_+$ there exist the unique solutions $\phi_\pm(x, \lambda; v)$ of (7) bound by the conditions

$$\phi_\pm(x, \lambda; v) = \phi_\pm^0(x, \lambda) + o(1), \quad x \to \pm\infty,$$

where

$$\phi_+^0(x, \lambda) = \begin{pmatrix} \dfrac{\sqrt{-1}(\zeta(\lambda) - \lambda)}{m} \\ 1 \end{pmatrix} \exp(\sqrt{-1}\zeta(\lambda)x) \tag{8}$$

$$\phi_-^0(x, \lambda) = \begin{pmatrix} 1 \\ \dfrac{\sqrt{-1}(\zeta(\lambda) - \lambda)}{m} \end{pmatrix} \exp(-\sqrt{-1}\zeta(\lambda)x). \tag{9}$$

We call the solutions $\phi_\pm(x, \lambda; v) = {}^t(\phi_{\pm 1}(x, \lambda; v), \phi_{\pm 2}(x, \lambda; v))$ the Dirac-Jost solutions. The Dirac-Jost solutions $\phi_\pm(x, \lambda; v)$ are analytic in $\lambda \in \mathcal{R}_+$.

Put

$$y^\sharp = {}^t(\overline{y_2}, \overline{y_1})$$

for $y = {}^t(y_1, y_2) \in \boldsymbol{C}^2$. We have

$$W_D[\phi_+(x, \xi; v), \phi_+(x, \xi; v)^\sharp] = \frac{2\sigma(\xi)(\sigma(\xi) - \xi)}{m^2}$$

for $\xi \in I_m$, where $W_D[\phi, \psi] = \phi_1\psi_2 - \phi_2\psi_1$ is the Wronskian for $\phi = {}^t(\phi_1, \phi_2)$, $\psi = {}^t(\psi_1, \psi_2)$. Since $\phi_+(x, \xi; v)$ and $\phi_+(x, \xi; v)^\sharp$ are linearly independent solutions of (7) for $\xi \in I_m$, there exist the coefficients $A(\xi; v)$, $B(\xi; v)$ such that

$$\phi_-(x, \xi; v) = A(\xi; v)\phi_+(x, \xi; v)^\sharp + B(\xi; v)\phi_+(x, \xi; v). \tag{10}$$

One has

$$A(\xi; v) = \frac{m^2 W_D[\phi_-(x, \xi; v), \phi_+(x, \xi; v)]}{2\sigma(\xi)(\xi - \sigma(\xi))}, \tag{11}$$

$$B(\xi; v) = \frac{m^2 W_D[\phi_-(x, \xi; v), \phi_+(x, \xi; v)^\sharp]}{2\sigma(\xi)(\xi - \sigma(\xi))}$$

and

$$|A(\xi;v)|^2 = 1 + |B(\xi;v)|^2.$$

By (11), $A(\xi;v)$ can be extended to the analytic function $A(\lambda;v)$, $\lambda \in \mathcal{R}_+$. The coefficients

$$R_\pm(\xi;v) = \frac{\pm B(\pm\xi;v)}{A(\xi;v)}, \quad \xi \in I_m$$

and

$$T_\pm(\lambda;v) = \frac{1}{A(\lambda;v)}, \quad \lambda \in \mathcal{R}_+$$

are called the reflection coefficients and the transmission coefficients respectively. We have

$$|R_\pm(\xi;v)| < 1, \quad \xi \in I_m$$

and

$$R_\pm(\xi;v) = O(\frac{1}{|\xi|}), \quad |\xi| \to \infty.$$

If $A(\lambda_*;v) = 0$ for $\lambda_* \in \mathcal{R}_+$ then, by (10), $\phi_\pm(x,\lambda_*;v)$ are square integrable by their asymtotic property (8) and (9). Since $D(v)$ is symmetric, $\lambda_* \in \boldsymbol{R} \cap \mathcal{R}_+$, i.e., $-m < \lambda_* < m$ follows. The integral representation

$$A(\lambda;v) = \frac{\lambda}{\zeta(\lambda)}(1 + \int_0^\infty F(x)\exp(2\sqrt{-1}\zeta(\lambda)x)dx) \tag{12}$$

is valid for a real valued integrable function $F(x)$. By (12), $A(-\lambda;v) = -A(\lambda;v)$ follows. Hence, the zeros of $A(\lambda;v)$ are simple and distributed symmetrically in the interval $(-m, m)$, especially $A(0;v) = 0$. Let the zeros of $A(\lambda;v)$ be $0, \pm\kappa_1, \cdots, \pm\kappa_N$, where $0 < \kappa_1 < \cdots < \kappa_N < m$. On the other hand, one verifies that there exist the real constants D_j, $j = 0, 1, \cdots, N$ such that

$$\phi_+(x, \pm\kappa_j;v) = D_j\phi_-(x, \pm\kappa_j;v), \quad j = 0, 1, \cdots, N. \tag{13}$$

We define the norming constant Γ_j, $j = 0, 1, \cdots, N$ by

$$\Gamma_j = \begin{cases} \dfrac{1}{\displaystyle\int_{-\infty}^{\infty} \|\phi_+(x,0;v)\|^2 dx} = \dfrac{\sqrt{-1}D_0}{2A'(0;v)} & j = 0, \\ \dfrac{2}{\displaystyle\int_{-\infty}^{\infty} \|\phi_+(x,\pm\kappa_j;v)\|^2 dx} = \dfrac{\sqrt{-1}mD_j}{\sqrt{m^2-\kappa_j^2}A'(\pm\kappa_j;v)} & 1 \le j \le N \end{cases} \tag{14}$$

where $\| * \|$ is the Euclidean norm. The above definition of Γ_j for $1 \le j \le N$ seems to be curious. But it is convenient for the inverse problem. Hence we employ this definition according with that of [10]. We call the collection

$$\Sigma_D(v) = \{R_+(\xi;v); 0 = \kappa_0 < \kappa_1 < \cdots < \kappa_N; \Gamma_0, \Gamma_1, \cdots, \Gamma_N\}$$

the scattering data of $D(v)$.

4. Representation of Miura transformation

Let $\phi(x)$ be a solution of (7) then

$$D(v)^2\phi(x) = \lambda^2\phi(x)$$

follows. Hence, by the operator identity (2), we have

$$\begin{pmatrix} H(u_-) & 0 \\ 0 & H(u_+) \end{pmatrix} {}^tT\phi(x) = \zeta(\lambda)^{2t}T\phi(x).$$

This implies

$$H(u_\pm)(\phi_1(x) \pm \phi_2(x)) = \zeta(\lambda)^2(\phi_1(x) \pm \phi_2(x)).$$

Hence one verifies

$$m\frac{\phi_{+1}(x,\lambda;v) \pm \phi_{+2}(x,\lambda;v)}{m \pm \sqrt{-1}(\zeta(\lambda)-\lambda)} = \exp(-\sqrt{-1}\zeta(\lambda)x) + o(1), \quad x \to \infty.$$

Therefore, by the uniqueness of Schrödinger-Jost solutions, we have

$$f_+(x,\zeta(\lambda);u_\pm) = m\frac{\phi_{+1}(x,\lambda;v) \pm \phi_{+2}(x,\lambda;v)}{\sqrt{-1}(\zeta(\lambda)-\lambda) \pm m} \tag{15}$$

$$f_-(x,\zeta(\lambda);u_\pm) = m\frac{\phi_{-1}(x,\lambda;v) \pm \phi_{-2}(x,\lambda;v)}{m \pm \sqrt{-1}(\zeta(\lambda)-\lambda)}. \tag{16}$$

By (10), (15) and (16), one verifies

$$f_-(x,\sigma(\xi);u_\pm) = \frac{\sigma(\xi) \pm \sqrt{-1}m}{\xi}A(\xi;v)\overline{f_+(x,\sigma(\xi);u_\pm)} \pm B(\xi;v)f_+(x,\sigma(\xi);u_\pm).$$

Hence

$$a(\zeta(\lambda);u_\pm) = \frac{\zeta(\lambda) \pm \sqrt{-1}m}{\lambda}A(\lambda;v), \quad \lambda \in \mathcal{R}_+ \tag{17}$$

and

$$b(\sigma(\xi);u_\pm) = \pm B(\xi;v), \quad \xi \in I_m$$

follow. Hence, we have

$$\begin{aligned} r_+(\sigma(\xi);u_\pm) &= \frac{b(\sigma(\xi);u_\pm)}{a(\sigma(\xi);u_\pm)} \\ &= \frac{\pm\xi}{\sigma(\xi) \pm \sqrt{-1}m}\frac{B(\xi;v)}{A(\xi;v)} \\ &= \pm\frac{\xi}{\sigma(\xi) \pm \sqrt{-1}m}R_+(\xi;v), \quad \xi \in I_m. \end{aligned}$$

Next we consider the zeros of $a(\zeta(\lambda);u_{\pm})$. Note (12) and put

$$A^0(\lambda;v)=\frac{\zeta(\lambda)}{\lambda}A(\lambda;v)=1+\int_0^\infty F(x)\exp(2\sqrt{-1}\zeta(\lambda)x)dx.$$

Then, since $\lambda=0,\pm\kappa_1,\cdots,\pm\kappa_N$ are simple zero of $A(\lambda;v)$, we have

$$A^0(0;v)\neq 0$$

and

$$A^0(\pm\kappa_j;v)=0,\quad j=1,2,\cdots,N.$$

By (17), we have

$$a(\zeta(\lambda);u_{\pm})=\frac{\zeta(\lambda)\pm\sqrt{-1}m}{\zeta(\lambda)}A^0(\lambda;v).$$

Since $\zeta(0)=\sqrt{-1}m$, one verifies that

(18) $$a(\zeta(\pm\kappa_j);u_{\pm})=0,\quad j=1,2,\cdots,N,$$

(19) $$a(\zeta(0);u_+)\neq 0$$

and

(20) $$a(\zeta(0);u_-)=0.$$

By (18), $\kappa_j^2-m^2$, $j=1,2,\cdots,N$ turn out to be the discrete eigenvalues of $H(u_{\pm})$. Thus it suffices to determine the norming constants of $H(u_{\pm})$ for our purpose. By (15) and (16), we have

(21) $$f_+(x,\sqrt{-1}\eta_j;u_{\pm})=\pm D_j f_-(x,\sqrt{-1}\eta_j;u_{\pm}),\quad j=0,1,\cdots,N,$$

where D_j, $j=1,2,\cdots,N$ are defined by (13) and

$$\eta_j=\sqrt{m^2-\kappa_j^2},\quad j=1,2,\cdots,N.$$

One verifies

$$\frac{d}{d\zeta}a(\zeta;u_{\pm})|_{\zeta=\sqrt{-1}\eta_j}=\frac{\eta_j A'(\kappa_j;v)}{\eta_j\mp m},\quad j=1,2,\cdots,N,$$

where $A'(\kappa_j;v)=\frac{d}{d\lambda}A(\kappa_j;v)$. By (6), we have

$$\gamma_j(\pm)=\frac{m\mp\eta_j}{m}\Gamma_j,\quad j=1,2,\cdots,N,$$

where $\gamma_j(\pm)$ are the norming constants of the eigenfunctions $f_+(x,\sqrt{-1}\eta_j;u_{\pm})$, $j=1,2,\cdots,N$, respectively.

Note that $-m^2$ is the discrete eigenvalue of $H(u_-)$ by (20), while $-m^2$ is not the discrete eigenvalue of $H(u_+)$ by (19). Hence, next we calculate the norming constant $\gamma_0(-)$ of $f_+(x,\sqrt{-1}m;u_-)$. By (17) and (21), one has

(22) $$\gamma_0(-)=\frac{-D_0}{\sqrt{-1}a'(\sqrt{-1}m;u_-)}.$$

On the other hand, one has

$$\frac{d}{d\zeta}a(\zeta;u_-)|_{\zeta=\sqrt{-1}m} = \frac{A^0(0;v)}{\sqrt{-1}m} = \frac{d}{d\lambda}A(\lambda;v)|_{\lambda=0}.$$

Hence, by (14) and (22), we have

$$\gamma_0(-) = \frac{-D_0}{\sqrt{-1}A'(0;v)} = 2\Gamma_0.$$

Thus we have the following.

Theorem. *Suppose that the absolutely continuous function $v(x)$ belongs to $\boldsymbol{D}_1^1(\boldsymbol{R};m)$ for some positive real number m. Let*

$$\Sigma_D(v) = \{R(\xi;v); 0 = \kappa_0 < \kappa_1 < \cdots < \kappa_N; \Gamma_0, \Gamma_1, \cdots, \Gamma_N\}$$

be the scattering data of the 1-dimensional Dirac operator $D(v)$, then the scattering data of the 1-dimensional Schrödinger operators $H(u_\pm)$ with the potentials $u_\pm(x)$ defined by the Miura transformations (1) *are given by*

$$\Sigma_H(u_+) = \{\frac{\xi}{\sigma(\xi)+\sqrt{-1}m}R(\xi;v); -\eta_1^2 < \cdots < -\eta_N^2; \frac{m-\eta_1}{m}\Gamma_1, \cdots, \frac{m-\eta_N}{m}\Gamma_N\}$$

and

$$\begin{aligned}\Sigma_H(u_-) = \{&\frac{-\xi}{\sigma(\xi)-\sqrt{-1}m}R(\xi;v);\\ &-\eta_0^2 < -\eta_1^2 < \cdots < -\eta_N^2; 2\Gamma_0, \frac{m+\eta_1}{m}\Gamma_1, \cdots, \frac{m+\eta_N}{m}\Gamma_N\},\end{aligned}$$

where $\sigma(\xi) = \sqrt{\xi^2 - m^2}$ for $\xi \in I_m$ and $\eta_j = \sqrt{m^2 - \kappa_j^2}$ for $j = 0, 1, \cdots, N$, particularly, $-\eta_0^2 = -m^2$.

Remark. The correspondences $u_+(x) \mapsto u_-(x)$ and $u_-(x) \mapsto u_+(x)$ are nothing but the Darboux transformations, i.e.,

$$\begin{aligned}H(u_+) + m^2 &= A_- \cdot A_+,\\ H(u_-) + m^2 &= A_+ \cdot A_-,\end{aligned}$$

where $A_\pm = \pm\partial + v(x)$. Hence, as the application of Theorem, we can easily prove Crum's algorithm for removing and adding eigenvalues of the 1-dimensional Schrödinger operator (see e.g. [2, pp. 167–173]). In particular, we have immediately

$$\begin{aligned}r_+(\sigma;u_+) &= \frac{\sqrt{-1}\eta_0 - \sigma}{\sqrt{-1}\eta_0 + \sigma}r_+(\sigma;u_-),\\ \text{Spec } H(u_+) &= \text{Spec } H(u_-) \setminus \{-\eta_0^2\}\end{aligned}$$

and

$$\gamma_j(+) = \frac{\eta_0 - \eta_j}{\eta_0 + \eta_j}\gamma_j(-), \quad j = 1, 2, \cdots, N,$$

where Spec A denotes the set of the point spectrum of the operator A.

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