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Recursion Operator and the Trace Formula for 1-dimensional Schrödinger Operator

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Abstract

By using the recursion operator $R(u) = \partial^{-1}(2^{-1}u' + u\partial - 4^{-1}\partial^3)$, various kind of the trace formulae for the 1-dimentional Schödinger operator $H(u) = -\partial^2 + u(x)$ are proved. A notion of rank of the function with respect to $R(u)$ is introduced. Moreover, an alternative proof of the characterization theorem of the reflectionless potentials is given.

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In [4], Deift and Trubowitz derived the trace formula

$$(1)\quad i\pi^{-1}\int_{-\infty}^{\infty}\xi r_{\pm}(\xi;u)f_{\pm}(x,\xi;u)^2 d\xi - \sum_{j=1}^{N} 2c_j(\pm)\beta_j f_{\pm}(x,i\beta_j;u)^2 = 2^{-1}u(x)$$

for the 1-dimensional Schrödinger operator

$$H(u) = -\partial_x^2 + u(x), \partial_x = d/dx, -\infty < x < \infty$$

with the twice differentiable real valued potential $u(x)$ such that $u'(x)$ and $u''(x)$ are in $L^1(\mathbf{R})$, and

$$\int_{-\infty}^{\infty}|x||u(x)|dx < \infty,$$

where $f_\pm(x,\xi;u)$ and $r_\pm(\xi;u)$ are the Jost solutions and the reflection coefficients; $-\beta_j^2$ and $c_j(\pm)$ are the discrete eigenvalues of $H(u)$ and the normalizing coefficients of the eigenfunctions $f_\pm(x,i\beta_j;u)$ respectively ($j=1,2,\cdots,N$).

The purpose of the present paper is to study the problems related to the trace formula (1) by using the operator

$$R(u)=\partial_x^{-1}(2^{-1}u'(x)+u(x)\partial_x-4^{-1}\partial_x^3).$$

$R(u)$ is usually called the recursion operator, since it is related to the recursion relation

$$Z_{n+1}(u)=R(u)Z_n(u),\quad n=0,1,2,\cdots$$

with $Z_0(u)=1$, which is called the Lenard relation, where $u(x)$ is an infinitely differentiable function. By giving appropriate meaning of the operator ∂_x^{-1}, we can determine uniquely the functions $Z_n(u)$ as the differential polynomials of $u(x)$. The general representations for $Z_n(u)$ in terms of the scattering data and the eigenfunctions are known;

$$(2)\qquad i\pi^{-1}\int_{-\infty}^{\infty}\xi^{2n-1}r_\pm(\xi;u)f_\pm(x,\xi;u)^2d\xi + (-1)^n\sum_{j=1}^{N}2c_j(\pm)\beta_j^{2n-1}f_\pm(x,i\beta_j;u)^2=Z_n(u),\quad n\in\mathrm{N}.$$

The formulae (2) are usually derived by the method of asymptotic expansion of an appropriate quantity associated with scattering data. See [2], [8], [14] and references cited in them. In reflectionless case, a simple proof was obtained in [7]. On the other hand, in view of (2), one may conceive that the formulae

$$(3)\qquad i\pi^{-1}\int_{-\infty}^{\infty}\xi^{-1}r_\pm(\xi;u)f_\pm(x,\xi;u)^2d\xi + \sum_{j=1}^{N}2c_j(\pm)\beta_j^{-1}f_\pm(x,i\beta_j;u)^2=Z_0(u)=1$$

are also valid. However this is not true in general. In fact, if the potential $u(x)$ is reflectionless, i.e., $r_\pm(\xi;u)\equiv 0$, then the formula

$$(4)\qquad \sum_{j=1}^{N}2c_j(\pm)\beta_j^{-1}f_\pm(x,i\beta_j;u)^2+f_\pm(x,0;u)^2=1$$

are valid. See e.g. [9:§6]. On the other hand, in [3], Deift, Lund and Trubowitz proved the formulae

$$i\pi^{-1}\int_{-\infty}^{\infty}\xi^{-1}r_{\pm}(\xi;u)f_{\pm}(x,\xi;u)^{2}d\xi=1 \tag{5}$$

for the rapidly decreasing real valued potential $u(x)$ such that $H(u)$ has no bound states and $r_{\pm}(0;u) = -1$. It is pointed out by many authors that there exists a close relation between spectral theory and constrained harmonic motion with the constraint defined by (4) or (5). See e.g. [9] and [3]. Therefore it seems to be worthwhile to give an unified understanding of (4) and (5). Accordingly the first problem of the present paper is to prove the following.

Theorem 1. *Let $u(x)$ be the twice differentiable real potential such that $u'(x)$ and $u''(x)$ are in $L^1(\mathbf{R})$, and*

$$\int_{-\infty}^{\infty}|x||u(x)|dx<\infty.$$

Suppose that $H(u)$ has N bound states $-\beta_j^2$, $j=1,2,\cdots,N$, where $0<\beta_1<\beta_2<\cdots<\beta_N$, and the normalization coefficients $c_j(\pm)$, $j=1,2,\cdots,N$. Then the formulae

$$\begin{aligned}(6)\qquad & i\pi^{-1}\int_{-\infty}^{\infty}\xi^{-1}r_{\pm}(\xi;u)f_{\pm}(x,\xi;u)^{2}d\xi \\ & +\sum_{j=1}^{N}2c_j(\pm)\beta_j^{-1}f_{\pm}(x,i\beta_j;u)^2+(1+r_{\pm}(0;u))f_{\pm}(x,0;u)^2=1\end{aligned}$$

are valid, where the integrals in (6) are interpreted as principal values.

Thus, by Theorem 1, the formulae (3) turn out to be valid for the potential $u(x)$ such that

$$r_{\pm}(0;u)=-1. \tag{7}$$

The condition (7) is known to be quite generic one, while condition

$$|r_{\pm}(0;u)|<1$$

is unstable. See [4] for details.

On the other hand, in [3], they simultaneously proved the formula

$$(8)\qquad -i\pi^{-1}\int_{-\infty}^{\infty}\xi^{-1}r_{\pm}(\xi;u)f'_{\pm}(x,\xi;u)^2d\xi=2^{-1}u(x)$$

for the real valued rapidly decreasing potential $u(x)$ such that $H(u)$ has no bound states and satisfies (7). The second aim of the present paper is to prove the following formulae which are generalization of (8).

Theorem 2. *If the potential $u(x)$ satisfies the conditions in Theorem 1, then the formulas*

$$(9)\qquad -i\pi^{-1}\int_{-\infty}^{\infty}\xi^{-1}r_{\pm}(\xi;u)f'_{\pm}(x,\xi;u)^2d\xi-\sum_{j=1}^{N}2c_j(\pm)\beta_j^{-1}f'_{\pm}(x,i\beta_j;u)^2$$
$$-(1+r_{\pm}(0;u))f'_{\pm}(x,0;u)^2=2^{-1}u(x)$$

are valid. Moreover if the real potential $u(x)$ is rapidly decreasing then

$$(10)\qquad -i\pi^{-1}\int_{-\infty}^{\infty}\xi^{2n-1}r_{\pm}(\xi;u)f'_{\pm}(x,\xi;u)^2d\xi$$
$$-(-1)^n\sum_{j=1}^{N}2c_j(\pm)\beta_j^{2n-1}f'_{\pm}(x,i\beta_j;u)^2$$
$$=-Z_{n+1}(u(x))+u(x)Z_n(u(x))-2^{-1}\partial_x^2Z_n(u(x))$$

are valid for all $n\in\mathbf{N}$.

The third aim of this paper is to give elementary proof of already known results in the soliton theory such as the formula (2) and the characterization theorem of the reflectionless potential. We shall do this by enhancing the role of the recursion operator $R(u)$ without using the inverse scattering theory and the asymptotic expansion.

The contents of the present paper are as follows. In section 1, we give a brief sketch of scattering data of $H(u)$. In section 2, we prove Theorem 1. In section 3, we give a simple proof of the formulas (2) for $n\geq 2$ by using the recursion operator $R(u)$. In section 4, we prove Theorem 2. In section 5, we give an elementary proof of the characterization theorem of the reflectionless potentials.

§1. Scattering data.

In this section, we will briefly explain the scattering data of $H(u)$. We refer to [1], [4] and [6] for details.

Let $f_\pm(x,\xi;u)$ be those solutions of the eigenvalue problem

$$H(u)f = -f'' + u(x)f = \xi^2 f \tag{1.1}$$

which behave like $e^{\pm i\xi x}$ as $x \to \pm\infty$ respectively. If the potential $u(x)$ is a real valued measurable function and satisfies

$$\int_{-\infty}^{\infty} |x||u(x)|dx < \infty$$

then such solutions uniquely exist for all ξ, $\Im\xi \geq 0$, and are analytic in $\Im\xi > 0$ for any x. They are called the *Jost solutions*. Since $e^{\mp i\xi x} f_\pm(x,\xi;u) - 1$ belong to H^{2+}, the Hardy space of functions $h(\xi)$ analytic in $\Im\xi > 0$ with

$$\sup_{t>0} \int_{-\infty}^{\infty} |h(s+it)|^2 ds < \infty,$$

the integral representations

$$f_\pm(x,\xi;u) = e^{\pm i\xi x}(1 \pm \int_0^{\pm\infty} B_\pm(x,y)e^{\pm 2i\xi y}dy) \tag{1.2}$$

hold. $B_\pm(x,y)$ satisfy

$$|B_\pm(x,y)| \leq e^{\gamma_\pm(x)}\eta_\pm(x+y),$$

where

$$\gamma_\pm(x) = \int_x^{\pm\infty} (t-x)|u(t)|dt$$

and

$$\eta_\pm(x) = \pm\int_x^{\pm\infty} |u(x)|dt.$$

Since $u(x)$ is real, $f_+(x,\pm\xi;u), \xi \in \mathbf{R}\backslash\{0\}$ are linearly independent solutions. Hence there uniquely exist the functions $a_\pm(\xi;u)$ and $b_\pm(\xi;u)$ such that

$$f_-(x,\xi;u) = a_+(\xi;u)f_+(x,-\xi;u) + b_+(\xi;u)f_+(x,\xi;u). \tag{1.3}$$

Similarly, there exist $a_-(\xi;u)$ and $b_-(\xi;u)$ such that

$$f_+(x,\xi;u) = a_-(\xi;u)f_-(x,-\xi;u) + b_-(\xi;u)f_-(x,\xi;u).$$

Therefore

$$a_\pm(\xi;u) = (\pm 2i\xi)^{-1}W(f_\mp(x,\xi;u), f_\pm(x,\xi;u))$$

and

$$b_\pm(\xi;u) = (\pm 2i\xi)^{-1}W(f_\pm(x,-\xi;u), f_\mp(x,\xi;u))$$

follow, where $W(f,g) = fg' - f'g$ is the Wronskian. Hence $a_\pm(\xi;u)$ are analytic in $\Im\xi > 0$. Moreover if ξ is real then we have

$$|a_\pm(\xi;u)|^2 = 1 + |b_\pm(\xi;u)|^2.$$

Hence the functions

$$r_\pm(\xi;u) = b_\pm(\xi;u)/a_\pm(\xi;u)$$

are defined for $\xi \in \mathbf{R} \setminus \{0\}$ and continuous, which are called the right and left reflection coefficients. We have

$$\overline{r_\pm(\xi;u)} = r_\pm(-\xi;u). \tag{1.4}$$

Moreover if the potential $u(x)$ satisfies

$$\int_{-\infty}^{\infty} x^2|u(x)|dx < \infty$$

then $r_\pm(\xi;u)$ are continuous even at $\xi = 0$, and there are two possibilities:

$$r_\pm(0;u) = -1 \tag{1.5}$$

or

$$|r_\pm(\xi;u)| < 1, \quad \xi \in \mathbf{R}. \tag{1.6}$$

Furthermore $u(x)$ belongs to the Schwarz space S of rapidly decreasing functions, if and only if $r_\pm(\xi;u)$ belong to S. The functions $a_\pm(\xi;u)$ have only finite number of simple zeros $i\beta_j, 1 \le j \le N$ on the purely imaginary axis in $\Im\xi > 0$, where $0 < \beta_1 < \beta_2 < \cdots < \beta_N$. Since $f_\pm(x,i\beta_j;u)$ decay exponentially as $x \to \pm\infty$ and $W(f_\mp(x,i\beta_j;u), f_\pm(x,i\beta_j;u)) = 0$, $f_\pm(x,i\beta_j;u)$ belong to $L^2(\mathbf{R})$, that is, it follows that $H(u)$ has the N bound states $-\beta_1^2, \cdots, -\beta_N^2$. Put

$$c_j(\pm) = (\int_{-\infty}^{\infty} f_\pm(x, i\beta_j; u)^2 dx)^{-1},$$

which are called the normalization coefficients. The triplets

$$\Sigma_\pm = \{r_\pm(\xi; u); -\beta_1^2, \cdots, -\beta_N^2; c_1(\pm), \cdots, c_N(\pm)\}$$

are called the right and left *scattering data.* The scattering data uniquely determines the potential $u(x)$ provided

$$\int_{-\infty}^{\infty} |x||u(x)|dx < \infty.$$

Here we explain Crum's algorithm for removing eigenvalueas (see [4: p172]): If $u(x)$ satisfies

$$\int_{-\infty}^{\infty} |x||u(x)|dx < \infty,$$

and $H(u)$ has the N bound states $-\beta_1^2, \cdots, -\beta_N^2$, where $0 < \beta_1 < \cdots < \beta_N$. Put

$$u^*(x) = u(x) - 2\partial_x^2 \log f_+(x, i\beta_N; u)$$

then $H(u^*)$ has $N-1$ bound states $-\beta_1^2, \cdots, -\beta_{N-1}^2$, and

$$r_\pm(\xi; u^*) = -\frac{\xi - i\beta_N}{\xi + i\beta_N} r_\pm(\xi; u)$$

hold. The above procedure obtaining the potential $u^*(x)$ from the original one $u(x)$ and/or $u^*(x)$ itself are called the *Crum transformation.* By N times repeated applications of the Crum transformation, we can construct the potential $u_0(x)$ such that $H(u_0)$ has no bound states and

$$r_\pm(\xi; u_0) = (-1)^N \prod_{j=1}^{N} \frac{\xi - i\beta_j}{\xi + i\beta_j} r_\pm(\xi; u). \tag{1.7}$$

Now suppose $u(x) \in C^\infty(\mathbf{R})$ and let $Q_n(x; u), n = 0, 1, 2, \cdots$ be the infinite sequence of the functions defined by the recursion relation

$$\partial_x Q_{n+1}(x; u) = L(u) Q_n(x; u)$$

with $Q_0(x;u) \equiv 1$, where

$$L(u) = \partial_x R(u) = 2^{-1}u'(x) + u(x)\partial_x - 4^{-1}\partial_x^3.$$

Then, $Q_n(x;u)$ are known to be polynomials of $u, u', \cdots, u^{(2n-2)}$ with constant coefficients. While an arbitrary constant appears when we integrate $\partial_x Q_n(x;u)$ to obtain $Q_n(x;u)$ itself, we can define uniquely $Q_n(x;u)$ by putting it zero. Hence we can determine uniquely the infinite sequence of differential polynomials $Q_n(x;u)$. We denote them by $Z_n(u(x))$;

$$Z_n(u(x)) = Q_n(x;u).$$

Here we give the first two of them:

$$Z_1(u) = 2^{-1}u(x), \qquad Z_2(u) = 8^{-1}(3u^2 - u'').$$

Note that if $u(x)$ is in the Schwarz space S then

$$Z_{n+1}(u(x)) = \int_{-\infty}^{x} L(u(y))Z_n(u(y))dy, \quad n = 0, 1, 2, \cdots$$

follows. The ordinary differential equations

$$\partial_x(Z_{N+1}(u) + \sum_{j=1}^{N} \alpha_j Z_j(u)) = 0$$

are called the N-th stationary KdV equations, which play important roles in the soliton theory. We will also use the notations

$$X_n(u(x)) = \partial_x Z_n(u(x)), \quad n \in \mathrm{N}.$$

The differential polynomials $X_n(u)$ are related to the theory of semicommutative differential operators originated by Burchnall and Chaundy.

§2. Proof of Theorem 1.

In this section we shall prove Theorem 1. Put

$$\phi(x) = i\pi^{-1}\int_{-\infty}^{\infty} \xi^{-1} r_+(\xi;u) f_+(x,\xi;u)^2 d\xi + \sum_{j=1}^{N} 2c_j(+)\beta_j^{-1} f_+(x, i\beta_j; u)^2.$$

One easily verifies that the integral of the above converges as principal value. Noting that $u(x)$ is twice differentiable and eliminating $f''_+(x,\xi;u)$ by (1.1), one can check that $\phi(x)$ is three times differentiable. Accordingly, by differentiating $\phi(x)$ three times and using the formula (1), we have

$$\phi'''(x) - 4u(x)\phi'(x) - 2u'(x)\phi(x) = -2u'(x). \tag{2.1}$$

One can readily check that the constant function 1 solves (2.1). On the other hand let $f_j(x), j = 1,2$ be the fundamental system of solutions of

$$-f'' + u(x)f = 0 \tag{2.2}$$

then $f_1(x)^2, f_1(x)f_2(x)$ and $f_2(x)^2$ are the fundamental system of solutions of

$$g''' - 4u(x)g' - 2u'(x)g = 0$$

(see e.g. [12],[13],[14]). Put

$$f_1(x) = f_+(x,0;u).$$

Note that $f_1(x)$ tends to 1 as $x \to \infty$. Hence

$$f_2(x) = f_+(x,0;u)\int_a^x f_+(y,0;u)^{-2}dy$$

solves (2.2) for sufficiently large a, and

$$W(f_1, f_2) = 1$$

follows. Therefore there exist $c_j, j = 1,2,3$ such that

$$\phi(x) = 1 + \sum_{j=1}^{3} c_j g_j(x), \tag{2.3}$$

where $g_1(x) = f_1(x)^2, g_2(x) = f_1(x)f_2(x)$ and $g_3(x) = f_2(x)^2$. Since $f_1(x)$ tends to 1 as $x \to \infty$, we have

$$\lim_{x\to\infty} \frac{g_j(x)}{x^{j-1}} = 1$$

Now we investigate the asymptotic behaviour of $\phi(x)$ as $x \to \infty$. Since $f_+(x,\xi;u) \sim e^{i\xi x}$ as $x \to \infty$, we have

$$\phi(x) \sim i\pi^{-1}\int_{-\infty}^{\infty} \xi^{-1} r_+(\xi;u) e^{2i\xi x} d\xi + \sum_{j=1}^{N} 2c_j(+)\beta_j^{-1} e^{-2\beta_j x}$$

as $x \to \infty$. Put

$$r_+(\xi;u) = P_1(\xi) + iP_2(\xi),$$

where $P_j(\xi), j = 1,2$ are real valued. Then we have

$$\int_{-\infty}^{\infty} \xi^{-1} r_+(\xi;u) e^{2i\xi x} d\xi = R_1(x) + iR_2(x),$$

where

$$R_j(x) = \int_{-\infty}^{\infty} \xi^{-1} P_j(\xi) e^{2i\xi x} d\xi, \quad j = 1,2.$$

From (1.3), $P_1(\xi) = P_1(-\xi)$ and $P_2(\xi) = -P_2(-\xi)$ follow. Hence we have

$$R_1(x) = 2i\int_0^{\infty} \xi^{-1} P_1(\xi) \sin \xi x d\xi,$$

$$R_2(x) = 2i\int_0^{\infty} \xi^{-1} P_2(\xi) \cos \xi x d\xi.$$

On the other hand, by (1.3), we have

$$P_1(0) = r_+(0;u), \qquad P_2(0) = 0.$$

Taking into consideration the above, we rewrite $R_1(x)$ as follows:

$$R_1(x) = 2i\int_0^1 \xi^{-1}(P_1(\xi) - r_+(0;u)) \sin 2\xi x d\xi$$

$$+2ir_+(0;u)\int_0^1 \xi^{-1} \sin 2\xi x d\xi + 2i\int_1^{\infty} \xi^{-1} P_1(\xi) \sin \xi x d\xi.$$

Since $P_1(\xi)$ is differentiable at $\xi = 0$ and

$$P_1(0) - r_+(0;u) = 0,$$

$\xi^{-1}(P_1(\xi) - r_+(0;u))$ is bounded in the compact interval [0, 1]. Moreover $\xi^{-1}P_1(\xi)$ is summable in the interval $[1,\infty)$. Hence, by the well known formula

$$\lim_{A\uparrow\infty}\int_0^A \gamma^{-1} \sin \gamma d\gamma = 2^{-1}\pi$$

and the Riemann-Lebesgue theorem, we have

$$R_1(x) \sim ir_+(0;u)\pi$$

as $x \to \infty$. Similarly one verifies that $R_2(x)$ tends to zero as $x \to \infty$. Therefore we proved

$$\phi(x) \sim -r_+(0;u)$$

as $x \to \infty$. Hence, from (2.3) and (2.4),

$$\phi(x) = 1 - (1 + r_+(0;u))f_+(x,0;u)^2$$

follows. Thus we proved (6) for +. The proof for − is similar. This completes the proof of Theorem 1.

§3. Proof of (2) for $n \geq 2$.

In this section we will give a simple proof of (2) for $n \geq 2$ assuming the original formula (1). The method for the proof is a quite simple inductive argument based on the recursion operator $R(u)$. This method was already carried out by Gardner et al in [7] for the reflectionless potential.

Put

$$\phi_n(x;\pm) = i\pi^{-1}\int_{-\infty}^{\infty} \xi^{2n-1} r_\pm(\xi;u) f_\pm(x,\xi;u)^2 dxi, \quad n = 0,1,2,\cdots \tag{3.1}$$

and

$$\psi_{nj}(x;\pm) = (-1)^n \beta_j^{2n-1} f_\pm(x, i\beta_j;u)^2, \quad n = 0,1,2,\cdots$$

First we have

Lemma 3.1.

$$\pm\int_{\mp\infty}^{x} L(u(x))\phi_n(x;\pm)dx = \phi_{n+1}(x;\pm), \quad n = 0,1,2,\cdots \tag{3.2}$$

and

$$\pm\int_{\mp\infty}^{x} L(u(x))\psi_{nj}(x;\pm)dx = \psi_{n+1j}(x;\pm), \quad n = 0,1,2,\cdots \tag{3.3}$$

are valid.

PROOF. First we prove (3.2). By direct calculation we have

$$L(u)f_\pm(x,\xi;u)^2 = 2\xi^2 f_\pm(x,\xi;u)f'_\pm(x,\xi;u). \tag{3.4}$$

Hence

$$\begin{aligned}
&\pm\int_{\mp\infty}^{x} L(u(y))\phi_n(y;\pm)dy \\
&= \pm i\pi^{-1}\int_{\mp\infty}^{x} dy\int_{-\infty}^{\infty}\xi^{2n-1}r_\pm(\xi;u)L(u(y))f_\pm(y,\xi;u)^2 d\xi \\
&= \pm 2i\pi^{-1}\int_{\mp\infty}^{x} dx\int_{-\infty}^{\infty}\xi^{2n+1}r_\pm(\xi;u)f_\pm(x,\xi;u)f'_\pm(x,\xi;u)d\xi \\
&= \pm i\pi^{-1}\int_{-\infty}^{\infty}\xi^{2n+1}r_\pm(\xi;u)f_\pm(x,\xi;u)^2 d\xi \\
&= \phi_{n+1}(x;\pm).
\end{aligned}$$

follows. Second we prove (3.3). Similarly to (3.4), one verifies immediately

$$L(u)\psi_{nj}(x;\pm) = (-1)^{n+1}2\beta_j^{2n+1}f_\pm(x,i\beta_j;u)f'_\pm(x,i\beta_j;u).$$

Since $f_\pm(x,i\beta_j;u)$ exponentially tend to zero as $|x|\to\infty$,

$$\begin{aligned}
\pm\int_{-\infty}^{x} L(u(y))\psi_{nj}(y;\pm)dy &= (-1)^{n+1}\beta_j^{2n+1}f_\pm(x,i\beta_j;u)^2 \\
&= \psi_{n+1j}(x;\pm)
\end{aligned}$$

hold. Q.E.D.

Here we prove (2) for $n\geq 2$. The trace formula (1) can be expressed as

$$\phi_1(x;\pm) + 2\sum_{j=1}^{n}\psi_{1j}(x;\pm) = Z_1(u). \tag{3.5}$$

Assme that

$$\phi_{n-1}(x;\pm) + 2\sum_{j=1}^{N}\psi_{n-1j}(x;\pm) = Z_{n-1}(u)$$

holds. Then, by lemma 1, we have

$$\pm\int_{\mp\infty}^{x} L(u(y))\{\phi_{n-1}(y;\pm) + 2\sum_{j=1}^{N}\psi_{n-1j}(y;\pm)\}dy$$

$$= \phi_n(x;\pm) + 2\sum_{j=1}^{N} \psi_{nj}(x;\pm)$$

$$= \pm \int_{\mp\infty}^{x} L(u(y))Z_{n-1}(u(y))dy = Z_n(u).$$

This completes the proof of (2) for $n \geq 2$.

Remark. The formula (1) itself can be derived by elementary contour integral (see [4;p195]). Consequently, all of the trace formulae (2) can be derived by the quite elementary method without making use of the inverse scattering theory and the asymptotic expansion. Compare this proof with the one in [15].

§4. Proof of Theorem 2.

Differentiate twice the both sides of (2), then by (2), (1.1) and the Lenard relation, we have

(4.1)
$$\partial_x^2 Z_n(u) = 2i\pi^{-1}\int_{-\infty}^{\infty} \xi^{2n-1} r_\pm(x,\xi;u)^2 f'_\pm(x,\xi;u)^2 d\xi$$
$$+ (-1)^n \sum_{j=1}^{N} 4c_j(\pm)\beta_j^{2n-1} f'_\pm(x,i\beta_j;u)^2 - 2Z_{n+1}(u) + 2u(x)Z_n(u),$$

for $n \geq 1$. This implies (10). On the other hand, similarly to the above, by differentiating twice the both side of (5), eliminating $f''_\pm$ by (1.1) and taking into consideration the trace formula (1), we have

$$-i\pi^{-1}\int_{-\infty}^{\infty} \xi^{-1} r_\pm(\xi;u) f'_\pm(x,\xi;u)^2 d\xi - \sum_{j=1}^{N} 2c_j(\pm)\beta_j^{-1} f'_\pm(x,i\beta_j;u)^2$$
$$- (1 + r_\pm(0;u)) f'_\pm(x,0;u)^2 = 2^{-1}u(x).$$

This completes the proof of Theorem 2.

§5. Application of (2).

The following results are classical ones in the soliton theory:

(A) *Suppose that $u(x)$ is the reflectionless potential with the N discrete eigenvalues $-\beta_1^2, \cdots, -\beta_N^2$, and the corresponding normalization coefficients $c_1(\pm), \cdots, c_N(\pm)$. Define $\mu_j(u), j = 1, 2, \cdots, N$ by*

$$\prod_{j=1}^{N}(X + \beta_j^2) = X^N + \sum_{j=1}^{N} \mu_j(u) X^{j-1},$$

that is, $\mu_j(u), j = 1, 2, \cdots, N$ are the elementary symmetric polynomials of $\beta_j^2, j = 1, 2, \cdots, N$. Then $u(x)$ solves the ordinary differential equation

$$Z_{N+1}(u) + \sum_{j=1}^{N} \mu_j(u) Z_j(u) = 0.$$

(B) *Conversely if $u(x)$ is the rapidly decreasing solution of the N-th stationary KdV equation*

$$\partial_x(Z_{N+1}(u) + \sum_{j=1}^{N} \gamma_j Z_j(u)) = 0$$

then $u(x)$ is the reflectionless potential, i.e., $r_\pm(\xi; u) \equiv 0$, and

$$\sharp\sigma_p(H(u)) \leq N,$$

where $\sigma_p(*)$ is the point spectrum and $\sharp$ denotes cardinal.

In this section, we will explain the new method to prove (A) and (B) by applying the formula (2) and Crum's algorithm.

First we introduce the notion of the rank of the potential. Suppose $u(x) \in C^\infty(\mathrm{R})$. Let $V(u)$ be the vector space over $\mathbf{C}$ spanned by the infinite sequence of the differential polynomials $Z_0(u), Z_1(u), Z_2(u), \cdots$.

Definition. The potential $u(x)$ is said to be of finite rank if and only if the vector space $V(u)$ is finite dimensional. When $u(x)$ is of finite rank, we put

$$\operatorname{rank} u(x) = \dim V(u) - 1.$$

Then we have

Lemma 5.1. *Suppose* $0 \neq \operatorname{rank} u(x) = N < \infty$. *Then* $V(u)$ *is spanned by* $Z_0(u), Z_1(u), \cdots, Z_N(u)$. Conversely, if $Z_0(u), Z_2(u), \cdots, Z_N(u)$ *are linearly independent and* $Z_{N+1}(u)$ *is expressed as the linear combination of* $Z_0(u), Z_2(u), \cdots, Z_N(u)$ *then* $\operatorname{rank} u(x) = N$ *follows.*

Proof. Since $u \neq 0$ follows from the assumption, there exists $m \in$ N such that $Z_0(u), Z_1(u), \cdots, Z_m(u)$ are linearly independent and $Z_0(u)$, $Z_1(u), \cdots, Z_m(u), Z_{m+1}(u)$ are linearly dependent. This implies that we can uniquely express $Z_{m+1}(u)$ as

$$Z_{m+1}(u) = \sum_{\nu=0}^{m} c_\nu Z_\nu(u).$$

Hence we have

$$\begin{aligned} X_{m+2}(u) &= L(u) Z_{m+1}(u) \\ &= \sum_{\nu=0}^{m} c_\nu L(u) Z_\nu(u) \\ &= c_m X_{m+1}(u) + \sum_{\nu=0}^{m-1} c_\nu X_{\nu+1}(u) \\ &= c_m \sum_{\nu=0}^{m} c_\nu X_\nu(u) + \sum_{\nu=0}^{m-1} c_\nu X_{\nu+1}(u) \\ &= \sum_{\nu=2}^{m} (c_m c_\nu + c_{\nu-1}) X_\nu(u) + c_m c_1 X_1(u) \end{aligned}$$

This implies that $Z_{m+2}(u)$ can be expressed as the linear combination of $Z_0(u), Z_1(u), \cdots, Z_m(u)$. Similarly to the above, one verifies that $Z_{m+j}(u), j \geq 1$ can be expressed as the linear combination of $Z_0(u), Z_1(u), \cdots, Z_m(u)$. Hence $m = N$ follows. The converse statement can be proved by the similar argument. Q.E.D.

By Lemma 5.1, it follows that if $0 \neq \operatorname{rank} u(x) = N < \infty$ then there uniquely exist $\mu_0(u), \mu_1(u), \cdots, \mu_N(u)$ such that

$$Z_{N+1}(u) + \sum_{j=0}^{N} \mu_j(u) Z_j(u) = 0.$$

We call $\mu_j(u), j = 0, 1, \cdots, N$ the characteristic coefficients of $u(x)$.

Remark. By Novikov's theorem [5], it follows that the periodic potential $u(x)$ is finitezonal if and only if $u(x)$ is of finite rank. The calculation of the rank and the characteristic coefficients will be carried out for various potentials in the forthcoming paper.

Now we can express the assertion (A) as follows, which is a slight refinement of (A).

Theorem 3. *If $u(x)$ is the reflectionless potential with N discrete eigenvalues $-\beta_1^2, -\beta_2^2, \cdots, -\beta_N^2$ then* rank $u(x) = N, \mu_0(u) = 0$ *and* $\mu_1(u), \cdots, \mu_N(u)$ *are the elementary symmetric polynomials of $\beta_1^2, \beta_2^2, \cdots, \beta_N^2$, i.e.*

$$P(X) = \prod_{j=1}^{N}(X + \beta_j^2) = X^N + \sum_{j=1}^{N} \mu_j(u) X^{j-1}. \tag{5.1}$$

PROOF. First we prove that $Z_0(u), Z_1(u), \cdots, Z_N(u)$ are linearly independent. Suppose that there exist $\lambda_n \in \mathbf{C}, n = 1, 2, \cdots, N$ such that

$$\sum_{n=0}^{N} \lambda_n Z_n(u) = 0. \tag{5.2}$$

One readily verifies $\lambda_0 = 0$, because $u \in S$. By (2), we have

$$\sum_{n=1}^{N} \lambda_n Z_n(u) = \sum_{j=1}^{N} c_j(+)(\sum_{n=1}^{N}(-1)^n \beta_j^{2n-1}\lambda_n) f_+(x, i\beta_j; u)^2.$$

Since $f_+(x, i\beta_j; u) \sim e^{-\beta_j x}$ as $x \to \infty$ and $0 < \beta_1 < \beta_2 < \cdots < \beta_N$,

$$e^{2\beta_1 x} \sum_{n=1}^{N} \lambda_n Z_n(u) \sim c_1(+) \sum_{n=1}^{N}(-1)^n \beta_1^{2n-1}\lambda_n, x \to \infty$$

follows. By (5.2), we have

$$\sum_{n=1}^{N}(-1)^n \beta_1^{2n-1}\lambda_n = 0.$$

Similarly to the above, one verifies

$$\sum_{n=1}^{N}(-1)^n\beta_j^{2n-1}\lambda_n = 0, j = 2, \cdots, N.$$

This implies $\lambda_1 = \lambda_2 = \cdots = \lambda_N = 0$. Hence $Z_0(u), Z_1(u), \cdots, Z_N(u)$ are linearly independent. Next we have

$$\begin{aligned} &Z_{N+1}(u) + \sum_{i=1}^{N}\mu_i(u)Z_i(u) \\ &= -\sum_{j=1}^{N} c_j(+)\beta_j f_+(x, i\beta_j; u)^2((-\beta_j^2)^N + \sum_{i=1}^{N}\mu_i(u)(-\beta_j^2)^{i-1}) \\ &= -\sum_{j=1}^{N} c_j(+)\beta_j P(-\beta_j^2) f_+(x, i\beta_j; u)^2. \end{aligned}$$

Since $P(-\beta_j^2) = 0, j = 1, 2, \cdots, N$ by (5.1), $Z_1(u), \cdots, Z_N(u), Z_{N+1}(u)$ are linearly dependent. Q.E.D.

Next, to show the converse of Theorem 3, we show the following.

Lemma 5.2. *Suppose that* $g(\xi) \in L^1(\mathbf{R})$,

$$\int_{-\infty}^{\infty} |x||u(x)| < \infty,$$

and

$$\int_{-\infty}^{\infty} g(\xi) f_\pm(x, \xi; u)^2 d\xi \equiv 0$$

then $g(\xi) \equiv 0$ *follows.*

Proof. By (1.2) and the convolution formula, we have

$$f_+(x, \xi; u)^2 = e^{2i\xi x}(1 + \int_0^{\infty} K(x, y)e^{2i\xi y}dy),$$

where

$$K(x, y) = 2B_+(x, y) + \int_0^{y} B_+(x, z - y)B_+(x, z)dz.$$

Hence one verifies

$$\int_{-\infty}^{\infty} g(\xi) f_+(x, \xi; u)^2 d\xi$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} g(\xi)e^{2i\xi x}(1+\int_0^{\infty} K(x,y)e^{2i\xi y}dy)d\xi \\
&= \widehat{g}(x) + \int_0^{\infty} K(x,y)\int_{-\infty}^{\infty} g(\xi)e^{2i\xi(x+y)}d\xi dy \\
&= \widehat{g}(x) + \int_0^{\infty} K(x,y)\widehat{g}(x+y)dy \\
&= \widehat{g}(x) + \int_x^{\infty} K(x,y-x)\widehat{g}(y)dy = 0,
\end{aligned}
$$

where

$$\widehat{g}(x) = \int_{-\infty}^{\infty} g(\xi)e^{2i\xi x}d\xi.$$

Since $\widehat{g}(x)$ solves the homogeneous Volterra integral equation of the second kind, $\widehat{g}(x) \equiv 0$ follows. Q.E.D.

Next we have

Lemma 5.3. *Suppose $u(x) \in C^{\infty}(\mathbf{R})$ then*

$$V(u+\lambda) = V(u)$$

holds true for arbitrary $\lambda \in \mathbf{C}$.

PROOF. Put

$$A_n(u) = \sum_{j=0}^{n}(Z_j(u)\partial_x - 2^{-1}X_j(u))H(u)^{n-j}$$

then one easily verifies

$$[A_n(u), H(u)] = 2X_{n+1}(u),$$

where$[A,B] = AB - BA$ (cf. [8] and [13]). First we show that there exist $\alpha_j^{(n)}(\lambda) \in \mathbf{C}, j = 0,1,\cdots,n$ such that

$$Z_n(u+\lambda) = \sum_{j=0}^{n} \alpha_j^{(n)}(\lambda)Z_j(u). \tag{5.3}$$

We prove this by induction. If $n=1$ then one can check readily that (5.3) holds. We assume that (5.3) holds for any $j \leq n$. Suppose that f is a nontrivial solution of

$$H(u+\lambda)f = -f'' + (u+\lambda)f = 0.$$

By the assumption we have

$$A_n(u+\lambda)=\sum_{j=0}^{n}((Z_j(u+\lambda)\partial_x-2^{-1}X_j(u+\lambda))H(u+\lambda)^{n-j}$$

$$=\sum_{j=0}^{n}\sum_{k=0}^{j}\alpha_k^{(j)}(\lambda)(Z_k(u)\partial_x-2^{-1}X_k(u))H(u+\lambda)^{n-j}.$$

On the other hand we have

$$2X_{n+1}(u+\lambda)f=[A_n(u+\lambda),H(u+\lambda)]f$$
$$=-H(u+\lambda)A_n(u+\lambda)f.$$

Moreover, by slight calculation, we have

$$A_n(u+\lambda)f=(Z_n(u+\lambda)\partial_x-2^{-1}X_n(u+\lambda))f$$
$$=\sum_{j=0}^{n}\alpha_j^{(n)}(\lambda)(Z_j(u)\partial_x-2^{-1}X_j(u))f.$$

This implies

$$2X_{n+1}(u+\lambda)=-H(u+\lambda)\sum_{j=0}^{n}\alpha_j^{(n)}(\lambda)(Z_j(u)f'-2^{-1}X_j(u)f).$$

By calculating the right hand side of the above and eliminating f'' and f''' by

$$f''=(u+\lambda)f,$$
$$f'''=u'f+(u+\lambda)f',$$

we have

$$2X_{n+1}(u+\lambda)f=-\sum_{j=0}^{n}\alpha_j^{(n)}(\lambda)(-2L(u)Z_j(u)-2\lambda X_j(u))f$$
$$=2\sum_{j=0}^{n}\alpha_j^{(n)}(\lambda)(X_{j+1}(u)+\lambda X_j(u))f.$$

Thus we proved that $X_{n+1}(u+\lambda)$ can be expressed as the linear combination of $X_1(u), \cdots, X_{n+1}(u)$. This implies that $Z_{n+1}(u+\lambda)$ is the linear combination of $Z_0(u), Z_1(u), \cdots, Z_{n+1}(u)$. Thus (5.3) is proved. By (5.3), we have

$$V(u+\lambda) \subset V(u).$$

This implies also

$$V(u) = V((u+\lambda)-\lambda) \subset V(u+\lambda).$$

Hence $V(u) = V(u+\lambda)$ follows. Q.E.D.

Corollary. *If $u(x) \in C^\infty$ is of finite rank then*

$$\text{rank } u(x) = \text{rank } (u(x)+\lambda)$$

holds for arbitrary $\lambda \in \mathbf{C}$.

Next we have

Lemma 5.4. *Suppose that the rapidly decreasing real valued potential $u(x)$ is of finite rank and $H(u)$ has the N bound states $-\beta_1^2, \cdots, -\beta_N^2$, where $0<\beta_1<\cdots<\beta_N$. Put*

$$u^*(x) = u(x) - 2\partial_x^2 \log f_+(x, i\beta_N; u)$$

then $u^*(x)$, which is in S, is of finite rank and $H(u^*)$ has $N-1$ bound states $-\beta_1^2, \cdots, -\beta_{N-1}^2$.

PROOF. By [4], it turns out that $u^*(x) \in S$ and $H(u^*)$ has $N-1$ bound states $-\beta_1^2, \cdots, -\beta_{N-1}^2$. Hence it suffices to show that u^* is of finite rank. By Lemma 5.3, $u(x)+\beta_N^2$ is of finite rank. On the other hand, by [11] (see also [10]), we have

$$\text{rank } (u^*(x)+\beta_N^2) \leq 1 + \text{rank } (u(x)+\beta_N^2).$$

This implies that $u^*(x)$ is of finite rank. Q.E.D.

Corollary. *Suppose that the rapidly decreasing real valued potential $u(x)$ is of finite rank and $H(u)$ has the N bound states $-\beta_1^2, \cdots, -\beta_N^2$. Let $u_0(x)$ be the potential obtained by N times repeated applications of the Crum* *transformation such that $H(u_0)$ has no bound states. Then $u_0(x)$ is of finite rank.*

Then we can prove the following, which corresponds to the assertion (B).

Theorem 4. *If the rapidly decreasing real valued potential $u(x)$ is of finite rank then $u(x)$ is reflectionless.*

Proof. Let $u_0(x)$ be the potential obtained by N times applications of the Crum transformatin such that $H(u_0)$ has no bound states. Then, by Corollary, $u_0(x)$ is of finite rank. Hence there exist $\alpha_1, \cdots, \alpha_N \in \mathbf{R}$ such that

$$Z_{N+1}(u_0) + \sum_{j=1}^{N} \alpha_j Z_j(u_0) = 0.$$

Therefore, by (2), we have

$$i\pi^{-1} \int_{-\infty}^{\infty} Q(\xi) r_{\pm}(\xi; u_0) f_{\pm}(x, \xi; u_0)^2 d\xi = 0,$$

where

$$Q(\xi) = \xi^{2N+1} + \sum_{j=1}^{N} \alpha_j \xi^{2j-1}.$$

Since $u_0(x) \in S, r_{\pm}(\xi; u_0) \in S$ follows. Hence $Q(\xi) r_{\pm}(\xi; u_0) \in S$ also valid. Therefore, by Lemma 5.2, we have

$$Q(\xi) r_{\pm}(\xi; u_0) \equiv 0,$$

that is, $r_{\pm}(\xi; u_0) \equiv 0$ follows. By (1.7), it follows that $r_{\pm}(\xi; u)$ itself identically vanish. Q.E.D.

References

[1] K. Chadan and P. C. Sabatier, Inverse problems in quntum scattering theory, Springer, New York 1977.

[2] D. V. Chudnovsky, Riemann monodromy problem, isomonodromy deformation equations and completely integrable systems, appeareed in *Bifurcation phenomena in mathematical physics and related topics* (edited by C. Bardos and D. Bessis), 385–447, D. Reidel 1980.

[3] P. Deift, F. Lund and E. Trubowitz, Nonlinear wave equations and constrained harmonic motion, Comm. Math. Phys., **74** (1980) 141–188.

[4] P. Deift and E. Trubowitz, Inverse scattering on the line, Comm. Pure Appl. Math., **32** (1979) 121–251.

[5] B. A. Dubrovin, V. B. Matveev and S. P. Novikov, Non-linear equations of the Korteweg-de Vries type, finite zone linear operators, and Abelian varieties, Russian Math. Surveys, **31** (1976) 59–146.

[6] L. D. Faddeev, Properties of the S-matrix of the one-dimensional Schrödinger equation, AMS transl. **65** (1967) 139–166.

[7] C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura, Korteweg-de Vries equation and generalization VI. Methods for exact solution, Comm. Pure Appl. Math. **27** (1974) 97–133.

[8] B. M. Levitan, Inverse Sturm-Liouville problems, VNU science press, Utrecht 1987.

[9] J. Moser, Integrable Hamiltonian systems and spectral theory, Lezioni Fermiane, Pisa 1981.

[10] M. Ohmiya, On the Darboux transformation of the second order ordinary differential operator, Proc. Japan Acad., **64** ser A (1988) 327–330.

[11] M. Ohmiya, On the Darboux transformation of the second order differential operator of Fuchsian type on the Riemann sphere, Osaka J. Math., **25** (1988) 607–632.

[12] M. Ohmiya, On the Deift-Trubowitz trace formula for the 1-dimensional Schrödinger operator, Proc. Japan Acad., **65** ser A (1989) 203–206.

[13] M. Ohmiya, On the Deift-Trubowitz trace formula for the 1-dimensional Schrödinger operator with integrable potential, to appear in Funkcialaj Ekvacioj.

[14] S. Tanaka and E. Date, KdV equation, Kinokuniya, Tokyo 1979 (in Japanese).

[15] V. E. Zakharov and L. D. Faddeev, The Korteweg-de Vries equation : a completely integrable Hamiltonian system, Functional Anal. its Appl., 5 (1972) 280–287.