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Theory of (Vector Valued) Fourier Hyperfunctions. Their Realization as Boundary Values of (Vector Valued) Slowly Increasing Holomorphic Function, (V)

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Abstract

We realize partial mixed Fourier hyperfunctions and Fréchet-space-valued partial mixed Fourier hyperfunctions as boundary values of (Fréchet-space-valued) partially slowly increasing holomorphic functions. Then we prove the equivalence of the above and the correspondant realized independently by the duality method.

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Introduction

This paper is the last part of this series of papers, which includes Chapters 11 and 12. For the outline of this paper, see "Contents" in the part of this series of papers [37]. Here we note that "isomorphisms" usually mean topological ones without explicit mention for the contrary. For references we refer to the lists of references at the end of papers [37], [38], [40], [41], and this one.

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Chapter 11. Case of sheaves $\mathcal{O}^*$, $\mathcal{A}^*$, $\mathcal{O}_*$ and $\mathcal{A}_*$

11.1 The Oka-Cartan-Kawai Theorem B

In this section we prove the Oka-Cartan-Kawai Theorem B for the sheaves $\mathcal{O}^*$ and $\mathcal{O}_*$.

For a 3-tuple $n = (n_1, n_2, n_3) = (n_1, n')$ of nonnegative integers with

$|n| = n_1 + n_2 + n_3 \neq 0$, where $n' = (n_2, n_3)$, we denote by $\boldsymbol{K}^n$ the product space $\boldsymbol{C}^{n_1} \times \boldsymbol{F}^{n'} = \boldsymbol{C}^{n_1} \times \tilde{\boldsymbol{C}}^{n_2} \times \boldsymbol{E}^{n_3}$ and by $\boldsymbol{Y}^n$ the product space $\boldsymbol{R}^{n_1} \times \boldsymbol{D}^{n'} = \boldsymbol{R}^{n_1} \times \boldsymbol{D}^{n_2} \times \boldsymbol{D}^{n_3}$. We put $\boldsymbol{C}^{|n|} = \boldsymbol{C}^{n_1} \times \boldsymbol{C}^{n_2} \times \boldsymbol{C}^{n_3}$. We denote then $z = (z', z'', z''') \in \boldsymbol{C}^{|n|}$ so that $z' = (z_1, \cdots, z_{n_1})$, $z'' = (z_{n_1+1}, \cdots, z_{n_1+n_2})$ and $z''' = (z_{n_1+n_2+1}, \cdots, z_{|n|})$.

For a subset F of $\boldsymbol{K}^n$, we denote by $\mathrm{int}(F)$ its interior and by F^a its closure in $\boldsymbol{K}^n$.

Definition 11.1.1 (the sheaf $\mathcal{O}^*$ of germs of partially slowly increasing holomorphic functions). We define the sheaf $\mathcal{O}^*$ over $\boldsymbol{K}^n$ to be the sheaf $\{\mathcal{O}^*(\Omega); \Omega$ is an open set in $\boldsymbol{K}^n\}$, where the section module $\mathcal{O}^*(\Omega)$ on an open set Ω in $\boldsymbol{K}^n$ is the space of all holomorphic functions $f(z)$ on $\Omega \cap \boldsymbol{C}^{|n|}$ such that, for every positive number ε and for every compact set K in Ω, the estimate $\sup\{|f(z)|e(-\varepsilon(|z''| + |z'''|)); z \in K \cap \boldsymbol{C}^{|n|}\} < \infty$ holds. Here $e(t)$ denotes the function $e^t = \exp(t)$ of $t \in \boldsymbol{C}$.

If we define a seminorm $\|f\|_{K,\varepsilon}$ of $\mathcal{O}^*(\Omega)$ by the relation $\|f\|_{K,\varepsilon} = \sup\{|f(z)|e(-\varepsilon(|z''| + |z'''|)); z \in K \cap \boldsymbol{C}^{|n|}\}$ for a compact set K in Ω and a positive number ε, $\mathcal{O}^*(\Omega)$ becomes a nuclear FS-space with respect to the topology defined by the family of seminorms $\{\|f\|_{K,\varepsilon}; K$ is a compact set in Ω and ε is a positive number$\}$.

Definition 11.1.2 (the sheaf $\mathcal{O}_*$ of germs of partially rapidly decreasing holomorphic functions). We define the sheaf $\mathcal{O}_*$ over $\boldsymbol{K}^n$ to be the sheaf $\{\mathcal{O}_*(\Omega); \Omega$ is an open set in $\boldsymbol{K}^n\}$, where the section module $\mathcal{O}_*(\Omega)$ on an open set Ω in $\boldsymbol{K}^n$ is the space of all holomorphic functions $f(z)$ on $\Omega \cap \boldsymbol{C}^{|n|}$ such that, for every compact set K in Ω, there exists some positive constant δ so that the estimate $\sup\{|f(z)|e(\delta(|z''| + |z'''|)); z \in K \cap \boldsymbol{C}^{|n|}\} < \infty$ holds.

$\mathcal{O}_*(\Omega)$ becomes a nuclear FS-space with respect to the topology defined by the family of seminorms $\{\|f\|_{K,-\delta}; K$ is an arbitrary compact set in Ω and δ is some positive constant depending on K). Let $\mathcal{O}_*(K)$ be the space of all partially rapidly decreasing holomorphic functions on a certain neighborhood of K. Then $\mathcal{O}_*(K)$ becomes a nuclear DFS-space.

It is easy to see that $\mathcal{O}^*|_{\boldsymbol{C}^{|n|}} = \mathcal{O}_*|_{\boldsymbol{C}^{|n|}} = \mathcal{O}$ holds.

Definition 11.1.3. An open set Ω in $\boldsymbol{K}^n$ is said to be an $\mathcal{O}^*$-pseudoconvex open set if it satisfies the conditions:

(1) $\sup\{|\mathrm{Im}\ z''|, |\mathrm{Im}\ z'''| - |\mathrm{Re}\ z'''|; z = (z', z'', z''') \in \Omega \cap \boldsymbol{C}^{|n|}\} < \infty$.

(2) There exists a C^∞-plurisubharmonic function $\varphi(z)$ on $\Omega \cap \boldsymbol{C}^{|n|}$ having the following two properties:

(i) The closure of $\Omega_t = \{z \in \Omega \cap \boldsymbol{C}^{|n|}; \varphi(z) < t\}$ in $\boldsymbol{K}^{|n|}$ is a compact subset of Ω for every real number t.

(ii) $\varphi(z)$ is bounded on $L \cap \boldsymbol{C}^{|n|}$ for every compact subset L of V.

Then we construct a soft resolution of the sheaf $\mathcal{O}^*$ and prove the Oka-Cartan-Kawai Theorem B using it.

First we mention the definition of the sheaf $L^* = L^*_{2,\mathrm{loc}}$ of germs of partially slowly increasing locally square intergrable functions over $\boldsymbol{K}^{|n|}$.

Definition 11.1.4. We define the sheaf L^* over $\boldsymbol{K}^n$ to be the sheaf $\{L^*(\Omega); \Omega$ is an open set in $\boldsymbol{K}^n\}$, where the section module $L^*(\Omega)$ on an open set Ω in $\boldsymbol{K}^n$ is the space of all $f \in L_{2,\mathrm{loc}}(\Omega \cap \boldsymbol{C}^{|n|})$ such that, for every positive number ε and for every relatively compact open subset ω of Ω, $e(-\varepsilon\|z\|)f(z)|_\omega \in L_2(\omega \cap \boldsymbol{C}^{|n|})$ holds.

Then L^* becomes a soft FS*-sheaf.

Definition 11.1.5(the sheaf $\mathscr{L}^{*,p,q}$). We define the sheaf $\mathscr{L}^{*,p,q}$ over $\boldsymbol{K}^n$ to be the sheaf $\{\mathscr{L}^{*,p,q}(\Omega); \Omega$ is an open set in $\boldsymbol{K}^n\}$, $(p, q \geqq 0)$, where, for an open set Ω in $\boldsymbol{K}^n$, the section module $\mathscr{L}^{*,p,q}(\Omega)$ is the space of all $f \in L^{*,p,q}(\Omega)$ such that $\bar{\partial} f \in L^{*,p,q+1}(\Omega)$ holds. We put $\mathscr{L}^* = \mathscr{L}^{*,0,0}$.

Then $\mathscr{L}^{*,p,q}$ becomes a soft FS*-sheaf with respect to the graph topology of the operator $\bar{\partial}$. Then we have the following.

Theorem 11.1.6(Hörmander-Kaneko). *Put* $X = \mathrm{int}(\{z = (z', z'', z''') \in \boldsymbol{C}^{|n|}; |\mathrm{Im}\, z''| < 1 + |\mathrm{Re}\, z''|/\sqrt{3}, |\mathrm{Im}\, z'''| < 1 + |\mathrm{Re}\, z'''|/\sqrt{3}\}^a)$, *and let* Ω *be an arbitrary* $\mathcal{O}^*$*-pseudoconvex open set in* X. *Then, for every* $f \in \mathscr{L}^{*,p,q+1}(\Omega)$ *with* $\bar{\partial} f = 0$, *there exists a solution* $u \in \mathscr{L}^{*,p,q}(\Omega)$ *which satisfies the equation* $\bar{\partial} u = f$. *Here* $p, q \geqq 0$.

Proof. We can prove this by a similar way to Kaneko [17], Theorem 8.6.6, (p.175). Q.E.D.

Theorem 11.1.7(the Dolbeault-Grothendieck resolution). *The sequence of sheaves over* $\boldsymbol{K}^n$

$$0 \longrightarrow \mathcal{O}^{*,p} \longrightarrow \mathscr{L}^{*,p,0} \xrightarrow{\bar{\partial}} \mathscr{L}^{*,p,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathscr{L}^{*,p,|n|} \longrightarrow 0$$

is exact, $(p \geqq 0)$.

Proof. Since every point of $\boldsymbol{K}^n$ has a fundamental system of neighborhoods composed of open sets which are transforms of $\mathcal{O}^*$-pseudoconvex open sets contained in X of Theorem 11.1.6 by certain regular inhomogeneous linear transformations, the conclusion follows from Theorem 11.1.6 and Hörmander [4], Theorem 4.2.5 and Corollary 4.2.6, (pp.86-87). Q.E.D.

Corollary 1. *For an arbitrary open set* Ω *in* $\boldsymbol{K}^n$, *we have the following isomorphism*:

$$H^q(\Omega, \mathcal{O}^{*,p}) \cong \{f \in \mathscr{L}^{*,p,q}(\Omega); \bar{\partial} f = 0\}/\{\bar{\partial} g; g \in \mathscr{L}^{*,p,q-1}(\Omega)\}, \qquad (p \geqq 0, q \geqq 1).$$

Now, we can prove the Oka-Cartan-Kawai Theorem B.

Theorem 11.1.8(the Oka-Cartan-Kawai Theorem B). *Let* X *be as in Theorem*

11.1.6. *Let Ω be an open set in $\boldsymbol{K}^n$ which is the transform of an arbitrary $\mathcal{O}^*$-pseudoconvex open set in X by a certain regular inhomogeneous linear transformation. Then we have* $H^q(\Omega, \mathcal{O}^{*,p}) = 0$, $(p \geqq 0, q \geqq 1)$.

Proof. It follows from Theorem 11.1.6 and Corollary 1 to Theorem 11.1.7.

Q.E.D

Now we define the sheaf $\mathscr{E}^*$ of germs of partially slowly increasing C^∞-functions over $\boldsymbol{K}^n$.

Definition 11.1.9. We define the sheaf $\mathscr{E}^*$ over $\boldsymbol{K}^n$ to be the sheaf $\{\mathscr{E}^*(\Omega); \Omega$ is an open set in $\boldsymbol{K}^n\}$, where, for an open set Ω in $\boldsymbol{K}^n$, the section module $\mathscr{E}^*(\Omega)$ is defined as follows:

$\mathscr{E}^*(\Omega) = \{f \in \mathscr{E}(\Omega \cap \boldsymbol{C}^{|n|})$; for every $\varepsilon > 0$, every compact set K in Ω and every $\alpha \in \bar{\mathbf{N}}^{2|n|}$, the estimate $\sup\{|f^{(\alpha)}(z)| e(-\varepsilon(|z''| + |z'''|)); z = (z', z'', z''') \in K \cap \boldsymbol{C}^{|n|}\} < \infty$ holds$\}$.

Then the sheaf $\mathscr{E}^*$ becomes a soft nuclear Fréchet sheaf. Then we have the following.

Theorem 11.1.10(the Dolbeault-Grothendieck resolution). *The sequence of sheaves over* $\boldsymbol{K}^n$

$$0 \longrightarrow \mathcal{O}^{*,p} \longrightarrow \mathscr{E}^{*,p,0} \xrightarrow{\bar{\partial}} \mathscr{E}^{*,p,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathscr{E}^{*,p,|n|} \longrightarrow 0$$

is exact, $(p \geqq 0)$.

Proof. It goes in a similar way to Hörmander [4], Therem 2.3.3, (p.32) by using a certain weight functions as used in Kaneko [17], Theorem 8.6.6, (p.175).

Q.E.D.

Corollary 1. *For an arbitrary open set Ω in $\boldsymbol{K}^n$, we have the following isomorphism:*

$$H^q(\Omega, \mathcal{O}^{*,p}) \cong \{f \in \mathscr{E}^{*,p,q}(\Omega); \bar{\partial} f = 0\} / \{\bar{\partial} g; g \in \mathscr{E}^{*,p,q-1}(\Omega)\}, \qquad (p \geqq 0, q \geqq 1).$$

Corollary 2. *Let Ω be as in Theorem* 11.1.8. *Then, for every $f \in \mathscr{E}^{*,p,q+1}(\Omega)$ with $\bar{\partial} f = 0$, there exists a solution $u \in \mathscr{E}^{*,p,q}(\Omega)$ which satisfies the equation $\bar{\partial} u = f$. Here we assume p, $q \geqq 0$.*

Proof. It follows from Theorem 11.1.8 and the above Corollary 1.

Q.E.D

The above Corollary 2 gives some estimate and existence theorems for the $\bar{\partial}$-operator.

Now we define the sheaf $L_* = L_{*,2,\mathrm{loc}}$ of germs of partially rapidly decreasing locally square integrable functions over $\boldsymbol{K}^n$.

Definition 11.1.11. We define the sheaf L_* over $\boldsymbol{K}^n$ to be the sheaf $\{L_*(\Omega); \Omega$ is an open set in $\boldsymbol{K}^n\}$, where, for an open set Ω in $\boldsymbol{K}^n$, the section module $L_*(\Omega)$ is defined as follows:

$L_*(\Omega) = \{f \in L_{2,\mathrm{loc}}(\Omega \cap \boldsymbol{C}^{|n|})$; for every relatively compact set ω in Ω, there exists some $\delta > 0$ such that $e(\delta \| z \|) f(z)|_\omega \in L_2(\omega \cap \boldsymbol{C}^{|n|})$ holds$\}$.

Then the sheaf L_* becomes a soft FS*-sheaf.

Definition 11.1.12(the sheaf $\mathscr{L}_*^{p,q}$). We define the sheaf $\mathscr{L}_*^{p,q}$ over $\boldsymbol{K}^n$ to be the sheaf $\{\mathscr{L}_*^{p,q}(\Omega); \Omega$ is an open set in $\boldsymbol{K}^n\}$, $(p, q \geqq 0)$, where, for an open set Ω in $\boldsymbol{K}^n$, the section module $\mathscr{L}_*^{p,q}(\Omega)$ is defined as follows:

$$\mathscr{L}_*^{p,q}(\Omega) = \{f \in L_*^{p,q}(\Omega);\ \bar{\partial} f \in L_*^{p,q+1}(\Omega)\}.$$

Especially, we put $\mathscr{L}_* = \mathscr{L}_*^{0,0}$.

Then the sheaf $\mathscr{L}_*^{p,q}$ becomes a soft FS*-sheaf with respect to the graph topology of the $\bar{\partial}$-operator. Then we have the following.

Theorem 11.1.13(Hörmander-Kaneko). *For certain positive numbers a, $(1 > a > 0)$ and $b > 0$, put $X = \mathrm{int}(\{z = (z', z'', z''') \in \boldsymbol{C}^{|n|}; |\mathrm{Im}\, z''| < 1 + |\mathrm{Re}\, z''|/\sqrt{3}, |\mathrm{Im}\, z''|^2 < 1/2 + |\mathrm{Re}\, z''|^2, |\mathrm{Im}\, z'''| < 1 + |\mathrm{Re}\, z'''|/\sqrt{3}, |\mathrm{Im}\, z'''|^2 < a^2 |\mathrm{Re}\, z'''|^2 + b^2\}^a)$. Let Ω be an arbitrary $\mathcal{O}^*$-pseudoconvex open set in X. Assume $f \in \mathscr{L}_*^{p,q+1}(\Omega)$ with $\bar{\partial} f = 0$. Then, for every open set $\omega \subset\subset \Omega$, there exists a solution $u \in \mathscr{L}_*^{p,q}(\omega)$ which satisfies the equation $\bar{\partial} u = f$ on ω. Here assume $p, q \geqq 0$.*

Theorem 11.1.14(the Dolbeault-Grothendieck resolution). *The sequence of sheaves over $\boldsymbol{K}^n$*

$$0 \longrightarrow \mathcal{O}_*^p \longrightarrow \mathscr{L}_*^{p,0} \xrightarrow{\bar{\partial}} \mathscr{L}_*^{p,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathscr{L}_*^{p,|n|} \longrightarrow 0$$

is exact, $(p \geqq 0)$.

Proof. Since the exactness of the above sequence is equivalent to the local solvability of the $\bar{\partial}$-equation, we have the conclusion by a similar way to Theorem 11.1.7 using Theorem 11.1.13 and Hörmander [4], Theorem 4.2.5 and Corollary 4.2.6, (pp.86–87). Q.E.D

Corollary 1. *For an arbitrary open set Ω in $\boldsymbol{K}^n$, we have the isomorphism*

$$H^q(\Omega, \mathcal{O}_*^p) \cong \{f \in \mathscr{L}_*^{p,q}(\Omega);\ \bar{\partial} f = 0\}/\{\bar{\partial} g;\ g \in \mathscr{L}_*^{p,q-1}(\Omega)\}, \qquad (p \geqq 0,\ q \geqq 1).$$

Then we have the following.

Theorem 11.1.15. *Let X be as in Theorem* 11.1.13. *Let K be a compact subset in $\boldsymbol{K}^n$ which is the transform of a compact subset in X having a fundamental system of neighborhoods composed of $\mathscr{O}^*$-pseudoconvex open sets in X by a certain regular inhomogeneous linear transformation. Then we have* $H^q(K, \mathscr{O}_*^p) = 0$, $(p \geqq 0, q \geqq 1)$.

Proof. We can prove this by using Theorem 11.1.13 and Corollary 1 to Theorem 11.1.14. Q.E.D.

Next, we mention the Grauert Theorem.

Theorem 11.1.16(the Grauert Theorem). *Let S be an arbitrary subset of $\boldsymbol{Y}^n$ and U a certain neighborhood of S in $\boldsymbol{K}^n$. Then there exists an $\mathscr{O}^*$-pseudoconvex open neighborhood V with $S \subset V \subset U$. Namely, S has a fundamental system of neighborhoods composed of $\mathscr{O}^*$-pseudoconvex open sets.*

Theorem 11.1.17(Malgrange). *For an arbitrary subset S of $\boldsymbol{Y}^n$, we have* $H^q(S, \mathscr{A}^{*,p}) = 0$, $(p \geqq 0, q \geqq 1)$. *Here we put* $\mathscr{A}^* = \mathscr{O}^*|_{\boldsymbol{Y}^n}$.

Theorem 11.1.18(Malgrange). *For an arbitrary compact subset K of $\boldsymbol{Y}^n$, we have* $H^q(K, \mathscr{A}_*^p) = 0$, $(p \geqq 0, q \geqq 1)$. *Here we put* $\mathscr{A}_* = \mathscr{O}_*|_{\boldsymbol{Y}^n}$.

11.2 The Malgrange Theorem

In this section we prove the following Malgrange Theorem for the sheaf $\mathscr{O}^*$.

Theorem 11.2.1(the Malgrange Theorem). *Let Ω be an arbitrary open set in $\boldsymbol{K}^n$ such that there exists an open neighborhood $\tilde{\Omega}$ of Ω with* $H^{|n|}(\tilde{\Omega}, \mathscr{O}^*) = 0$. *Then we have* $H^{|n|}(\Omega, \mathscr{O}^*) = 0$.

Proof. We use the notations in Theorems 11.1.7 and 11.1.14. By Corollary 1 to Theorem 11.1.7, we have only to prove the exactness of the sequence

$$\mathscr{L}^{*,0,|n|-1}(\Omega) \xrightarrow{\bar{\partial}} \mathscr{L}^{*,0,|n|}(\Omega) \longrightarrow 0.$$

Here we consider its dual sequence

$$\mathscr{L}^{0,1}_{*,c}(\Omega) \xleftarrow{-\bar{\partial}^{\Omega}} \mathscr{L}^{0,0}_{*,c}(\Omega) \longleftarrow 0.$$

Then, by virtue of the Serre-Komatsu Duality Theorem for FS*-spaces, it suffices to show the injectiveness and the closedness of the range of $-\bar{\partial}^{\Omega} = (\bar{\partial}^{\Omega})'$. Since $\bar{\partial}^{\Omega}$ is elliptic, its injectivity is an immediate consequence of the unique continuation property. Now we prove the closedness of its range. This is surely true if Ω is replaced by the open set $\tilde{\Omega}$ in the assumption of this theorem because $H^{|n|}(\tilde{\Omega}, \mathscr{O}^*) = 0$. Then we consider the commutative diagram:

$$\begin{array}{ccccc} \mathscr{L}^{0,1}_{*,c}(\Omega) & \xleftarrow{-\bar{\partial}^{\Omega}} & \mathscr{L}^{0,0}_{*,c}(\Omega) & \longleftarrow & 0. \\ {\scriptstyle i}\downarrow & & \downarrow & & \\ \mathscr{L}^{0,1}_{*,c}(\tilde{\Omega}) & \xleftarrow{-\bar{\partial}^{\tilde{\Omega}}} & \mathscr{L}^{0,0}_{*,c}(\tilde{\Omega}) & \longleftarrow & 0. \end{array}$$

where the map i is the natural injection. By the above remark, $-\bar{\partial}^{\tilde{\Omega}}$ is of closed range. Then $\mathrm{Im}(-\bar{\partial}^{\Omega}) = \{i^{-1}(\mathrm{Im}(-\bar{\partial}^{\tilde{\Omega}}))\} \cap \{[\mathrm{Im}(-\bar{\partial}^{\Omega})]^a\}$ is closed, where $[\ ]^a$ is the closure of the set $[\ ]$. Therefore $-\bar{\partial}^{\Omega}$ is of closed range. This completes the proof. Q.E.D.

Corollary. Flabby $\dim \mathcal{O}^* \leqq |n|$.

11.3 The Serre Duality Theorem

In this section we prove the Serre Duality Theorem for the sheaves $\mathcal{O}^*$ and $\mathcal{O}_*$.

Theorem 11.3.1. *Let Ω be an open set in $\boldsymbol{K}^n$ such that* $\dim H^p(\Omega, \mathcal{O}^*) < \infty$ *holds for $p \geqq 1$. Then we have the isomorphism* $[H^p(\Omega, \mathcal{O}^*)]' \cong H_c^{|n|-p}(\Omega, \mathcal{O}_*)$, $(0 \leqq p \leqq |n|)$.

Proof. We can prove this in a similar way to Theorem 1.4.1, using the dual complexes

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathscr{L}^{*,0,0}(\Omega) & \xrightarrow{\bar{\partial}} & \mathscr{L}^{*,0,1}(\Omega) & \xrightarrow{\bar{\partial}} & \cdots & \xrightarrow{\bar{\partial}} & \mathscr{L}^{*,0,|n|}(\Omega) & \longrightarrow & 0. \\ & & \updownarrow & & \updownarrow & & & & \updownarrow & & \\ 0 & \longleftarrow & \mathscr{L}^{0,|n|}_{*,c}(\Omega) & \xleftarrow{-\bar{\partial}} & \mathscr{L}^{0,|n|-1}_{*,c}(\Omega) & \xleftarrow{-\bar{\partial}} & \cdots & \xleftarrow{-\bar{\partial}} & \mathscr{L}^{0,0}_{*,c}(\Omega) & \longleftarrow & 0. \end{array}$$

Q.E.D.

Remark. The theorem is also true for such an open set Ω that every $\bar{\partial}$ is of closed range in the above diagram used in the proof of Theorem 11.3.1.

11.4 The Martineau-Harvey Theorem (in case of sheves $\mathcal{O}^*$ and $\mathcal{O}_*$)

In this section we prove the Martineau-Harvey Theorem for the sheaves $\mathcal{O}^*$ and $\mathcal{O}_*$.

Theorem 11.4.1 (the Martineau-Harvey Theorem). *Let K be a compact set $\boldsymbol{K}^n$. Further we assume the following* (i) *and* (ii):

(i) $H^p(K, \mathcal{O}_*) = 0$, $(p \geqq 1)$.

(ii) *V is an open neighborhood of K and satisfies* $H^p(V, \mathcal{O}^*) = 0$, $(p \geqq 1)$.

Then we have the following:

(1) $H^p_K(V, \mathcal{O}^*) = 0$, $(p \neq |n|)$.

(2) *If* $|n| \geqq 2$, *we have algebraic isomorphisms*

$$H^{|n|}_K(V, \mathcal{O}^*) \cong H^{|n|-1}(V \setminus K, \mathcal{O}^*) \cong \mathcal{O}_*(K)'.$$

(3) *If* $|n| = 1$, *we have topological isomorphisms*

$$H^1_K(V, \mathcal{O}^*) \cong \mathcal{O}^*(V \setminus K)/\mathcal{O}^*(V) \cong \mathcal{O}_*(K)'.$$

Proof. It goes in a similar way to Ito [43]. Q.E.D.

As a corollary, we mention a fact in the theory of complex functions of several variables obtained in the proof of Theorem 11.4.1.

Corolloary 1(Hartogs). *Assume* $|n| \geqq 2$ *and let* K *and* V *be as in Theorem* 11.4.1. *Then we have the isomorphism* $\mathcal{O}^*(V \setminus K) \cong \mathcal{O}^*(V)$.

We mention the important fact necessary for the proof of Theorem 11.4.1 in the following Proposition.

Proposition 11.4.2. *Let* K *and* V *be as in Theorem* 11.4.1. *Then we have the following isomorphisms*:

(1) $H^1_c(V \setminus K, \mathcal{O}_*) \cong H^0(K, \mathcal{O}_*) \cong \mathcal{O}_*(K)$.

(2) $H^p_c(V \setminus K, \mathcal{O}_*) \cong H^p_c(V, \mathcal{O}_*)$.

11.5 The Sato Theorem (the case of the sheaf $\mathcal{O}^*$)

In this section we prove the pure-codimensionality of $\boldsymbol{Y}^n$ with respect to the sheaf $\mathcal{O}^*$. Then we realize partial mixed Fourier hyperfunctions as "boundary values" of partially slowly increasing holomorphic functions or as (relative) cohomology classes of partially slowly increasing holomorphic functions.

Theorem 11.5.1(the Sato Theorem).

(1) $\boldsymbol{Y}^n$ *is purely* $|n|$*-codimensional with respect to the sheaf* $\mathcal{O}^*$. *Namely, for every* $p \neq |n|$, $\mathscr{H}^p_{\boldsymbol{Y}^n}(\mathcal{O}^*) = 0$ *holds.*

(2) *The presheaf* $\{H^{|n|}_\Omega(V, \mathcal{O}^*);\ \Omega$ *is an open set in* $\boldsymbol{Y}^n\}$ *over* $\boldsymbol{Y}^n$ *is a flabby sheaf. Here the section module* $H^{|n|}_\Omega(V, \mathcal{O}^*)$ *is a relative cohomology group with coefficients in the sheaf* $\mathcal{O}^*$ *and* V *is an open set in* $\boldsymbol{K}^n$ *which contains* Ω *as its closed subset. We denote this sheaf by* $\mathscr{H}^{|n|}_{\boldsymbol{Y}^n}(\mathcal{O}^*) = \mathrm{Dist}^{|n|}(\boldsymbol{Y}^n, \mathcal{O}^*)$.

(3) *The sheaf* $\mathscr{H}^{|n|}_{\boldsymbol{Y}^n}(\mathcal{O}^*)$ *is isomorphic to the sheaf* $\mathscr{BP}$ *of partial mixed Fourier hyperfunctions defined in Ito* [11]. *Thereby we identify two realizations.*

Proof. It goes in a similar way to Theorem 9.6.1. Q.E.D.

Corollary. *Let* Ω *be an arbitrary open set in* $\boldsymbol{Y}^n$ *and* V *an open neighborhood of* Ω *such that* $H^p(V, \mathcal{O}^*) = 0$, $(p \geqq 1)$ *holds. Then we have the following*:

(1) *If* $|n| \geqq 2$, $H^{|n|}_\Omega(V, \mathcal{O}^*) \cong H^{|n|-1}(V \setminus \Omega, \mathcal{O}^*)$.

(2) *If* $|n| = 1$, $H^1_\Omega(V, \mathcal{O}^*) \cong \mathcal{O}^*(V\backslash\Omega)/\mathcal{O}^*(V)$.

Theorem 11.5.2. *We use notations similar to Theorem* 1.6.2. *Then we have the algebraic isomorphism*

$$H^{|n|}_\Omega(V, \mathcal{O}^*) \cong \mathcal{O}^*(V\sharp\Omega)/\Sigma^{|n|}_{j=1}\ \mathcal{O}^*(V\sharp_j\Omega).$$

Here we put

$$V_0 = V,\ V_j = V\backslash\{z\in V;\ \mathrm{Im}\ z_j = 0\}^a,\quad (j = 1, 2, \cdots, |n|),$$

$$V\sharp\Omega = \cap^{|n|}_{j=1} V_j,$$

$$V\sharp_j\Omega = \cap_{i\neq j} V_i.$$

We consider the mapping

$$b: \mathcal{O}^*(V\sharp\Omega) \longrightarrow H^{|n|}_\Omega(V, \mathcal{O}^*) \cong \mathscr{BP}(\Omega)$$

determined by the above Theorem. Then, for $f\in\mathcal{O}^*(V\sharp\Omega)$, we call $b(f)\in\mathscr{BP}(\Omega)$ the boundary value of f and f a defining function of $b(f)$. We have $b^{-1}(0) = \Sigma^{|n|}_{j=1}\ \mathcal{O}^*(V\sharp_j\Omega)$.

Chapter 12. Case of the sheaf ${}^E\mathcal{O}^*$

12.1 The Oka-Cartan-Kawai Theorem B

In this section we construct a soft resolution of the sheaf ${}^E\mathcal{O}^*$ and prove the Oka-Cartan-Kawai Theorem B for the sheaf ${}^E\mathcal{O}^*$. In this chapter we always assume that E is a Fréchet space whose topology is defined by a family $\mathscr{T} = \mathscr{T}_E$ of continuous seminorms of E.

At first we define sheaves ${}^E\mathcal{O}^*$ and ${}^E\mathscr{E}^*$.

Definition 12.1.1 (the sheaf ${}^E\mathcal{O}^*$ of germs of partially slowly increasing E-valued holomorphic functions over $\boldsymbol{K}^n$). We define the sheaf ${}^E\mathcal{O}^*$ over $\boldsymbol{K}^n$ to be the sheaf $\{\mathcal{O}^*(\Omega; E);\ \Omega$ is an open set in $\boldsymbol{K}^n\}$, where, for an open set Ω in $\boldsymbol{K}^n$, the section module $\mathcal{O}^*(\Omega; E)$ is defined as follows:

$\mathcal{O}^*(\Omega; E) = \{f\in\mathcal{O}(\Omega\cap\boldsymbol{C}^{|n|}; E)$; for every positive ε, every compact set K in Ω and every $q\in\mathscr{T}$, $\sup\{q(f(z))e(-\varepsilon(|z''| + |z'''|));\ z\in K\cap\boldsymbol{C}^{|n|}\} < \infty$ holds$\}$.

We call this sheaf ${}^E\mathcal{O}^*$ the sheaf of germs of partially slowly increasing E-valued holomorphic functions.

Definition 12.1.2 (the sheaf ${}^E\mathscr{E}^*$ of germs of partially slowly increasing E-valued $\boldsymbol{C}^\infty$-functuions). We define the sheaf ${}^E\mathscr{E}^*$ over $\boldsymbol{K}^n$ to be the sheaf $\{\mathscr{E}^*(\Omega; E);\ \Omega$ is an open set in $\boldsymbol{K}^n\}$, where, for an open set Ω in $\boldsymbol{K}^n$, the section module

$\mathscr{E}^*(\Omega; E)$ is defined as follows:

$\mathscr{E}^*(\Omega; E) = \{f \in \mathscr{E}(\Omega \cap \boldsymbol{C}^{|n|}; E)$; for every positive ε, every compact set K in Ω, every $\alpha \in \bar{\boldsymbol{N}}^{2|n|}$ and every $q \in \mathscr{T}$, $\sup\{q(f^{(\alpha)}(z))e(-\varepsilon(|z''| + |z'''|)); z \in K \cap \boldsymbol{C}^{|n|}\} < \infty$ holds$\}$.

Then the sheaf ${}^E\mathscr{E}^*$ is a soft Fréchet sheaf and we have the isomorphism

$$\mathscr{E}^*(\Omega; E) \cong \mathscr{E}^*(\Omega) \hat{\otimes} E$$

for every open set Ω in $\boldsymbol{K}^n$.

Then we have the following.

Theorem 12.1.3(Hörmander). *Let Ω be as in Theorem* 11.1.8. *Then, for every $f \in \mathscr{E}^{*,p,q+1}(\Omega; E)$ with $\bar{\partial} f = 0$, the equation $\bar{\partial} u = f$ has a solution $u \in \mathscr{E}^{*,p,q}(\Omega; E)$, $(p, q \geqq 0)$.*

Theorem 12.1.4(the Dolbeault-Grothendieck resolution of ${}^E\mathscr{O}^{*,p}$). *Let E be a Fréchet space. The sequence of sheaves over $\boldsymbol{K}^n$*

$$0 \longrightarrow {}^E\mathscr{O}^{*,p} \longrightarrow {}^E\mathscr{E}^{*,p,0} \xrightarrow{\bar{\partial}} {}^E\mathscr{E}^{*,p,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} {}^E\mathscr{E}^{*,p,|n|} \longrightarrow 0$$

is exact. Here we assume $p \geqq 0$.

Proof. It goes in a similar way to Theorem 11.1.7. Q.E.D.

Corollary. *We use the notations in Theorem* 12.1.4. *Then, for an open set Ω in $\boldsymbol{K}^n$, we have the isomorphism*

$$H^q(\Omega, {}^E\mathscr{O}^{*,p}) \cong \{f \in \mathscr{E}^{*,p,q}(\Omega; E); \bar{\partial} f = 0\} / \{\bar{\partial} g; g \in \mathscr{E}^{*,p,q-1}(\Omega; E)\},$$

$(p \geqq 0$ *and* $q \geqq 1)$.

Now we prove the Oka-Cartan-Kawai Theorem B for the sheaf ${}^E\mathscr{O}^*$.

Theorem 12.1.5(the Oka-Cartan-Kawai Theorem B). *Let E be a Fréchet space and Ω be as in Theorem* 11.1.8. *Then we have $H^q(\Omega, {}^E\mathscr{O}^{*,p}) = 0$, $(p \geqq 0$ and $q \geqq 1)$.*

Proof. It goes in a similar way to Theorem 10.2.1. Q.E.D.

12.2 The Malgrange Theorem

In this section, we prove the Malgrange Theorem for the sheaf ${}^E\mathscr{O}^*$.

The 12.2.1(the Malgrange Theorem). *Let Ω be as in Theorem* 11.2.1. *Then we have $H^{|n|}(\Omega, {}^E\mathscr{O}^*) = 0$.*

Proof. It goes in a similar way to Theorem 10.3.1. Q.E.D.

Corollary. Flabby dim ${}^{E}\mathcal{O}^* \leqq |n|$.

12.3 The Serre Duality Theorem

In this section we prove the Serre Duality Theorem for the sheaves ${}^{E}\mathcal{O}^*$ and $\mathcal{O}_*$.

Theorem 12.3.1. *Let Ω be as in Theorem* 11.3.1. *Then we have the isomorphism* $H^p(\Omega, {}^{E}\mathcal{O}^*) \cong L(H_c^{|n|-p}(\Omega, \mathcal{O}_*), E)$, $(0 \leqq p \leqq |n|)$.

Proof. It goes in a similar way to Theorem 10.4.1. Q.E.D.

Here we note that an analogous fact to the Remark at the end of section 9.4 is also true in this case.

12.4 The Martineau-Harvey Theorem

In this section we prove the Martineau-Harvey Theorem for the sheaves ${}^{E}\mathcal{O}^*$ and $\mathcal{O}_*$.

Theorem 12.4.1 (the Martineau-Harvey Theorem). *Let K and V be as in Theorem* 11.4.1. *Then we have the following*:

(1) $H_K^p(V, {}^{E}\mathcal{O}^*) = 0$, $(p \neq |n|)$.

(2) *If* $|n| \geqq 2$, *we have algebraic isomorphisms*

$$H_K^{|n|}(V, {}^{E}\mathcal{O}^*) \cong H^{|n|-1}(V \backslash K, {}^{E}\mathcal{O}^*) \cong \mathcal{O}'_*(K; E) = L(\mathcal{O}_*(K), E).$$

(3) *If* $|n| = 1$, *we have topological isomorphisms*

$$H_K^1(V, {}^{E}\mathcal{O}^*) \cong \mathcal{O}^*(V \backslash K; E)/\mathcal{O}^*(V; E) \cong \mathcal{O}'_*(K; E).$$

Proof. It goes in a similar way to Therem 10.5.1. Q.E.D.

As a corollary, we mention a fact in the theory of complex functions of several variables obtained in the proof of Theorem 12.4.1.

Corollary (Hartogs). *Assume* $|n| \geqq 2$. *Let K and V be as in Theorem* 12.4.1. *Then we have the isomorphism* $\mathcal{O}^*(V \backslash K; E) \cong \mathcal{O}^*(V; E)$.

12.5 The Sato Theorem (the case of the sheaf ${}^{E}\mathcal{O}^*$)

In this section we prove the pure-codimensionality of $\boldsymbol{Y}^n$ with respect to the sheaf ${}^{E}\mathcal{O}^*$. Thereby we realize E-valued partial mixed Fourier hyperfunctions as "boundary values" of E-valued partially slowly increasing holomorphic functions or as (relative) cohomology classes of E-valued partially slowly increasing holomorphic functions. This is a realization of Fourier hyperfunctions of general type equivalent to the one discussed in Ito [11]. As special cases of this hyperfunctions, all types of Sato-Fourier hyperfunctions discussed in this series of papers (I) $\sim$ (V) are obtained.

Theorem 12.5.1 (the Sato Theorem).

(1) *Y^n is purely $|n|$-codimensional with respect to the sheaf ${}^E\mathcal{O}^*$. Namely. for every $p \neq |n|$, $\mathcal{H}^p_{Y^n}({}^E\mathcal{O}^*) = 0$ holds.*

(2) *The presheaf $\{H^{|n|}_\Omega(V, {}^E\mathcal{O}^*); \Omega$ is an open set in $Y^n\}$ over Y^n is a flabby sheaf. Here the section module $H^{|n|}_\Omega(V, {}^E\mathcal{O}^*)$ is a relative cohomology group with coefficients in the sheaf ${}^E\mathcal{O}^*$ and V is an open set in K^n which contains Ω as its closed subset. We denote this sheaf by $\mathcal{H}^{|n|}_{Y^n}({}^E\mathcal{O}^*) = \mathrm{Dist}^{|n|}(Y^n, {}^E\mathcal{O}^*)$.*

(3) *The sheaf $\mathcal{H}^{|n|}_{Y^n}({}^E\mathcal{O}^*)$ is isomorphic to the sheaf $\mathcal{M} = {}^E(\mathcal{BP})$ of E-valued partial mixed Fourier hyperfunctions or Fourier hyperfunctions of general type defined in Ito* [11]. *Thereby we identify two realizations.*

Proof. It goes in a similar way to Theorem 10.6.1. Q.E.D.

Corolloary. *Let Ω and V be as in Corollary to Theorem* 11.5.1. *Then we have the following:*

(1) *If $|n| \geqq 2$, $H^{|n|}_\Omega(V, {}^E\mathcal{O}^*) \cong H^{|n|-1}(V \backslash \Omega, {}^E\mathcal{O}^*)$.*

(2) *If $|n| = 1$, $H^1_\Omega(V, {}^E\mathcal{O}^*) \cong \mathcal{O}^*(V \backslash \Omega; E)/\mathcal{O}^*(V; E)$.*

Theorem 12.5.2. *We use notations similar to Theorem* 11.5.2. *Then we have the algebraic isomorphisms*

$$H^{|n|}_\Omega(V, {}^E\mathcal{O}^*) \cong H^{|n|}(\mathcal{U}, \mathcal{U}', {}^E\mathcal{O}^*)$$
$$\cong \mathcal{O}^*(V\sharp\Omega; E)/\Sigma^{|n|}_{j=1}\, \mathcal{O}^*(V\sharp_j\Omega; E).$$

We consider the mapping

$$b: \mathcal{O}^*(V\sharp\Omega; E) \longrightarrow H^{|n|}_\Omega(V, {}^E\mathcal{O}^*) = \mathcal{BP}(\Omega; E)$$

determined by the above Theorem. Then, for $f \in \mathcal{O}^*(V\sharp\Omega; E)$, we call $b(f) \in \mathcal{BP}(\Omega; E)$ the boundary value of f and f an defining function of $b(f)$. We have $b^{-1}(0) = \Sigma^{|n|}_{j=1}\, \mathcal{O}^*(V\sharp_j\Omega; E)$.

References

[1]-[40], See References in Y. Ito [37], [38], [40] and [42].

[42] Y. Ito, Theory of (Vector Valued) Fourier Hyperfunctions. Their Realization as Boundary Values of (Vector Valued) Slowly Increasing Holomorphic Functions, (IV), J. Math. Tokushima Univ., **24** (1990), 13–21.

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