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Darboux Transformation and Λ-operator

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Abstract

Firstly, the fundamental equality of Darboux transformation of the differential operator $H(u) = -\partial^2 + u$ is proved based on the Kupershmidt-Wilson factorization of the associated Λ-operator $\Lambda(u) = \partial^{-1} \cdot (2^{-1}u' + u\partial - 4^{-1}\partial^3)$. Secondly, elementary algebraic properties of the Darboux transformation are studied with the aid of Λ-operator. Finally, as an application of the fundamental equality of Darboux transformation, solutions of the higher order KdV equation are constructed.

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1. Introduction

In this paper we study the Darboux transformation of the differential operator

$$H(u) = -\partial^2 + u(x),$$

where $u(x)$ is a meromorphic function defined in the region Ω of the complex plane and $\partial = {}' = d/dx$. The Darboux transformation of $H(u)$ is defined as follows; Suppose $\phi(u) \in \ker H(u) \setminus \{0\}$ and put $q(x) = \partial \log \phi = \phi'(x)/\phi(x)$ then $H(u)$ is factorized as $H(u) = A_+ \cdot A_-$, where $A_\pm = \pm\partial + q(x)$, and $A \cdot B$ denotes the product of the operators A and B. Interchanging the factors $A_\pm$, we obtain the another operator $\hat{H}(u) = A_- \cdot A_+$. One immediately verifies

$$\hat{H}(u) = -\partial^2 + u(x) - 2\partial^2 \log \phi(x).$$

The operator $\hat{H}(u)$ and its coefficient

$$u^*(x) = u(x) - 2\partial^2 \log \phi(x)$$

are called the Darboux transformation of $H(u)$ and $u(x)$ respectively.

The idea of this approach was originated by Darboux [3]. After that, Burchnall and Chaundy [1] developed a similar method in their study of semi-commutative operators. Subsequently, Crum [2] used this method as an algorithm for adding or removing eigenvalues of Sturm-Liouville operator. Recently, there are many applications of the method of Darboux transformation to the soliton theory [6].

The first aim of the present paper is to clarify certain algebraic properties of Darboux transformation of $H(u)$ by using the Λ-operator

$$\Lambda(u) = \partial^{-1} \cdot \left(\frac{1}{2} u'(x) + u(x)\partial - \frac{1}{4}\partial^3\right).$$

More precisely, we investigate several criteria for $u(x)$ such that $u^*(x)$ is the rational function of $u(x)$ and its derivatives.

On the other hand, the Λ-operator generates the infinite sequence of the differential polynomials $Z_n(u)$, $n \in \boldsymbol{Z}_+$, which are called the KdV polynomials. In [8, p623, Theorem 3.2], the fundamental equality of the Darboux transformation

$$B_- Z_n(u) = B_+ Z_n(u^*)$$

is proved, where $B_\pm = \pm\,\partial + 2q(x)$. The second aim of the present paper is to give the alternative proof of the above equality based on the Kupershmidt-Wilson factorization of $\Lambda(u)$.

In [8], rational function solutions of the nonstationary higher order KdV equation are constructed as an application of the fundamental equality of the Darboux transformation. The third aim of the present paper is to generalize this method and construct some class of solutions of the nonstationary higher order KdV equation.

The contents of this paper are as follows. Section 2 is devoted to preliminaries such as KdV polynomials, the notion of Λ-rank and Λ-algorithm. In section 3, the fundamental equality of Darboux transformation is proved with the aid of the Kupershmidt-Wilson factorization of $\Lambda(u)$. In section 4, the notion of rational Darboux transformation is introduced. In section 5, the rational Darboux transformation with spectral parameter is discussed when $\mathrm{rank}_\Lambda u(x)$ is finite. In section 6, we study the Λ-rank of Darboux transformation. In section 7, solutions of the nonstationary higher order KdV equation are constructed. A part of the present work is announced in [11].

2. Preliminaries

In this section, the necessary materials are summarized from [10].

Suppose that $u(x)$ is a meromorphic function defined in the region Ω. Throughout the paper, we assume that $u(x)$ is not constant in Ω. Let $\mathscr{A}_u$ be the differential algebra of differential polynomials in derivatives $u^{(j)} = (d/dx)^j u(x)$, $j \in \boldsymbol{Z}_+$ of the meromorphic function $u(x)$ with constant coefficients. Note that $\mathscr{A}_u$ can be simultaneously regarded as an integral domain, i.e, a commutative unitary ring without zero devisor. We can uniquely define the infinite sequence of differential polynomials $Z_n(u) \in \mathscr{A}_u$, $n \in \boldsymbol{Z}_+$ by the reccurence relation

$$Z_n(u) = \Lambda(u) Z_{n-1}(u), \qquad n \in \boldsymbol{N}$$

with $Z_0(u) = 1$. They are called the KdV polynomials. Let $V(u)$ be the vector space over the complex number field $\boldsymbol{C}$ spanned by $Z_n(u)$, $n \in \boldsymbol{Z}_+$;

$$V(u) = \bigcup_{n \in \boldsymbol{Z}_+} \boldsymbol{C} Z_n(u).$$

When $\dim_{\boldsymbol{C}} V(u) < \infty$, the Λ-rank of $u(x)$ is defined by

$$\operatorname{rank}_\Lambda u(x) = \dim_{\boldsymbol{C}} V(u) - 1.$$

If $n = \operatorname{rank}_\Lambda u(x) < \infty$, $V(u)$ is spanned by $Z_0(u), Z_1(u), \cdots, Z_n(u)$;

$$V(u) = \bigoplus_{j=0}^{n} \boldsymbol{C} Z_j(u).$$

Therefore there uniquely exist $a_\nu(u)$, $\nu = 0, 1, \cdots, n$ such that

$$Z_{n+1}(u(x)) = \sum_{\nu=0}^{n} a_\nu(u) Z_\nu(u(x)).$$

We call $a_\nu(u)$, $\nu = 0, 1, \cdots, n$ the Λ-characteristic coefficients of $u(x)$. On the other hand, the following expansion formula of the KdV polynomial holds: Define the coefficients $\alpha_\nu^{(n)}$, $\nu = 0, 1, \cdots, n$ by the recurrence relation

$$\alpha_j^{(n)} = \begin{cases} 1, & j = n \\ \alpha_{j-1}^{(n-1)} + \alpha_j^{(n-1)}, & j = 1, 2, \cdots, n-1 \\ \dfrac{(2n)!}{2^{2n}(n!)^2}, & j = 0 \end{cases}$$

then

$$Z_n(u(x) + \lambda) = \sum_{j=0}^{n} \alpha_j^{(n)} Z_j(u(x)) \lambda^{n-j} \tag{1}$$

holds. By (1), one has

$$V(u+\lambda) = V(u), \qquad \lambda \in \boldsymbol{C}.$$

Therefore, if $n = \operatorname{rank}_\Lambda u(x) < \infty$ then

$$\operatorname{rank}_\Lambda u(x) = \operatorname{rank}_\Lambda (u(x) - \lambda) \tag{2}$$

holds for any $\lambda \in \boldsymbol{C}$. Hence there exist the Λ-characteristic coefficients $a_\nu(u-\lambda)$, $\nu = 0, 1, \cdots, n$ of $u(x) - \lambda$. The coefficients $a_\nu(u-\lambda)$, $\nu = 0, 1, \cdots, n$ are the polynomials of degree $n - \nu + 1$;

$$a_\nu(u-\lambda) = -\alpha_\nu^{(n+1)}\lambda^{n-\nu+1} + \sum_{\mu=\nu}^{n} \alpha_\nu^{(\mu)} a_\mu(u)\lambda^{\mu-\nu}.$$

Moreover put

$$F(x, \lambda) = Z_n(u(x) - \lambda) - \sum_{\nu=1}^{n} a_\nu(u-\lambda) Z_{\nu-1}(u(x) - \lambda), \tag{3}$$

then $F(x, \lambda)$ is not identically zero for any $\lambda \in \boldsymbol{C}$. Put

$$\Delta(\lambda; u) = F_x(a, \lambda)^2 - 2F(a, \lambda)F_{xx}(a, \lambda) + 4(u(a) - \lambda)F(a, \lambda)^2$$

and

$$\Gamma(u) = \{\lambda \in \boldsymbol{C} \mid \Delta(\lambda; u) = 0\}, \tag{4}$$

then $\Gamma(u) \neq \emptyset$. Moreover, since $\Delta(\lambda; u)$ is the polynomial of degree $2n+1$, $\#\Gamma(u) \leq 2n+1$ follows. If $\lambda_j \in \Gamma(u)$, $j = 0, 1, \cdots, 2n$ then

$$f(x, \lambda_j) = F(x, \lambda_j)^{\frac{1}{2}}$$

are the corresponding eigenfunctions of the eigenvalue problem of

$$(H(u) - \lambda_j) f(x) = 0.$$

The Λ-algorithm is such a method to study the problem related to the spectrum of the operator $H(u)$ on the basis of the algebraic properties of the corresponding Λ-operator $\Lambda(u)$ as above.

3. The proof of the fundamental equality

In what follows, we fix the fundamental system $f_\nu(x)$, $\nu = 1, 2$ of the solutions to the differential equation

$$H(u)f(x) = -f''(x) + u(x)f(x) = 0.$$

Let $\alpha \in \boldsymbol{C}^* = \boldsymbol{C} \cup \{\infty\}$ and put

$$q(x;\alpha)=\begin{cases}\partial\log(f_1(x)+\alpha f_2(x)), & \alpha\in\boldsymbol{C}\\ \partial\log f_2(x), & \alpha=\infty.\end{cases}$$

Then $H(u)$ is factorized as

$$H(u)=A_+(\alpha)\cdot A_-(\alpha),$$

where

$$A_\pm(\alpha)=\pm\partial+q(x;\alpha).$$

The Darboux transformation $\hat{H}(u;\alpha)$ of $H(u)$ is defined by

$$\hat{H}(u;\alpha)=A_-(\alpha)\cdot A_+(\alpha).$$

Put

$$u_\alpha^*=u^*(x;\alpha)=u(x)-2q'(x;\alpha),$$

then $\hat{H}(u;\alpha)=H(u_\alpha^*)$ follows. On the other hand, put

$$B_\pm(\alpha)=\pm\partial+2q(x;\alpha),$$

then Kupershmidt and Wilson [5] discovered that the Λ-operators $\Lambda(u)$ and $\Lambda(u_\alpha^*)$ are factorized (K-W factorization) as

$$\Lambda(u)=\frac{1}{4}\partial^{-1}\cdot B_+(\alpha)\cdot\partial\cdot B_-(\alpha) \tag{5}$$

and

$$\Lambda(u_\alpha^*)=\frac{1}{4}\partial^{-1}\cdot B_-(\alpha)\cdot\partial\cdot B_+(\alpha). \tag{6}$$

The linear operators $B_\pm(\alpha)$ can be formally regarded as the Fréchet derivatives of the Miura transformations $q^2+q'=u$ and $q^2-q'=u_\alpha^*$. In [7], the study on the Miura transformation from this standpoint of view is developed by the second author of the present work.

First we show the following.

Lemma 1. *For any* $\alpha\in\boldsymbol{C}^*$, *the identities*

$$B_+(\alpha)1=B_-(\alpha)1 \tag{7}$$

and

$$B_+(\alpha)\cdot\partial^{-1}\cdot B_-(\alpha)=B_-(\alpha)\cdot\partial^{-1}\cdot B_+(\alpha) \tag{8}$$

are valid.

PROOF. One readily verifies

$$B_{\pm}(\alpha)1 = 2q(x;\alpha).$$

Hence (7) follows. Moreover, by direct calculation, we have

$$B_{\pm}(\alpha)\cdot\partial^{-1}\cdot B_{\mp}(\alpha) = -\partial + 4q(x;\alpha)\partial^{-1}\cdot q(x;\alpha).$$

Hence (8) follows. This completes the proof.

The following identities were obtained by the first author in [8]. Since it plays fundamental roles in the study of Darboux transformation, we want to call it *the fundamental equality*. Here we give a simplified alternative proof based on the K-W factorization.

Theorem 2. *The identities*

$$B_{-}(\alpha)Z_n(u) = B_{+}(\alpha)Z_n(u_\alpha^*) \tag{9}$$

hold for any $\alpha\in\boldsymbol{C}^*$ *and* $n\in\boldsymbol{Z}_+$.

PROOF. Note that $Z_0(u) = Z_0(u_\alpha^*) = 1$. By lemma 1 and K-W factorization (5) and (6), we have

$$\begin{aligned}
B_{-}(\alpha)Z_n(u) &= B_{-}(\alpha)\cdot\Lambda(u)^n Z_0(u)\\
&= \left(\frac{1}{4}\right)^n B_{-}(\alpha)\cdot(\partial^{-1}\cdot B_{+}(\alpha)\cdot\partial\cdot B_{-}(\alpha))^n Z_0(u)\\
&= \left(\frac{1}{4}\right)^n (B_{-}(\alpha)\cdot\partial^{-1}\cdot B_{+}(\alpha)\cdot\partial)^n(B_{-}(\alpha)1)\\
&= \left(\frac{1}{4}\right)^n (B_{+}(\alpha)\cdot\partial^{-1}\cdot B_{-}(\alpha)\cdot\partial)^n(B_{+}(\alpha)1)\\
&= \left(\frac{1}{4}\right)^n B_{+}(\alpha)\cdot(\partial^{-1}\cdot B_{-}(\alpha)\cdot\partial\cdot B_{+}(\alpha))^n Z_0(u_\alpha^*)\\
&= B_{+}(\alpha)\cdot\Lambda(u_\alpha^*)^n Z_0(u_\alpha^*)\\
&= B_{+}(\alpha)Z_n(u_\alpha^*).
\end{aligned}$$

This completes the proof.

4. Rational Darboux transformation

We say that the operator $H(u)$ admits the *rational Darboux transformation* if and only if there exist $\phi(x)\in\ker H(u)\setminus\{0\}$ such that

$$u^*(x) = u(x) - 2\partial^2 \log \phi(x) \in \mathscr{K}_u,$$

where $\mathscr{K}_u$ is the field of quotients of the integral domain $\mathscr{A}_u$.

Since the Darboux transformation of $u(x)$ is the 1-parameter family $u^*_\alpha = u^*(x;\alpha) = u(x) - 2q'(x;\alpha)$, we consider the set

$$\chi(u) = \{\alpha \in \boldsymbol{C}^* \mid u^*(x;\alpha) \in \mathscr{K}_u\}.$$

While $\chi(u)$ itself depends on choice of the fundamental system $f_\nu(x)$, $\nu = 1, 2$ of solutions, the cardinal number $\#\chi(u)$ does not depend on it. Let $\mathscr{R}_k(\Omega)$ be the set of all meromorphic functions $u(x)$ defined in Ω such that $\#\chi(u) \geq k$ for $k \in \boldsymbol{N}$. Moreover let $\mathscr{R}_\infty(\Omega)$ be the set of all meromorphic functions $u(x)$ defined in Ω such that $\chi(u) = \boldsymbol{C}^*$. In this section we investigate several criteria for $u(x) \in \mathscr{R}_\infty(\Omega)$.

Since

$$\begin{aligned} u^*(x;\alpha) &= u(x) - 2q'(x;\alpha) \\ &= -u(x) + 2q(x;\alpha)^2, \end{aligned}$$

it follows that $u^*(x;\alpha) \in \mathscr{K}_u$ if and only if $q(x;\alpha) \in \mathscr{K}_u$.

Put

$$\eta(x;u) = \frac{f_2(x)}{f_1(x)}.$$

Of course, $\eta(x;u)$ depends on choice of the fundamental system $f_\nu(x)$, $\nu = 1, 2$. However, the property "$\eta(x;u) \in \mathscr{K}_u$" does not depend on it.

First we have the following, which is the most fundamental criterion for $u(x)$ to belong to $\mathscr{R}_\infty(\Omega)$.

Proposition 3. *The nontrivial meromorphic function $u(x)$ defined in Ω belongs to $\mathscr{R}_\infty(\Omega)$ if and only if $\eta(x;u) \in \mathscr{K}_u$.*

PROOF. One has

$$\eta'(x;u) = \frac{W(f_1, f_2)}{f_1(x)^2}, \tag{10}$$

where $W(f, g) = fg' - f'g$ is the Wronskian. Hence one verifies

$$f_1(x) = c\eta'(x;u)^{-\frac{1}{2}}$$

and

$$f_2(x) = c\eta'(x;u)^{-\frac{1}{2}}\eta(x;u),$$

where $c = W(f_1, f_2)^{\frac{1}{2}}$. Therefore we have

$$(11) \qquad q(x;\alpha) = \begin{cases} -\dfrac{\eta''(x;u)}{2\eta'(x;u)} + \dfrac{\alpha\eta'(x;u)}{1+\alpha\eta(x;\alpha)}, & \alpha \in \boldsymbol{C} \\ -\dfrac{\eta''(x;u)}{2\eta'(x;u)} + \dfrac{\eta'(x;u)}{\eta(x;u)}, & \alpha = \infty. \end{cases}$$

Firstly assume that $\eta(x;u) \in \mathscr{K}_u$. Then, by (11), $q(x;\alpha) \in \mathscr{K}_u$ are valid for any $\alpha \in \boldsymbol{C}^*$. Hence $u^*(x;\alpha) \in \mathscr{K}_u$ holds for any $\alpha \in \boldsymbol{C}^*$, i.e, $u(x) \in \mathscr{R}_\infty(\Omega)$. Secondly assume that $u(x) \in \mathscr{R}_\infty(\Omega)$. Then $q(x;\alpha) \in \mathscr{K}_u$ holds for any $\alpha \in \boldsymbol{C}^*$. Particularly, $q(x;\alpha)$, $\alpha = 0, 1, \infty$ belong to $\mathscr{K}_u$. Hence

$$\frac{q(x;\infty) - q(x;0)}{q(x;1) - q(x;0)} = 1 + \frac{1}{\eta(x;u)} \in \mathscr{K}_u$$

follows, i.e, $\eta(x;u) \in \mathscr{K}_u$. This completes the proof.

Note that the above proof simultaneously implies the following.

Corollary 4. *If $k \geq 3$ then $\mathscr{R}_k(\Omega) = \mathscr{R}_\infty(\Omega)$ are valid.*

On the other hand, if one cannot decide whether $\eta(x;u)$ itself belongs to $\mathscr{K}_u$ or not, the following criterion is effective.

Proposition 5. *If $u(x) \in \mathscr{R}_2(\Omega) \setminus \{0\}$ and $\eta'(x;u) \in \mathscr{K}_u$ then $u(x) \in \mathscr{R}_\infty(\Omega)$ holds.*

PROOF. By the assumption and (10), $f_1(x)^2 \in \mathscr{K}_u$ follows. Suppose that $\alpha, \beta \in \chi(u)$ and $\alpha \neq \beta$. We can assume $\alpha, \beta \neq \infty$ without loss of generality. By direct caluculation, we have

$$q(x;\alpha) - q(x;\beta) = \frac{(\alpha - \beta)W(f_1, f_2)}{f_1(x)^2(1+\alpha\eta(x;u))(1+\beta\eta(x;u))}.$$

Hence $(1+\alpha\eta(x;u))(1+\beta\eta(x;u)) \in \mathscr{K}_u$ follows. Therefore, by differentiating $(1+\alpha\eta)(1+\beta\eta)$, one verifies

$$(\alpha + \beta + \alpha\beta\eta(x;u))\eta'(x;u) \in \mathscr{K}_u.$$

Since $\eta'(x;u) \in \mathscr{K}_u \setminus \{0\}$, $\eta(x;u) \in \mathscr{K}_u$ follows. Hence, by Proposition 4, $u(x) \in \mathscr{R}_\infty(\Omega)$ is valid. This completes the proof.

The following is useful for investigating the rational Darboux transformation of $H(u)$ with spectral parameter when $\operatorname{rank}_\Lambda u(x) < \infty$, which is discussed in the next section.

Proposition 6. *The nontrivial meromorphic function $u(x)$ defined in Ω belongs to $\mathscr{R}_\infty(\Omega)$ if and only if there exists $F(x) \in \mathscr{K}_u \setminus \{0\}$ such that*

(12) $$F'(x)^2 - 2F(x)F''(x) + 4u(x)F(x)^2 = 0$$

and $\partial^{-1}(F(x)^{-1}) \in \mathscr{K}_u$.

PROOF. First suppose $u(x) \in \mathscr{R}_\infty(\Omega)$. Then, by Proposition 3, $\eta(x; u) \in \mathscr{K}_u$ holds. Since $f_1(x), f_2(x)$ are linearly independent, $\eta'(x; u)$ does not identically vanish, we can set

$$F(x) = \frac{W(f_1, f_2)}{\eta'(x; u)}.$$

$F(x) \in \mathscr{K}_u$ holds. By (10), $F(x)^{\frac{1}{2}} \in \ker H(u)$ follows. Then, one verifies (12) by direct calculation. Moreover we have

$$\partial^{-1}(F(x)^{-1}) = \frac{1}{W(f_1, f_2)} \partial^{-1}(\eta'(x; u))$$

$$= \frac{\eta(x; u)}{W(f_1, f_2)} + \text{Const.}$$

Hence $\partial^{-1}(F(x)^{-1}) \in \mathscr{K}_u$ follows. Conversely put

$$f_1(x) = F(x)^{\frac{1}{2}}$$

and

$$f_2(x) = F(x)^{\frac{1}{2}} \partial^{-1}(F(x)^{-1}),$$

then, by (12), $f_\nu(x)$, $\nu = 1, 2$ turn out to be the fundamental system of solutions to $H(u)f = 0$. Since

$$\eta(x; u) = \frac{f_2(x)}{f_1(x)} = \partial^{-1}(F(x)^{-1}) \in \mathscr{K}_u,$$

$u(x) \in \mathscr{R}_\infty(\Omega)$ follows. This completes the proof.

Next we investigate some elementary examples. Put

$$u_\xi(x) = \frac{\xi(\xi - 1)}{x^2}, \qquad \xi \in \boldsymbol{C} \setminus \{0, 1\},$$

and let us consider the Euler differential equation

$$-f''(x) + u_\xi(x) f(x) = 0.$$

The fundamental system $f_\nu(x)$, $\nu = 1, 2$ of solutions of this equation are as follows; if $\xi \neq \frac{1}{2}$ then $f_1(x) = x^\xi$ and $f_2(x) = x^{1-\xi}$; if $\xi = \frac{1}{2}$ then $f_1(x) = x^{\frac{1}{2}}$ and

$f_2(x) = x^{\frac{1}{2}} \log x$. Hence we have

$$\eta(x; u_\xi) = \begin{cases} x^{1-2\xi}, & \xi \neq \frac{1}{2} \\ \log x, & \xi = \frac{1}{2}. \end{cases}$$

Moreover suppose $\xi \in \boldsymbol{Z} \setminus \{0, 1\}$ then, by direct calculation, we have

$$\eta(x; u_\xi) = x^{1-2\xi} = -\frac{1}{2}\left(\frac{1}{\xi(\xi-1)}\right)^{\xi-1} u_\xi'(x) u_\xi(x)^{\xi-2} \in \mathscr{K}_{u_\xi}.$$

Hence, if $\xi \in \boldsymbol{Z} \setminus \{0, 1\}$ then

$$u_\xi(x) = \frac{\xi(\xi-1)}{x^2} \in \mathscr{R}_\infty(\boldsymbol{C})$$

follows. Next suppose that $\xi = \dfrac{2\mu - 1}{2}$, $\mu \in \boldsymbol{Z} \setminus \{1\}$ then we have

$$\eta(x; u_\xi) = x^{-2\mu} = \left(\frac{4}{(2\mu-1)(2\mu-3)}\right)^{\mu} u_\xi(x)^\mu \in \mathscr{K}_{u_\xi}.$$

Hence $\mu \in \boldsymbol{Z} \setminus \{1\}$ then

$$u_\xi(x) = \frac{(2\mu-1)(2\mu-3)}{4x^2} \in \mathscr{R}_\infty(\boldsymbol{C})$$

follows. Moreover one easily verifies

$$u_{\frac{1}{2}}(x) = -\frac{1}{4x^2} \in \mathscr{R}_1(\boldsymbol{C}) \setminus \mathscr{R}_2(\boldsymbol{C}).$$

Furthermore, one can see easily that if $\xi \in \boldsymbol{C} \setminus \frac{1}{2}\boldsymbol{Z}$ then $u_\xi(x) \in \mathscr{R}_2(\boldsymbol{C}) \setminus \mathscr{R}_\infty(\boldsymbol{C})$ follows, where $\frac{1}{2}\boldsymbol{Z} = \{\frac{1}{2}n \mid n \in \boldsymbol{Z}\}$. The theory of Darboux transformation of the above rational functions are extensively studied by Duistermaat and Grünbaum [4].

5. Rational Darboux transformation with spectral parameter

In this section, we consider the Darboux transformation of $H(u)$ with spectral parameter λ, when $\mathrm{rank}_A u(x) < \infty$.

First we fix the fundamental system $f_\nu(x, \lambda)$, $\nu = 1, 2$ of the solutions of the eigenvalue problem

$$(H(u) - \lambda) f(x) = 0, \qquad \lambda \in \boldsymbol{C}.$$

Put

$$(13)\qquad q(x, \lambda; \alpha) = \begin{cases} \partial \log(f_1(x, \lambda) + \alpha f_2(x, \lambda)), & \alpha \neq \infty \\ \partial \log f_2(x, \lambda), & \alpha = \infty, \end{cases}$$

and

$$u^*_{\lambda,\alpha} = u^*(x, \lambda; \alpha) = u(x) - 2q'(x, \lambda; \alpha).$$

We call $H(u^*_{\lambda,\alpha})$ and $u^*_{\lambda,\alpha}$ the Darboux transformation of $H(u)$ and $u(x)$ with spectral parameter respectively.

Now suppose

$$n = \mathrm{rank}_\Lambda u(x) < \infty$$

and let $a_\nu(u-\lambda)$, $\nu = 0, 1, \cdots, n$ be the Λ-characteristic coefficients of $u(x) - \lambda$. Define $F(x, \lambda)$ by (3) and the set $\Gamma(u)$ by (4). As explained in Section 2, we can set as follows;

$$(14)\qquad f_\nu(x, \lambda_j) = \begin{cases} F(x, \lambda_j)^{\frac{1}{2}}, & \nu = 1 \\ F(x, \lambda_j)^{\frac{1}{2}} \partial^{-1}(F(x, \lambda_j)^{-1}), & \nu = 2 \end{cases}$$

for $\lambda_j \in \Gamma(u)$, $j = 0, 1, \cdots, 2n$. Note that $F(x, \lambda_j)$ are the differential polynomials, i.e, $F(x, \lambda_j) \in \mathscr{A}_u$, $j = 0, 1, \cdots, 2n$. One verifies

$$u^*(x, \lambda_j; 0) = u(x) - \frac{F''(x, \lambda_j)F(x, \lambda_j) - F'(x, \lambda_j)^2}{F(x, \lambda_j)^2}.$$

Thus we have the following.

Proposition 7. *Suppose* $n = \mathrm{rank}_\Lambda u(x) < \infty$ *then* $u(x) - \lambda_j \in \mathscr{R}_1(\Omega)$ *holds for any* $\lambda_j \in \Gamma(u)$, $j = 0, 1, \cdots, 2n$.

Since

$$\eta(x; u - \lambda_j) = \partial^{-1}(F(x, \lambda_j)^{-1})$$

and $\mathscr{K}_{u-\lambda} = \mathscr{K}_u$ for any $\lambda \in \boldsymbol{C}$, one has immediately the following from Proposition 3.

Proposition 8. *Suppose* $n = \mathrm{rank}_\Lambda u(x) < \infty$ *and* $\lambda_j \in \Gamma(u)$ *then* $u(x) - \lambda_j$ *belongs to* $\mathscr{R}_\infty(\Omega)$ *if and only if*

$$\partial^{-1}(F(x, \lambda_j)^{-1}) \in \mathscr{K}_u.$$

On the other hand, since

$$\eta'(x; u - \lambda_j) = F(x, \lambda_j)^{-1} \in \mathscr{K}_u,$$

the following fact follows from Proposition 5.

Proposition 9. *Suppose* $n = \mathrm{rank}_\Lambda u(x) < \infty$ *and* $u(x) - \lambda_j \in \mathscr{R}_2(\Omega)$ *for some* $\lambda_j \in \Gamma(u)$, *then* $u(x) - \lambda_j \in \mathscr{R}_\infty(\Omega)$ *follows.*

6. Λ-rank of $u^*(x, \lambda; \alpha)$

In this section we investigate $\mathrm{rank}_\Lambda u^*(x, \lambda_j; \alpha)$ when $\mathrm{rank}_\Lambda u(x) < \infty$.

Suppose $n = \mathrm{rank}_\Lambda u(x) < \infty$ and define $f_\nu(x, \lambda_j)$, $\nu = 1, 2$ by (14), which are the fundamental system of solutions of

$$(H(u) - \lambda_j) f(x) = 0, \quad \lambda_j \in \Gamma(u), \quad j = 0, 1, \cdots, 2n.$$

Moreover, define $q(x, \lambda_j; \alpha)$, $j = 0, 1, \cdots, n$ by (13). Put

$$B_\pm(\lambda_j, \alpha) = \pm \partial + 2q(x; \lambda_j, \alpha)$$

then the K-W factorizations

$$\Lambda(u - \lambda_j) = \frac{1}{4} \partial^{-1} \cdot B_+(\lambda_j, \alpha) \cdot \partial \cdot B_-(\lambda_j, \alpha)$$

and

$$\Lambda(u^*_{\lambda_j, \alpha} - \lambda_j) = \frac{1}{4} \partial^{-1} \cdot B_-(\lambda_j, \alpha) \cdot \partial \cdot B_+(\lambda_j, \alpha)$$

follow. Hence, by Theorem 2, we have

$$\begin{aligned} (15) \qquad & B_+(\lambda_j, \alpha)\Big(Z_{n+1}(u^*_{\lambda_j, \alpha} - \lambda_j) - \sum_{\nu=0}^{n} a_\nu(u - \lambda_j) Z_\nu(u^*_{\lambda_j, \alpha} - \lambda_j)\Big) \\ & = B_-(\lambda_j, \alpha)\Big(Z_{n+1}(u - \lambda_j) - \sum_{\nu=0}^{n} a_\nu(u - \lambda_j) Z_\nu(u - \lambda_j)\Big) = 0. \end{aligned}$$

Operating with $B_-(\lambda_j, \alpha) \cdot \partial$ on the both side of the above from the left, we have

$$\partial Z_{n+2}(u^*_{\lambda_j, \alpha} - \lambda_j) - \sum_{\nu=0}^{n} a_\nu(u - \lambda_j) \partial Z_{\nu+1}(u^*_{\lambda_j, \alpha} - \lambda_j) = 0.$$

This implies

$$Z_{n+2}(u^*_{\lambda_j, \alpha} - \lambda_j) - \sum_{\nu=0}^{n} a_\nu(u - \lambda_j) Z_{\nu+1}(u^*_{\lambda_j, \alpha} - \lambda_j) = \text{Const.}$$

Hence, by [9, p41, lemma 5.1],

$$V(u^*_{\lambda_j, \alpha} - \lambda_j) = \bigcup_{\nu \in \boldsymbol{Z}_+} \boldsymbol{C} Z_\nu(u^*_{\lambda_j, \alpha} - \lambda_j) = \bigcup_{\nu=0}^{n+1} \boldsymbol{C} Z_\nu(u^*_{\lambda_j, \alpha} - \lambda_j)$$

follows. Thus we proved

$$\dim_{\mathbf{C}} V(u^*_{\lambda_j,\alpha} - \lambda_j) \leq n + 2,$$

in other words, we have

$$\operatorname{rank}_\Lambda (u^*_{\lambda_j,\alpha} - \lambda_j) \leq 1 + \operatorname{rank}_\Lambda u(x) = n + 1.$$

On the other hand, by Darboux's lemma [12, p88, lemma 1],

$$(H(u^*_{\lambda_j,\alpha}) - \lambda_j) g(x, \lambda_j; \alpha) = 0,$$

hold, where

$$g(x, \lambda_j; \alpha) = \begin{cases} \dfrac{1}{f_1(x, \lambda_j) + \alpha f_2(x, \lambda_j)}, & \alpha \neq \infty \\ \dfrac{1}{f_2(x, \lambda_j)}, & \alpha = \infty. \end{cases}$$

By the definition of $u^*(x, \lambda_j; \alpha)$, one verifies

$$u(x) = u^*(x, \lambda_j; \alpha) - 2\partial^2 \log g(x, \lambda_j; \alpha),$$

i.e, $H(u)$ can be regarded as the Darboux transformation $H(u^*_{\lambda_j,\alpha})$ with spectral parameter. Hence, by the above discussion, one has

$$n = \operatorname{rank}_\Lambda (u - \lambda_j) \leq 1 + \operatorname{rank}_\Lambda (u_{\lambda_j,\alpha} - \lambda_j),$$

in other words,

$$n - 1 \leq \operatorname{rank}_\Lambda (u^*_{\lambda_j,\alpha} - \lambda_j).$$

From (2),

$$\operatorname{rank}_\Lambda u^*_{\lambda_j,\alpha} = \operatorname{rank}_\Lambda (u^*_{\lambda_j,\alpha} - \lambda_j)$$

follows. Thus we proved the following.

Theorem 10. *If* $n = \operatorname{rank}_\Lambda u(x) < \infty$ *then*

$$n - 1 \leq \operatorname{rank}_\Lambda u^*(x, \lambda_j; \alpha) \leq n + 1$$

holds for any $\lambda_j \in \Gamma(u)$, $j = 0, 1, \cdots, 2n$ *and* $\alpha \in \mathbf{C}^*$.

Crum's algorithm [2] is based on the above fact.

7. Solutions of KdV equation

In this section, we show that the function $u^*_{\lambda_j,\alpha}$ constructed in the preceding section solves the some nonstationary higher order KdV equation by applying the fundamental equality in Theorem 2.

Put

$$G(x, \lambda_j; \alpha) = Z_{n+1}(u^*_{\lambda_j,\alpha} - \lambda_j) - \sum_{\nu=0}^{n} a_\nu(u - \lambda_j) Z_\nu(u^*_{\lambda_j,\alpha} - \lambda_j),$$

then, by (15), $G(x, \lambda_j; \alpha)$ turns out to solve the first order differential equation

$$G'(x, \lambda_j; \alpha) + 2q(x, \lambda_j; \alpha) G(x, \lambda_j; \alpha) = 0.$$

Hence, if $\alpha \neq \infty$, we have

$$\partial \log ((f_1(x, \lambda_j) + \alpha f_2(x, \lambda_j))^2 G(x, \lambda_j; \alpha)) = 0.$$

This implies that there exist the constants $c_j(\alpha)$, $j = 0, 1, \cdots, 2n$, which rationally depend only on the parameter α, such that

$$G(x, \lambda_j; \alpha) = \frac{c_j(\alpha)}{(f_1(x, \lambda_j) + \alpha f_2(x, \lambda_j))^2}.$$

On the other hand, by direct calculation, one verifies

$$\frac{\partial}{\partial \alpha} u^*(x, \lambda_j; \alpha) = 4W(f_1, f_2) \frac{f_1'(x, \lambda_j) + \alpha f_2'(x, \lambda_j)}{(f_1(x, \lambda_j) + \alpha f_2(x, \lambda_j))^3}.$$

This implies

$$d_j(\alpha) \frac{\partial}{\partial \alpha} u^*_{\lambda_j,\alpha} = \frac{\partial}{\partial x} G(x, \lambda_j; \alpha),$$

where $d_j(\alpha) = -c_j(\alpha)/2W(f_1, f_2)$. By (1), one can explicitly calculate the coefficients $b_\nu(\lambda_j)$, $\nu = 0, 1, \cdots, n$, which are polynomials in one variable λ_j, such that

$$G(x, \lambda_j; \alpha) = Z_{n+1}(u^*_{\lambda_j,\alpha}) + \sum_{\nu=0}^{n} b_\nu(\lambda_j) Z_\nu(u^*_{\lambda_j,\alpha}).$$

Thus we proved the following.

Theorem 11. *If $n = \operatorname{rank}_\Lambda u(x) < \infty$ and $\lambda_j \in \Gamma(u)$, then $u^*_{\lambda_j,\alpha}$ solves the nonstationary higher order KdV equation*

$$d_j(\alpha) \frac{\partial}{\partial \alpha} u^*_{\lambda_j,\alpha} = \frac{\partial}{\partial x} \left(Z_{n+1}(u^*_{\lambda_j,\alpha}) + \sum_{\nu=0}^{n} b_\nu(\lambda_j) Z_\nu(u^*_{\lambda_j,\alpha})\right).$$

Moreover, if $\partial^{-1}(F(x, \lambda_j)^{-1}) \in \mathscr{K}_u$, then the solution $u^*_{\lambda_j,\alpha}$ belongs to $\mathscr{K}_u$ for any $\alpha \in \boldsymbol{C}^*$.

The above theorem is the generalization of [8, p626, Theorem 4.2].

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