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Nonexistence of a Projective Plane Minimally Immersed with Some Embeddedness

By

Toru ISHIHARA

Department of Mathematics and Comuputer Sciences,
Faculty of Integrated Arts and Sciences,
The University of Tokushima
Tokushima 770, *JAPAN*

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Abstract

Assume that M is diffeomorphic to a projective plane minus k points ($k \geqq 1$), In this paper, we prove that there is no complete minimal embedding of M into R^3. It is also shown that if $1 \leq k \leq 3$, it does not exists a complete minimal immersion of M into R^3 with paralled embedded ends.

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§1. Introduction

Assume that M is diffeomorphic to a projective plane minus k points. Set $M = P^2 - \{p_1, \cdots, p_k\}$. Let $\pi : \tilde{M} = \mathbf{C} \cup \{\infty\} - \{q_1, q_1', \cdots, q_k, q_k'\} \to M$ be its oriented two-sheeted covering with $\pi(q_i) = \pi(q_i') = p_i, 1 \leq i \leq k$. Put $I(z) = -1/\bar{z}$. Then the map I is the antipodal map of $\mathbf{C} \cup \{\infty\}$ and $\pi(p) = \pi(p')$for $p, p' \in \tilde{M}$ if and only if $p' = I(p)$. For a regular complete minimal immersion $\tilde{x} : \tilde{M} \to R^3$, there exists a regular complete minimal immersion $x : M \to R^3$ such that $\tilde{x} = x \cdot \pi$ if and only if $\tilde{x}(I(z)) = \tilde{x}(z)$ for each $z \in \tilde{M}$. In this case, we call the immersion $\tilde{x} : \tilde{M} \to R^3$ a double surface of $x : M \to R^3$. If an end q_i is an embedded end, q' is so and we may call the corresponding end p_i embedded. If all the ends q_i and q_i' are embedded and parallel, the minimal surface is called pseudo-embedded by Peng[8]. If it happens, the corresponding nonorientable minimal surface is said to be pseudo-embedded in the present paper. If $x : M \to R^3$ is embedded, $\tilde{x} : \tilde{M} \to R^3$ is pseudo-embedded. An end p_i is called a Catenoid(resp. planar) end if q_i is a Catenoid(resp. planar) end.

Lopez and Ros [5] proved that the plane and the Catenoid are the only embedded complete minimal surfaces of finite total curvature and genus zero in R^3. On the other hand, there are complete genus zero pseudo-embedded minimal surface with k ends (see [3],[6]) except when $k = 3, 4$ or 5 [see [2]).

In the nonorientable case, Meeks [6] showed that there is no complete pseudo-embedded minimal immersion of a projective plane with two embedded ends. In the present paper, we will prove

Theorem 1. *There is no complete minimal embedding of a projective plane into R^3 with k ends, $k \geq 1$.*

Theorem2. *Assume that M is diffeomorphic to a projective plane minus k points, $k \geq 1$. There does not exist a complete pseudo-embedded minimal immersion of M into R^3 in the following cases:*

(1)*All the ends are planar.*

(2)*One end is a Catenoid end and the othor are planar.*

(3)*The ends of M are two Catenoid ends or two Catenoid ends and one palar end.*

(4)*The ends of M are three Catenoid ends.*

Corollary. *There is no complete pseudo-embedded minimal immersion of a projective plane with three ends.*

In the present paper, we could not find any conmlete pseudo-embedded minimal immersion of a projective plane with k ends.

§2. Preliminaries

Let $\tilde{x} : N \to R^3$ be a complete pseudo-embedded genus zero minimal surface of finite total curvature in the Euclidean space $\mathbf{R}^3$. Denote by ω, g the holomorhic 1-form and the meromorphic function on N determined by the Weierstrass representation of $\tilde{x}$, respecatively. Modulo natural identification, g is the Gauss map of $\tilde{x}$. We have the representation

$$\tilde{x} = \text{Real} \int (\phi_1, \phi_2, \phi_3), \tag{1}$$

where $\phi_1 = \omega(1-g^2)/2, \phi_2 = i\omega(1+g^2)/2$, and $\phi_3 = \omega g$. It is evident that N is conformally equivalent to $\mathbf{C} \cup \{\infty\} - \{q_1, \cdots, q_l\}$. An end q_i is a Catenoid (resp. planar) end if and only if q_i is the regular (resp. branch) point of the Gauss map g. We can assume that the ends of N are poles and zeros of g. Assume that l_1 ends are poles and the rests are zeros. Put $l_2 = l - l_1$. The sum of order L_1 of the poles is equal to the sum of order L_2 of the zeros. Since all the ends are embedded, we have that $L_1 = L_2 = l - 1$. If we assume that t_1 zeros and t_2 poles are the catenoid ends, where $0 \leq t_1 \leq l_1$ and $0 \leq t_2 \leq l_2$, we get that $l_2 + t_1 - 1 \geq l_1$, $l_1 \geq l_2 - t_2 + 1$. Hence we get $t_1 + t_2 \geq 2$. Thus we have

Proposition 1. *A complete oriented genus zero pseudo-embedded minimal surface in R^3 has at least two Catenoids ends.*

As a corollary, we have that the case (1) of theorem 2 does not occur. Next we show

Propositon 2. *Let $x : M \to R^3$ be a regular complete pseudo-embedded minimal immersion. Then, the two-sheeted covering $\tilde{M}$ is conformally equivalent to $\mathbf{C} - \{0, a_1, -1/\bar{a}_1, \cdots, a_n, -1/\bar{a}_n, b_1, -1/\bar{b}_1, \cdots, b_m, -1/\bar{b}_m\}$, where $0, \infty, a_1, -1/\bar{a}_1, \cdots, a_n, -1/\bar{a}_n$ are Catenoid ends, $b_1, -1/\bar{b}_1, \cdots$, $b_m, -1/\bar{b}_m$ are planar and $n + m + 1 = k$. Moreover, the double surface $\tilde{x} : \tilde{M} \to R^3$ is given by (1) with*

$$\phi_1 = \frac{i\lambda(\bar{B}\alpha_1 - B\alpha_2)}{2},\ \phi_2 = -\frac{\lambda(\bar{B}\alpha_1 + B\alpha_2)}{2},\ \phi_3 = i\lambda\alpha_3, \tag{2}$$

where $B \in \mathbf{C}$ with $| B | = 1, \lambda \in R - \{0\}$ and

$$\begin{aligned} \alpha_1 &= \frac{\prod_{i=1}^n (\bar{c}_i z + 1)^2 \prod_{j=1}^m (\bar{b}_j z + 1)^2}{z^2 \prod_{i=1}^n (z - a_i)^2 \prod_{j=1}^m (z - b_j)^2} dz\ , \\ \alpha_2 &= \frac{\prod_{i=1}^n (z - c_i)^2 \prod_{j=1}^m (z - b_j)^2}{\prod_{i=1}^n (\bar{a}_i z + 1)^2 \prod_{j=1}^m (\bar{b}_j z + 1)^2} dz\ , \\ \alpha_3 &= \frac{\prod_{i=1}^n (z - c_i)(\bar{c}_i z + 1)}{z \prod_{i=1}^n (z - a_i)(\bar{a}_i z + 1)} dz\ . \end{aligned} \tag{3}$$

Here, each c_i is not an end or coinsides with some of b_j. In addition, the 1-forms α_1, α_2 are exact and there exist real numbers $\lambda_1, \cdots, \lambda_n$ such that

$$\alpha_3 = i \sum_{i=1}^n \frac{\lambda_i(\bar{a}_i z^2 + a_i)}{z(z - a_i)(\bar{a}_i z + 1)} dz. \tag{4}$$

Conversely, suppose that $B \in \mathbf{C}$ with $| B | = 1, \lambda \in R - \{0\}$ are given and 1-forms $\alpha_1, \alpha_2, \alpha_3$ are defined by (3). If α_1 or α_2 is exact and there exist real numbers $\lambda_1, \cdots, \lambda_n$ which satisfy (4), the associated minimal surface given by (1)with (2)is the double surface of a regular complete pseudo-embedded minimal immerion of M.

Proof. We may asuume that $0, a_1, \cdots, a_n, b_1, \cdots, b_m$ are zeros of the Gauss map g of the double surface $\tilde{x} : \tilde{M} \to R^3$. Then $\infty, -1/\bar{a}_1, \cdots, -1/\bar{a}_n$, $-1/\bar{b}_1, \cdots, -1/\bar{b}_m$ are its poles. It is known (see [1],[5]) that $\tilde{x} \cdot I = \tilde{x}$ if and only if

$$g \cdot I = -1/\bar{g}\,, I^*\omega = -\overline{g^2\omega}. \tag{5}$$

Now we can put

$$g = B\frac{z\prod_{i=1}^n(z-a_i)\prod_{j=1}^m(z-b_j)^2\prod_{l=1}^t(z-c_l)}{\prod_{i=1}^n(\bar{a}_iz+1)\prod_{j=1}^m(\bar{b}_jz+1)^2\prod_{l=1}^t(\bar{c}_lz+1)}\,, \tag{6}$$

where $\mid B \mid = 1$ and since all the ends are embedded, we have $deg(g) = 2k-1$. Hence we obtain $t = n$. An end is embedded if and only if

$$Max\{O(\phi_j), j = 1,2,3\} = 2\,,$$

where $O(\phi_j)$ is the order of the pole at the end. Thus using (5), we obtain

$$\omega = A\frac{\prod_{i=1}^m(\bar{b}_jz+1)^2\prod_{j=1}^n(\bar{c}_jz+1)^2}{z^2\prod_{i=1}^n(z-a_i)^2\prod_{j=1}^m(z-b_j)^2}dz\,, \tag{7}$$

where A satisfies $\overline{AB} = -AB$. Hence we can put $AB = i\lambda$. Thus we have the representation (2). Since the 1-forms ϕ_1, ϕ_2 ϕ_3 have no real periods on $\tilde{M}$, α_1, α_2 are exact and α_3 has no imaginary periods, that is, $\oint \alpha_3$ is a real number. As we have

$$I^*\alpha_1 = \bar{\alpha}_2\,, I^*\alpha_3 = -\bar{\alpha}_3$$

α_1 is exact if and only if α_2 is so, and α_3 has no imaginary periods if and only if there exist real numbers $\lambda_1, \cdots, \lambda_n$ which satisfy (4). Now, it is evident that the converse is true. Q.E.D.

§3. A proof of theorem 1

Assume that there is an embedding x of M into R^3. Then, $\tilde{x} : \tilde{M} \to R^3$ is a pseudo-embedded complete minimal immersion with $\tilde{x} \cdot I = \tilde{x}$. We may put $q_1 = 0$ and $q_1' = \infty$. We assume that $z = 0$ is a zero of g. For $z \in \tilde{M}$, put $z = re^{i\theta}, 0 \le r \le \infty, 0 \le \theta \le \pi$. We may suppose that the lines given by $\theta = 0$ and by $\theta = \pi$ do not pass the points $q_2, q_2', \cdots, q_k, q_k'$. We deconpose the minimal surface $\tilde{x}$ into four parts I_1, I_2, I_3 and I_4 such that $I_1 = \tilde{x}(re^{i\theta}), 0 \le r \le 1, 0 \le \theta \le \pi$, $I_2 = \tilde{x}(re^{i\theta}), 1 \le r \le \infty, 0 \le \theta \le \pi$, $I_3 = \tilde{x}(re^{i\theta}), 0 \le r \le 1, \pi \le \theta \le 2\pi$, and $I_4 = \tilde{x}(re^{i\theta}), 1 \le r \le \infty, \pi \le \theta \le 2\pi$. Let e_1 and e_3 be the edges of I_1 given by $0 \le r \le 1, \theta = 0$ and $0 \le r \le 1, \theta = \pi$ respectively. Similarly, e_2 and e_4 are the egdes of I_2 given by $1 \le r \le \infty, \theta = 0$ and $1 \le r \le \infty, \theta = \pi$ respectively. Let M_0 (resp. M_0') be the end corresponding to $q_1 = 0$ (resp. $q_1' = \infty$)(see §2 in [9]). The ends have simple forms as shown in [9]. In fact, by changing coordinate in the $\tilde{x}_1\tilde{x}_2$-plane we can get

$$\tilde{x}_3 = a\log y + b + \frac{c_1\tilde{x}_1}{y^2} + \frac{c_2\tilde{x}_2}{y^2} + O(\ y^{-2})$$

for suitable constants a, b, c_1, c_2, where we set $y = \sqrt{\tilde{x}_1^2 + \tilde{x}_2^2}$. Put $M_1 = M_0 \cap I_1, M_2 = M_0' \cap I_2, M_3 = M_0 \cap I_3, M_4 = M_0' \cap I_4$. We may consider that the image of M_2 coincides with that of M_3. Think of the parts I_1 and I_2. They have the common part $r = 1, 0 \leq \theta \leq \pi$. The edge e_2 (resp. e_4) lies exactly on the edge e_3 (resp. e_1). But these can be done when I_1 and I_2 intersect. We get a contradiction. Q.E.D.

§4. The cases (2) and (3) of theorem 2

In the present section we prove that The cases (1) and (2) of the thorem do not occur. We use the notations in Proposition 2. Assume that the immersion $x : M \to R^3$ has only one Catenoid end, that is, n = 0. Then $\alpha_3 = 1/z$ and it has an imaginary period. We get a contradiction. Next we suppose the case (2). Then we can put

$$\alpha_3 = \frac{(\bar{c}z+1)(z-c)}{z(z-a)(\bar{a}z+1)}dz = i\frac{\lambda_i(\bar{a}z^2+a_i)}{z(z-a)(\bar{a}z+1)}dz.$$

Hence, we can put, up rotation in R^3,

(8) $$\lambda_1 = \frac{1}{|\,a\,|}, \quad c = -i\frac{a}{|\,a\,|}.$$

Since Meeks [3] showed that there is no complete pseudo-embedded minimal immersion of a projective plane with two ends, we suppose the nonorientable minimal surface $x : M \to R^3$ has two Catenoids ends and one flat end. Then we can set

$$\alpha_2 = \frac{P^2}{Q^2}dz\ ,$$

where $P = (z-b)(z-c)$ and $Q = (\bar{a}z+1)(\bar{b}z+1)$. Lopez proved in [5] that α_3 is exact if and only if $F = PQ" - 2P'Q' = 0$ at $-1/\bar{a}$ and $-1/\bar{b}$. Hence we have $F = -6(\bar{a}z+1)(\bar{b}z+1)$, that is,

(9) $$\bar{a} + \bar{b} + \overline{ab}(b+c) \ = \ 0\ ,$$

(10) $$(b+c)(\overline{a+b}) + \overline{ab}bc + 3 \ = \ 0.$$

Set

$$a = r(\cos\theta + i\sin\theta), 0 \le r \le 2\pi, \quad b = x + iy.$$

Then $c = \sin\theta - i\cos\theta$. From (9), we get

$$x - ry = -r(1+t)\cos\theta, \quad rx + y = -r(1+t)\sin\theta, \tag{11}$$

where $t = x^2 + y^2$. It flows from (11) that t must sastisfy the equation

$$r^2t^2 - (1-r^2)t + r^2 = 0.$$

There exist positeve real numbers which satisfy the above equation if and only if

$$0 < r \le \frac{1}{\sqrt{3}}. \tag{12}$$

From (11), we get

$$x = -\frac{r(\cos\theta + r\sin\theta)(1+t)}{1+r^2}, \quad y = -\frac{r(\sin\theta - r\cos\theta)(1+t)}{1+r^2}.$$

Subsituting these into (10), we obatain

$$(3 + r^2) + (1 - r^2)t = 0.$$

This contradicts (12). Thus we show the desired result.

§5. The case (4) of theorem 2

Assume that M has three Catenoid ends, that is $n = 2$ and $m = 0$. Since the 1-form α_3 has no imaginary periods, there exist real numbers λ_1, λ_2 such that

$$\frac{\prod_{i=1}^2 (z - c_i)(\bar{c}_i z + 1)}{z\prod_{i=1}^2 (z - a_i)(\bar{a}_i z + 1)} dz = i\sum_{i=1}^2 \frac{\lambda_i(\bar{a}_i z^2 + a_i)}{z(z - a_i)(\bar{a}_i z + 1)} dz.$$

Hence if we set

$$\lambda_1 + \lambda_2 = 2\mu_1, \quad (\lambda_1 - \lambda_2) = 2\mu_2,$$

$$c_1 + c_2 = C_1, \ c_1c_2 = C_2, a_1 + a_2 = A_1, a_1a_2 = A_2, a_1 - a_2 = A_3,$$

we have

$$C_2 = -i2\mu_1 A_2 \tag{13}$$

$$(14)\qquad 1+|C_2|^2-|C_1|^2 = i2\mu_2(a_1\bar{a}_2-\bar{a}_1a_2),$$
$$(15)\qquad i(C_1-C_2\bar{C}_1) = \mu_1(A_1-A_2\bar{A}_1)+\mu_2(A_3+A_2\bar{A}_3).$$

Put $P=z^2-C_1z+C_2$ and $Q=\bar{A}_2z^2+\bar{A}_1z+1$. Then the 1-form $\alpha_2=\frac{P^2}{Q^2}dz$ is exact if and only if

$$(16)\qquad \bar{A}_1+\bar{A}_2C_1 = 0,$$
$$(17)\qquad \bar{A}_1C_1+\bar{A}_2C_2 = -3.$$

Up reparametrization in $\mathbf{C}\cup\{\infty\}$, we can assume

$$A_2=r,\quad r>0.$$

We set

$$A_1=x_1+iy_1,\quad A_3=x_2+iy_2.$$

Then, from (13) and (16), we have, respectively,

$$C_1=-\frac{\bar{A}_1}{r},\quad C_2=-i2\mu_1r.$$

Substituting these into (17), we have

$$(18)\qquad x_1^2-y_1^2=3r,\quad x_1y_1=\mu_1r^3.$$

Thus we obtain

$$(19)\qquad x_1^2=\frac{3r+D}{2},\quad y_1^2=\frac{-3r+D}{2},$$

where $D=r\sqrt{9+4\mu_1^2r^4}$. Using (13) and (16), we get

$$i\bar{C}_1=\mu_1(\bar{A}_1+\bar{A}_2A_1)-\mu_2(\bar{A}_3+\bar{A}_2A_3).$$

This equation gives

$$(20)\qquad \mu_2r(1+r)x_2 = \mu_1r(1+r)x_1-y_1,$$
$$(21)\qquad \mu_2r(1-r)y_2 = -x_1+\mu_1r(1-r)y_1.$$

If $\mu_2=0$, from (20), we have

$$y_1=\mu_1r(1+r)x_1.$$

Substituting this into the second equation of (18), we get $x_1^2=r^2/(1+r)$. Using (19), we obtain a contradiction

$$r^2+3+(1+r)D=0.$$

Hence $\mu_2 \neq 0$. If $r = 1$, from (21), we have $x_1 = 0$. This contradics the first equation of (18). The equation (14) is rewritten as

$$2\mu_2 r^2(x_1y_2 - x_2y_1) = x_1^2 + y_1^2 - (1 + 4\mu_1^2 r^2)r^2.$$

Using (20), (21) and (19), we obtain

$$(r^2 + 1)D = 4\mu_1^2 r^4(1 - r^2) - r^4 - 5r^2. \tag{22}$$

From this, we have

$$0 < r < 1. \tag{23}$$

Solving (23), we get

$$\mu_1^2 = \frac{r(11 - 6r^2 - r^4) + (1 + r^2)(3 - r^2)\sqrt{4 - 3r^2}}{8r^3(1 - r^2)^2}. \tag{24}$$

Substituting this into (22), we have

$$D = \frac{r^2(1 + r^2) + (3 - r^3)r\sqrt{4 - 3r^2}}{2(1 - r^2)}. \tag{25}$$

Since $A_2 = (A_1^2 - A_3^2)/4$, from (18) it follows

$$x_2^2 - y_2^2 = -r, \quad x_2y_2 = \mu_1 r^3. \tag{26}$$

Now we set

$$F = \mu_1 r^3 \mu_2^2 x_2 y_2. \tag{27}$$

Then it must be that $F > 0$. Using (20), (21), (23) and (24), we obatain a contradiction ;

$$F = -\frac{\mu_1^2 r^3(r^2 - 3)^2(r + \sqrt{4 - 3r^2})}{8}.$$

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