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# *Conformally Flat Finsler Structures*

By

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**Abstract**

In the present paper, we consider the conformal theory of Finsler manifolds. We find, under a certain condition, a conformally invariant Finsler connection and several conformally invariant tensors of a Finsler metric. Finally we come to show, in terms of the conformally invariant tensors, the necessary and sufficient condition for a Finsler manifold to be conformally flat.

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## Introduction

The present author has introduced, in his paper [1], the notions of a $(g, N)$-structure and its conformal changes where $g$ is a generalized Finsler metric and $N$ is a non-linear connection, and has found conformal curvature tensors and conformal torsion tensors of such a structure.

In the present paper, continuing the paper [1], we are mainly concerned with the $(g, N)$-structure where $g$ is a Finsler metric. First we establish the notions of flatness and conformal flatness of the structure. Next we find some conformally invariant Finsler tensors and a conformally invariant Finsler connection. Based on these results, we find the condition for the structure to be conformally flat in terms of these tensors. The main results are shown as Theorem 5.2 and Theorem 6.1. In the last section we also investigate the condition for a Finsler manifold to be a conformally flat Finsler manifold.

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## §1 $(L, N)$-structures

Let $M$ be an $n$-dimensional differentiable manifold and let us suppose that there are given on $M$ a Finsler metric $L(x, y)$ and a non-linear connection $N$

$=(N^i{}_j(x, y))$. We call such a struture an $(L, N)$-*structure*. Of course, the Finsler metric tensor is given by $g_{ij}(x, y)=\frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2(x, y)$.

Using the operator $X_i=\partial_i-N^m{}_i\dot{\partial}_m$, we put

$$F_j{}^i{}_k(x, y)=\frac{1}{2}g^{im}(X_j g_{mk}+X_k g_{mj}-X_m g_{jk}),\quad C_j{}^i{}_k=\frac{1}{2}g^{im}\dot{\partial}_m g_{jk}, \tag{1.1}$$

then the triplet $(F_j{}^i{}_k, N^i{}_j, C_j{}^i{}_k)$ defines a symmetric Finsler connection. We call such a connection an $(L, N)$-*connection* hereafter. We denote by $\nabla$ and $\dot{\nabla}$ respectively the $h$- and $v$-covariant derivatives with respect to this $(L, N)$-connection. According to Matsumoto [2], we write the $h$-torsion and $hv$-torsion of the $(L, N)$-connection as

$$R^i{}_{jk}=X_k N^i{}_j-X_j N^i{}_k,\quad P^i{}_{jk}=\dot{\partial}_k N^i{}_j-F_k{}^i{}_j, \tag{1.2}$$

and the three kinds of curvatures as

$$\begin{cases} R_h{}^i{}_{jk}=X_k F_h{}^i{}_j-X_j F_h{}^i{}_k+F_m{}^i{}_k F_h{}^m{}_j-F_m{}^i{}_j F_h{}^m{}_k+C_h{}^i{}_m R^m{}_{jk}, \\ P_h{}^i{}_{jk}=\dot{\partial}_k F_h{}^i{}_j-\nabla_j C_h{}^i{}_k+C_h{}^i{}_m P^m{}_{jk}, \\ S_h{}^i{}_{jk}=\dot{\partial}_k C_h{}^i{}_j-\dot{\partial}_j C_h{}^i{}_k+C_m{}^i{}_k C_h{}^m{}_j-C_m{}^i{}_j C_h{}^m{}_k. \end{cases} \tag{1.3}$$

Moreover we put

$$\begin{cases} K_h{}^i{}_{jk}=R_h{}^i{}_{jk}-C_h{}^i{}_m R^m{}_{jk}=X_k F_h{}^i{}_j-X_j F_h{}^i{}_k+F_m{}^i{}_k F_h{}^m{}_j-F_m{}^i{}_j F_h{}^m{}_k, \\ Q_h{}^i{}_{jk}=\nabla_j C_h{}^i{}_k-C_h{}^i{}_m P^m{}_{jk}=\dot{\partial}_k F_h{}^i{}_j-P_h{}^i{}_{jk}. \end{cases} \tag{1.4}$$

It is obvious that the relations

$$\nabla_k g_{ij}=0,\quad \dot{\nabla}_k g_{ij}=0 \tag{1.5}$$

hold. Applying the Ricci identities to (1.5), we have

$$R_{ijkh}=-R_{jikh},\ P_{ijkh}=-P_{jikh},\ S_{ijkh}=-S_{jikh}. \tag{1.6}$$

Due to the second Bianchi identity, we have

$$\nabla_j C_{khi}-\nabla_k C_{jhi}+C_{jhr}P^r{}_{ki}-C_{khr}P^r{}_{ji}-P_{jhki}+P_{khji}=0.$$

Now applying the so-called Christoffel process [2] with respect to $j$, $h$, $k$ to the above, we have

$$P_{hkji}=\nabla_h C_{kij}-\nabla_k C_{hij}+C_{hjr}P^r{}_{ki}-C_{kjr}P^r{}_{hi}.$$

Hence we obtain

$$P_{hkji} = Q_{kjhi} - Q_{jhki} \tag{1.7}$$

where we put $Q_{jhki} = g_{hm}Q_j{}^m{}_{ki}$. The relations (1.7) and $(1.4)_2$ lead us to

$$\dot{\partial}_k F_h{}^i{}_j = g^{im}(Q_{mjhk} - Q_{jhmk}) + Q_h{}^i{}_{jk}. \tag{1.8}$$

Moreover, by using the relation $Q_{hijk} = \nabla_j C_{khi} - C_{ihm}P^m{}_{jk}$ and (1.8), we can show easily

$$Q_{hijk} = Q_{ihjk}, \tag{1.9}$$

$$Q_h{}^k{}_{ji} = \frac{1}{2}(\dot{\partial}_i F_h{}^k{}_j + g^{km}g_{hr}\dot{\partial}_i F_m{}^r{}_j). \tag{1.10}$$

## §2 Flat $(L, N)$-structures

We begin with the definition of flatness of an $(L, N)$-structure.

**Definition.** **Let** $M$ be assumed to admit an $(L, N)$-structure. If, for any point $p$ of $M$, there exists a coordinate neighbourhood $(U, x^i)$ containing $p$ such that $X_k g_{ij} = 0$ holds in $U$, then the $(L, N)$-*structure* is said to be *flat*.

In connection with this definition we show first

**Theorem 2.1.** *An* $(L, N)$-*structure is flat if and only if the* $(L, N)$-*connection satisfies* $K_h{}^i{}_{jk} = 0$ and $Q_h{}^i{}_{jk} = 0$.

Proof. If an $(L, N)$-structure is flat, $F_j{}^i{}_k = 0$ holds in each coordinate neighbourhood $(U, x^i)$ which defines its flatness. Hence we have $K_h{}^i{}_{jk} = 0$. Moreover we have, in each $U$, $\partial_i g_{jk} - 2C_{jkm}N^m{}_i = 0$. Differentiating this with respect to $y^h$, we have

$$\partial_i C_{jkh} - \dot{\partial}_h C_{jkm}N^m{}_i - C_{jkm}\dot{\partial}_h N^m{}_i = 0,$$

which implies, in this case, $\nabla_i C_{jkh} - C_{jkm}P^m{}_{ih} = 0$. Namely, we have $Q_h{}^i{}_{jk} = 0$.

Conversely, assume that $K_h{}^i{}_{jk} = 0$ and $Q_h{}^i{}_{jk} = 0$ hold on $M$. Then, $F_j{}^i{}_k$ are functions of position alone because of (1.8). So, the condition $K_h{}^i{}_{jk} = 0$ implies that $M$ is covered by a system of local coordinate neighbourhoods $\{(U, x^i)\}$ such that $F_j{}^i{}_k = 0$ holds in each $(U, x^i)$. Hence $\nabla_k g_{ij} = 0$ means that $X_k g_{ij} = 0$ in each $U$, that is, the given $(L, N)$-structure is flat.

Moreover we show

**Theorem 2.2.** *An* $(L, N)$-*structure is flat if and only if the Finsler metric L is a locally Minkowski metric and the* $(L, N)$-*connection satisfies* $C_{ijm}P^m{}_{k0} = 0$, *where we put* $P^m{}_{k0} = P^m{}_{kr}y^r$.

Proof. If an $(L, N)$-structure is flat, $M$ is covered by a system of local coordinate neighbourhoods $\{(U, x^i)\}$ such that, in each $U$, $X_k g_{ij} = 0$ holds, that is, $\partial_k g_{ij} - 2C_{ijm}N^m{}_k = 0$ holds. On transvecting this with $y^i y^j$, we have $\partial_k L^2 = 0$, from which we find that $L$ is a locally Minkowski metric and $\partial_k g_{ij} = 0$ in each $U$. Thus we have $C_{ijm}N^m{}_k = 0$. On the other hand, we see that $F_j{}^i{}_k = 0$ in each $U$. Hence, from (1.2), we see that $N^m{}_k = P^m{}_{k0}$ in each $U$. Consequently we find that $C_{ijm}P^m{}_{k0} = 0$ holds on $M$.

Conversely, we assume that $L$ is a locally Minkowski metric and $C_{ijm}P^m{}_{k0} = 0$ holds. Then, $M$ is covered by a system of local coordinate neighbourhoods $\{(U, x^i)\}$ such that $\partial_k g_{ij} = 0$ holds in each $U$. In this case, $X_k g_{ij} = -2C_{ijr}N^r{}_k$ holds. And we have $F_j{}^i{}_k = -C_r{}^i{}_k N^r{}_j - C_r{}^i{}_j N^r{}_k + g^{im}C_{rjk}N^r{}_m$. On the other hand, the condition $C_{ijm}P^m{}_{k0} = 0$ is equivalent to $C_{ijr}N^r{}_k = C_{ijr}F_0{}^r{}_k$. Hence, we have

$$F_j{}^i{}_k = -C_r{}^i{}_k F_0{}^r{}_j - C_r{}^i{}_j F_0{}^r{}_k + g^{im}C_{rjk}F_0{}^r{}_m.$$

By transvecting this with $y^j$, we have $F_0{}^i{}_k = -C_r{}^i{}_k F_0{}^r{}_0$, from which we have $F_0{}^i{}_0 = 0$ and $F_0{}^i{}_k = 0$. Thus we have $C_{ijr}N^r{}_k = 0$. Hence, in each $(U, x^i)$ shown above, $X_k g_{ij} = 0$ holds. Namely, the given $(L, N)$-structure is flat.

**Theorem 2.3.** *If an $(L, N)$-structure is flat, the $(L, N)$-connection always satisfies $C_j{}^i{}_m R^m{}_{hk} = 0$.*

Proof. From our assumption, the manifold is covered by a system of local coordinate neighbourhoods $\{(U, x^i)\}$ such that, in each $U$, $\partial_h g_{ij} = 0$ and henceforth $\partial_h C_{ijk} = 0$ hold. In addition to this, from the proof of Theorem 2.2, we see that $C_{ijm}N^m{}_k = 0$ in each $U$. So, we have $C_{ijm}\partial_h N^m{}_k = 0$. Hence we see

$$\begin{aligned} C_{ijm}R^m{}_{hk} &= -C_{ijm}N^r{}_k \dot{\partial}_r N^m{}_h + C_{ijm}N^r{}_h \dot{\partial}_r N^m{}_k \\ &= N^r{}_k \dot{\partial}_r C_{ijm} N^m{}_h - N^r{}_h \dot{\partial}_r C_{ijm} N^m{}_k \\ &= 0. \end{aligned}$$

## §3 Berwaldian $(L, N)$-structures

Corresponding to Berwald spaces, we prepare the following:

**Definition.** With respect to an $(L, N)$-structure, if the connection coefficients $F_j{}^i{}_k$ of the $(L, N)$-connection depend on position alone, we call the structure a *Berwaldian $(L, N)$-structure.*

Now, paying attention to (1.8) and (1.10), we can conclude immediately

**Theorem 3.1.** *An $(L, N)$-structure is a Berwaldian $(L, N)$-structure if and only if $Q_h{}^i{}_{jk} = 0$.*

Moreover we can show

**Theorem 3.2.** *An* $(L, N)$*-structure is a Berwaldian* $(L, N)$*-structure if and only if the Finsler metric* $L$ *is a Berwald metric and the* $(L, N)$*-connection satisfies* $C_{ijm}P^m{}_{k0} = 0$.

Proof. Let us consider a Berwaldian $(L, N)$-structure. In this case, we have $\dot{\partial}_h F_j{}^i{}_k = 0$. Since $X_k g_{ir} y^r = \partial_k g_{ir} y^r$, we have $G^i (= \gamma_0{}^i{}_0) = F_0{}^i{}_0$. From these, we have $G_j{}^i{}_k \left( = \frac{1}{2} \dot{\partial}_j \dot{\partial}_k G^i \right) = F_j{}^i{}_k(x)$. Thus $L$ is a Berwald metric. Moreover we have $Q_h{}^i{}_{jk} = 0$, that is, $\nabla_k C_{ijr} = C_{ijm} P^m{}_{kr}$. Hence we see

$$C_{ijm} P^m{}_{k0} = \nabla_k C_{ijr} y^r = \nabla_k C_{rij} y^r = C_{rim} P^m{}_{kj} y^r = 0.$$

Conversely, we assume that $L$ is a Berwald metric and $C_{ijm}P^m{}_{k0} = 0$ holds. Since $P^i{}_{k0} = N^i{}_k - F_0{}^i{}_k$, we have $C_{ijm} N^m{}_k = C_{ijm} F_0{}^m{}_k$, from which $X_k g_{ij} = \partial_k g_{ij} - 2C_{ijm} F_0{}^m{}_k$. Thus we have

$$F_j{}^i{}_k = \gamma_j{}^i{}_k - g^{im}(C_{mkr} F_0{}^r{}_j + C_{jmr} F_0{}^r{}_k - C_{jkr} F_0{}^r{}_m),$$

from which we see $F_0{}^i{}_k = \gamma_0{}^i{}_k - C_k{}^i{}_r F_0{}^r{}_0$, $F_0{}^i{}_0 = \gamma_0{}^i{}_0$ and $F_0{}^i{}_k = \gamma_0{}^i{}_k - C_k{}^i{}_r \gamma_0{}^r{}_0$. The last relation leads us to $F_j{}^i{}_k = \Gamma^*_j{}^i{}_k$(Cartan). From our assumption, we have $\Gamma^*_j{}^i{}_k = \Gamma^*_j{}^i{}_k(x)$. Thus we have $F_j{}^i{}_k = F_j{}^i{}_k(x)$. Consequently, the proof is completed.

**Remark.** By means of the above proof, we can see, at the same time, that $F_j{}^i{}_k = \Gamma^*_j{}^i{}_k$(Cartan) if $C_{ijm} P^m{}_{k0} = 0$. In this case the $(L, N)$-connection $(F_j{}^i{}_k, N^i{}_j, C_j{}^i{}_k)$ coincides with $(\Gamma^*_j{}^i{}_k, G^i{}_j - D^i{}_j, C_j{}^i{}_k)$ where $D^i{}_j = -P^i{}_{j0}$ is the so-called deflection tensor satisfying $C_{ijm} D^m{}_k = 0$.

## §4 Conformal changes of an $(L, N)$-structure

Let $M$ be a manifold admitting an $(L, N)$-structure and let $\sigma(x)$ be a scalar field on $M$. Then $\tilde{L}(x, y) = e^{\sigma(x)} L(x, y)$ is also a Finsler metric. Here we consider the $(\tilde{L}, N)$-structure defined on $M$. The $(\tilde{L}, N)$-structure is called a *conformal change of the* $(L, N)$*-structure.*

Now, we know very well the conformal changes of a Riemann metric. So, in this section, we treat only the conformal changes of the $(L, N)$-structure where $L$ is a non-Riemannian Finsler metric. Paying attention to well-known Deicke's theorem, here, we assume more strictly that $C = \sqrt{C_m C^m} \neq 0$ where $C_i = C_i{}^m{}_m$. In the paper [1] we have already obtained that, as for the $(\tilde{L}, N)$-structure where $\tilde{L} = e^{\sigma(x)} L$, the following relations hold:

$$(4.1)\quad \begin{cases} \tilde{g}_{ij} = e^{2\sigma(x)}g_{ij},\ \tilde{g}^{ij} = e^{-2\sigma(x)}g^{ij},\ \tilde{R}^i{}_{jk} = R^i{}_{jk},\ \tilde{C}_j{}^i{}_k = C_j{}^i{}_k, \\ \tilde{F}_j{}^i{}_k = F_j{}^i{}_k + \sigma_j\delta^i_k + \sigma_k\delta^i_j - \sigma^i g_{jk}, \\ \tilde{P}^i{}_{jk} = P^i{}_{jk} - \sigma_j\delta^i_k - \sigma_k\delta^i_j + \sigma^i g_{jk}, \\ \tilde{K}_h{}^i{}_{jk} = K_h{}^i{}_{jk} + \delta^i_j\sigma_{kh} - \delta^i_k\sigma_{jh} - g_{hj}\sigma^i{}_k + g_{hk}\sigma^i{}_j \end{cases}$$

where we out $\sigma_i = \partial_i\sigma$, $\sigma^i = g^{im}\sigma_m$, $\sigma_{hk} = \nabla_k\sigma_h - \sigma_k\sigma_h + \frac{1}{2}\sigma_r\sigma^r g_{hk}$ and $\sigma^h{}_k = g^{hm}\sigma_{mk}$.

Now, it is seen from $(4.1)_6$ and $(4.1)_3$ that $\tilde{C}_m\tilde{P}^m{}_{k0} = C_m P^m{}_{k0} - \sigma_0 C_k + C_m\sigma^m y_k$ where put $\sigma_0 = \sigma_m y^m$. On the other hand, we get $\tilde{C}_k = C_k$, $\tilde{C}^k = e^{-2\sigma}C^k$, $\tilde{C}^2 = e^{-2\sigma}C^2$. Hence we have $\tilde{C}_m\tilde{P}^m{}_{r0}\tilde{C}^r = e^{-2\sigma}(C_m P^m{}_{r0}C^r - \sigma_0 C^2)$. Since $C^2 \neq 0$, we have $\sigma_0 = C_m P^m{}_{r0}C^r/C^2 - \tilde{C}_m\tilde{P}^m{}_{r0}\tilde{C}^r/\tilde{C}^2$.

Putting

$$(4.2)\qquad B = C_m P^m{}_{r0}C^r/C^2,\quad B_k = \dot{\partial}_k B,\quad B^k = g^{km}B_m,$$

we have

$$(4.3)\qquad \sigma_0 = B - \tilde{B},\quad \sigma_k(x) = B_k - \tilde{B}_k,$$

from which $\tilde{C}_j{}^i{}_m\tilde{P}^m{}_{k0} = C_j{}^i{}_m P^m{}_{k0} - (B - \tilde{B})C_j{}^i{}_k + C_j{}^i{}_m g^{mr}(B_r - \tilde{B}_r)y_k$. Hence, by putting

$$(4.4)\qquad Q^{*}_j{}^i{}_k = C_j{}^i{}_m P^m{}_{k0} - B\,C_j{}^i{}_k + C_j{}^i{}_m B^m y_k,$$

we obtain $\tilde{Q}^{*}_j{}^i{}_k = Q^{*}_j{}^i{}_k$. Namely, the tensor field $Q^{*}_j{}^i{}_k(x, y)$ is invariant under the conformal changes of the given $(L, N)$-structure.

Next, by means of $(4.1)_5$ and (4.3), we have

$$\dot{\partial}_h\tilde{F}_j{}^i{}_k = \dot{\partial}_h F_j{}^i{}_k + 2C^{im}{}_h(B_m - \tilde{B}_m)g_{jk} - 2g^{im}(B_m - \tilde{B}_m)C_{jkh}.$$

On the other hand, it is easily seen that $\tilde{C}^{im}{}_h\tilde{g}_{jk} = C^{im}{}_h g_{jk}$. So, if we put

$$(4.5)\qquad F^{*}_h{}^i{}_{jk} = \dot{\partial}_h F_j{}^i{}_k + 2C_h{}^i{}_m B^m g_{jk} - 2B^i C_{jkh},$$

then the tensor field $F^{*}_h{}^i{}_{jk}(x, y)$ is also invariant under the conformal changes of the given $(L, N)$-structure.

As for the tensor $Q_h{}^i{}_{jk}$, from $(4.1)_5$ and $(4.1)_6$, it follows that

$$\begin{aligned} \tilde{Q}_h{}^i{}_{jk} &= X_j C_h{}^i{}_k + \tilde{F}_m{}^i{}_j C_h{}^m{}_k - \tilde{F}_h{}^m{}_j C_m{}^i{}_k - \tilde{F}_k{}^m{}_j C_h{}^i{}_m - C_h{}^i{}_m\tilde{P}^m{}_{jk} \\ &= Q_h{}^i{}_{jk} + \sigma_m(\delta^i_j C_k{}^m{}_h - g^{im}C_{hjk} - \delta^m_h C_j{}^i{}_k + g_{hj}C^{im}{}_k). \end{aligned}$$

Using (4.3), we obtain that the tensor field $Q^{*}_h{}^i{}_{jk}(x, y)$ defined by

$$(4.6)\qquad Q^{*}_h{}^i{}_{jk} = Q_h{}^i{}_{jk} + B_m(\delta^i_j C_k{}^m{}_h - g^{im}C_{hjk} - \delta^m_h C_j{}^i{}_k + g_{hj}C^{im}{}_k)$$

is invariant under the conformal changes of the given $(L, N)$-structure.

Moreover, because of $\sigma_i = \sigma_i(x)$, (4.3) leads us to

$$\dot{\partial}_j \tilde{B}_k = \dot{\partial}_j B_k, \tag{4.7}$$

that is, the tensor field $\dot{\partial}_j B_k$ itself is invariant under the conformal changes of the given $(L, N)$-structure.

In addition to the above, we have

$$\tilde{\nabla}_j \tilde{B}_k = \nabla_j B_k - \nabla_j \sigma_k - \sigma_k B_j - \sigma_j B_k + \sigma_m B^m g_{jk} + 2\sigma_j \sigma_k - \sigma_m \sigma^m g_{jk},$$

from which we have

$$\nabla_j \sigma_k = \nabla_j B_k - \tilde{\nabla}_j \tilde{B}_k - \sigma_k B_j - \sigma_j B_k + \sigma_m B^m g_{jk} + 2\sigma_j \sigma_k - \sigma_m \sigma^m g_{jk}. \tag{4.8}$$

Since $\nabla_j \sigma_k = \nabla_k \sigma_j$, we have $\tilde{\nabla}_j \tilde{B}_k - \tilde{\nabla}_k \tilde{B}_j = \nabla_j B_k - \nabla_k B_j$. Namely, the tensor field defined by

$$\nabla_j B_k - \nabla_k B_j \tag{4.9}$$

is also invariant under the conformal changes of the given $(L, N)$-structure.

Finally, on account of (4.8) and (4.3), we have

$$\begin{aligned} \sigma_{kj} &= \nabla_j B_k - \tilde{\nabla}_j \tilde{B}_k - \sigma_k B_j - \sigma_j B_k + \sigma_m B^m g_{jk} + \sigma_j \sigma_k - \frac{1}{2}\sigma_m \sigma^m g_{jk} \\ &= \nabla_j B_k - \tilde{\nabla}_j \tilde{B}_k - B_j B_k + \tilde{B}_j \tilde{B}_k + \frac{1}{2} B_m B^m g_{jk} - \frac{1}{2} \tilde{B}_m \tilde{B}^m \tilde{g}_{jk}. \end{aligned}$$

Here we put

$$B_{kj} = \nabla_j B_k - B_j B_k + \frac{1}{2} B_m B^m g_{jk}, \tag{4.10}$$

then we have $\sigma_{kj} = B_{kj} - \tilde{B}_{kj}$, from which we have

$$\begin{aligned} \tilde{K}_h{}^i{}_{jk} = K_h{}^i{}_{jk} &+ \delta^i_j (B_{kh} - \tilde{B}_{kh}) - \delta^i_k (B_{jh} - \tilde{B}_{jh}) \\ &- g_{hj} g^{im} (B_{mk} - \tilde{B}_{mk}) + g_{hk} g^{im} (B_{mj} - \tilde{B}_{mj}). \end{aligned}$$

Thus the tensor field defined by

$$K^{*}_h{}^i{}_{jk} = K_h{}^i{}_{jk} + \delta^i_j B_{kh} - \delta^i_k B_{jh} - g_{hj} g^{im} B_{mk} + g_{hk} g^{im} B_{mj} \tag{4.11}$$

is also invariant under the conformal changes of the given $(L, N)$-structure. Summarizing up the above, we conclude

**Theorem 4.1.** *Let $L(x, y)$ be a Finsler metric satifying $C = \sqrt{C_m C^m} \neq 0$ and $N$ be a non-linear connection. With respect to the $(L, N)$-connection, let us put $B$*

$= C_m P^m{}_{r0} C^r / C^2$ *and* $B_k = \dot{\partial}_k B$. *Then the tensor fields* $Q^{*}_{h}{}^{i}{}_{k}$, $F^{*}_{h}{}^{i}{}_{jk}$, $Q^{*}_{h}{}^{i}{}_{jk}$, $K^{*}_{h}{}^{i}{}_{jk}$, *which are given respectively by* (4.4), (4.5), (4.6), (4.11), *and* $\dot{\partial}_j B_k$, $\nabla_j B_k - \nabla_k B_j$ *are all invariant under the conformal changes of the given* $(L, N)$*-structure.*

## §5 Conformally flat $(L, N)$-structures

The following classical definition is well-known: Let $L(x, y)$ be a Finsler metric defined on a manifold $M$. If, for any point $p$ of $M$, there exist a coordinate neighbourhood $(U, x^i)$ containing $p$ and a local scalar $\sigma(x)$ defined on $U$ such that $e^{\sigma(x)} L(x, y)$ is a locally Minkowski metric on $U$, $L$ is said to be a *conformally flat Finsler metric* and $M$ is said to be a *conformally flat Finsler manifold.* Now we set

**Definition.** Let $M$ be a manifold admitting an $(L, N)$-structure. If, for any point $p$ of $M$, there exist a coordinate neighbourhood $(U, x^i)$ containing $p$ and a local scalar $\sigma(x)$ defined on $U$ such that the $(e^{\sigma(x)} L, N)$-structure is flat on $U$, then the given $(L, N)$*-structure* is said to be *conformally flat.*

In connection with this notion, we show first

**Theorem 5.1.** *In order that an* $(L, N)$*-structure is conformally flat, it is necessary and sufficient that* $L(x, y)$ *is a conformally flat Finsler metric and* $C_j{}^i{}_m \tilde{P}^m{}_{k0} = 0$ *holds good. Here* $\tilde{P}^i{}_{jk}$ *is the hv-torsion of the locally defined* $(\tilde{L}, N)$*-connection where* $\tilde{L}(x, y) = e^{\sigma(x)} L(x, y)$ *is the locally Minkowski metric.*

Proof, The condition for an $(L, N)$-structure to be conformally flat is that, for any pont $p$ of $M$, there are a local coordinate neighbourhood $(U, x^i)$ containing $p$ and a local scalar $\sigma(x)$ defined on each $U$ such that the $(e^{\sigma(x)} L, N)$-structure is flat. The result of Theorem 2.2 shows us that the condition under consideration is that $e^{\sigma(x)} L$ is a locally Minkowski metric and, at the same time, $\tilde{C}_j{}^i{}_m \tilde{P}^m{}_{k0} = 0$ holds for the $(e^{\sigma(x)} L, N)$-connection. Since $\tilde{C}_j{}^i{}_k = C_j{}^i{}_k$, the condition becomes to that $L$ is a conformally flat Finsler metric and $C_j{}^i{}_m \tilde{P}^m{}_{k0} = 0$ holds. Accordingly the proof is completed.

Now we will rewrite the above condition in tensor forms. However, we have full knowledge of conformally flat Riemann metrics. So, as same as the case of §4, we assume here that the Finsler metric $L$ satisfies $C \neq 0$. Now, using the tensor fields which are defined in Theorem 4.1, we shall show

**Theorem 5.2.** *Let* $L(x, y)$ *be a Finsler metric satisfying* $C \neq 0$ *and* $N$ *be a non-linear connection. In order that the* $(L, N)$*-structure is conformally flat, it is necessary and sufficient that the* $(L, N)$*-connection satisfies*

$$\dot{\partial}_j B_k = 0, \quad \nabla_j B_k - \nabla_k B_j = 0, \quad K^{*}_{h}{}^{i}{}_{jk} = 0, \quad Q^{*}_{h}{}^{i}{}_{jk} = 0.$$

Proof. We suppose that the given $(L, N)$-structure is conformally flat. Then, Theorem 5.1 and Theorem 2.1 give us $\tilde{C}_j{}^i{}_m \tilde{P}^m{}_{k0} = 0$, $\tilde{K}_h{}^i{}_{jk} = 0$, $\tilde{Q}_h{}^i{}_{jk} = 0$. Hence we have $\tilde{B} = 0$, from which $\tilde{B}_j = 0$. Moreover, these results yield $\dot{\partial}_j \tilde{B}_k = 0$, $\tilde{\nabla}_j \tilde{B}_k - \tilde{\nabla}_k \tilde{B}_j = 0$, $\tilde{K}^{*}{}_h{}^i{}_{jk} = 0$, $\tilde{Q}^{*}{}_h{}^i{}_{jk} = 0$. Accordingly, it follows from Theorem 4.1 that $\dot{\partial}_j B_k = 0$, $\nabla_j B_k - \nabla_k B_j = 0$, $K^{*}{}_h{}^i{}_{jk} = 0$, $Q^{*}{}_h{}^i{}_{jk} = 0$.

Conversely, we assume that the above conditions are all satisfied. Here we put $\sigma_k = B_k$. Then the condition $\dot{\partial}_j B_k = 0$ yields $\sigma_k = \sigma_k(x)$, and the condition $\nabla_j B_k - \nabla_k B_j = 0$ yields $\partial_k \sigma_j = \partial_j \sigma_k$. Hence, for any point $p$ of $M$, there is a local coordinate neighbourhood $(U, x^i)$ containing $p$ such that $U$ admits a local scalar $\sigma(x)$ satisfying $\partial_k \sigma = \sigma_k = B_k$. Now we consider in each $U$ such a conformal change of the $(L, N)$-structure as $(L, N) \to (\tilde{L}, N)$ where $\tilde{L} = e^{\sigma(x)} L$. Then (4.3) leads us to $\tilde{B}_k = 0$. From this and Theorem 4.1, the condition $K^{*}{}_h{}^i{}_{jk} = 0$ yields $\tilde{K}_h{}^i{}_{jk} = 0$ and the condition $Q^{*}{}_h{}^i{}_{jk} = 0$ yields $\tilde{Q}_h{}^i{}_{jk} = 0$. Hence, from Theorem 2.1, it follows that the $(e^{\sigma(x)} L, N)$-structure is flat. Consequently the proof is completed.

**Remark.** According to Theorem 3.1, we may replace the condition $Q^{*}{}_h{}^i{}_{jk} = 0$ in this Theorem 5.2 by $F^{*}{}_h{}^i{}_{jk} = 0$.

**Corollary.** *Let L be a Finsler metric satisfying $C \neq 0$ and G be its Cartan's non-linear connection. The $(L, G)$-structure is conformally flat if and only if L is a locally Minkowski metric.*

Proof. With respect to the $(L, G)$-connecton, we have $F_j{}^i{}_k = \Gamma^{*}{}_j{}^i{}_k$(Cartan). So, $P^i{}_{jk}$ always satisfies $P^i{}_{j0} = 0$. Then we have $B = 0$. Hence, from Theorem 5.2, Theorem 2.1 and Theorem 2.2, it follows that the $(L, G)$-structure is conformally flat if and only if $K_h{}^i{}_{jk} = 0$ and $Q_h{}^i{}_{jk} = 0$, namely, $L$ is a locally Minkowski metric.

## §6 A conformally invariant Finsler connection

Continued from the preceding section, we are concerned with a Finsler metric satisfying $C \neq 0$.

By substituting $(4.3)_2$ into $(4.1)_5$, as same as Matsumoto [3], we can see that the quantity ${}^cF_j{}^i{}_k$ which is defined by

$$(6.1) \qquad {}^cF_j{}^i{}_k = F_j{}^i{}_k + B_j \delta^i_k + B_k \delta^i_j - g^{im} B_m g_{jk}$$

satisfies ${}^c\tilde{F}_j{}^i{}_k = {}^cF_j{}^i{}_k$, that is, ${}^cF_j{}^i{}_k$ is an invariant $h$-connection under the conformal changes of the given $(L, N)$-structure. Thus the triplet $({}^cF_j{}^i{}_k, N^i{}_j, C_j{}^i{}_k)$ defines a Finsler connection which is invariant under the conformal changes of the given $(L, N)$-structure. We call this connection the *conformal Finsler N-connection.* We denote hereafter by ${}^c\nabla$ and ${}^c\dot{\nabla}$ respectively the $h$- and $v$-covariant derivatives with repect to the conformal Finsler $N$-connection. Of course, ${}^c\dot{\nabla} = \dot{\nabla}$

holds good, and also

$$ {}^c\nabla_k g_{ij} = -2B_k g_{ij} \tag{6.2} $$

holds good. Moreover we denote by ${}^cQ_h{}^i{}_{jk}$, ${}^cK_h{}^i{}_{jk}$, ${}^cR_h{}^i{}_{jk}$ and ${}^cP_h{}^i{}_{jk}$ respectively the tensor fields $Q_h{}^i{}_{jk}$, $K_h{}^i{}_{jk}$, $R_h{}^i{}_{jk}$ and $P_h{}^i{}_{jk}$ with respect to the conformal Finsler $N$-connection. It is easy to verify

$$ {}^cQ_h{}^i{}_{jk} = Q_h^*{}^i{}_{jk}, \tag{6.3} $$

$$ {}^cK_h{}^i{}_{jk} = K_h^*{}^i{}_{jk} + \delta_h^i(\nabla_k B_j - \nabla_j B_k) + \delta_j^i(\nabla_k B_h - \nabla_h B_k) - \delta_k^i(\nabla_j B_h - \nabla_h B_j). \tag{6.4} $$

Also direct calculation immediately leads us to

$$ {}^c\nabla_h B_k - {}^c\nabla_k B_h = \nabla_h B_k - \nabla_k B_h, \tag{6.5} $$

$$ ({}^c\nabla_h{}^c\nabla_k - {}^c\nabla_k{}^c\nabla_h)g_{ij} = -2(\nabla_h B_k - \nabla_k B_h)g_{ij}. \tag{6.6} $$

Moreover, since ${}^c\dot{\nabla}_k g_{ij} = 0$, we have

$$ ({}^c\dot{\nabla}_h{}^c\nabla_k - {}^c\nabla_k{}^c\dot{\nabla}_h)g_{ij} = -2(\dot{\partial}_h B_k - C_k{}^m{}_h B_m)g_{ij}. \tag{6.7} $$

On the other hand, on account of Ricci identity for the general Finsler connection [2], it is seen that, in our case,

$$ ({}^c\nabla_h{}^c\nabla_k - {}^c\nabla_k{}^c\nabla_h)g_{ij} = -{}^cR_{ijkh} - {}^cR_{jikh}, $$

$$ ({}^c\dot{\nabla}_h{}^c\nabla_k - {}^c\nabla_k{}^c\dot{\nabla}_h)g_{ij} = -{}^cP_{ijkh} - {}^cP_{jikh} - {}^c\nabla_m g_{ij} C_k{}^m{}_h. $$

From (6.6) and (6.7), these equations can be rewritten in the forms

$$ 2(\nabla_h B_k - \nabla_k B_h)g_{ij} = {}^cR_{ijkh} + {}^cR_{jikh}, \tag{6.8} $$

$$ 2\dot{\partial}_h B_k g_{ij} = {}^cP_{ijkh} + {}^cP_{jikh}. \tag{6.9} $$

Now we show

**Theorem 6.1.** *Let $L(x, y)$ be a Finsler metric satisfying* $C \neq 0$ and $N$ *be a non-linear connection. In order that the $(L, N)$-structure is conformally flat, it is necessary and sufficient that the conformal Finsler N-connection satisfies*

$$ \begin{cases} C_j{}^i{}_m R^m{}_{hk} = 0, \quad {}^cQ_h{}^i{}_{jk} = 0, \quad {}^cK_h{}^i{}_{jk} = 0, \\ \qquad {}^cP_{ijkh} + {}^cP_{jikh} = 0. \end{cases} \tag{6.10} $$

Proof. If the given $(L, N)$-structure is conformally flat, then Theorem 5.2 gives us

$$ \dot{\partial}_j B_k = 0, \quad \nabla_j B_k - \nabla_k B_j = 0, \quad K_h^*{}^i{}_{jk} = 0, \quad Q_h^*{}^i{}_{jk} = 0. $$

Then, by means of (6.3), (6.4) and (6.9), we have

$$^cQ_h{}^i{}_{jk} = 0, \quad ^cK_h{}^i{}_{jk} = 0, \quad ^cP_{ijkh} + {}^cP_{jikh} = 0.$$

Moreover, due to Theorem 2.3 and (4.1), we have $C_j{}^i{}_m R^m{}_{hk} = 0$.

Conversely, let us assume that the condition (6.10) is satisfied. First, the condition $^cP_{ijkh} + {}^cP_{jikh} = 0$ and (6.9) yield $\dot{\partial}_h B_k = 0$. The conditions $C_j{}^i{}_m R^m{}_{hk} = 0$ and $^cK_h{}^i{}_{jk} = 0$ yield $^cR_h{}^i{}_{jk} = 0$. Then (6.8) gives us $\nabla_h B_k - \nabla_k B_h = 0$. And, therefore, (6.4) leads us to $K_h^*{}^i{}_{jk} = 0$. Moreover, the condition $^cQ_h{}^i{}_{jk} = 0$ and (6.3) also lead us to $Q_h^*{}^i{}_{jk} = 0$. Consequently the proof is completed.

## §7 Conformally flat Finsler manifolds

In this section we consider the case where a Finsler metric $L(x, y)$ is given but a non-linear connection $N$ is not assigned previously.

Concerning the given Finsler metric $L(x, y)$ and its conformal change $\tilde{L}(x, y) = e^{\sigma(x)} L(x, y)$, the respective Cartan's Finsler connections $(\Gamma_j^*{}^i{}_k, G^i{}_j, C_j{}^i{}_k)$ and $(\tilde{\Gamma}_j^*{}^i{}_k, \tilde{G}^i{}_j, \tilde{C}_j{}^i{}_k)$ have the relations

$$(7.1) \quad \begin{cases} \tilde{\Gamma}_j^*{}^i{}_k = \Gamma_j^*{}^i{}_k - \sigma_0 C_j{}^i{}_k + \sigma^m (C_m{}^i{}_k y_j + C_m{}^i{}_j y_k - C_{mjk} y^i) \\ \qquad - L^2 \sigma^m (C_r{}^i{}_k C_m{}^r{}_j + C_r{}^i{}_j C_m{}^r{}_k - C_m{}^i{}_r C_j{}^r{}_k) \\ \qquad + \sigma_j \delta^i_k + \sigma_k \delta^i_j - \sigma^i g_{jk}, \\ \tilde{G}^i{}_j = G^i{}_j + L^2 \sigma^m C_m{}^i{}_j + \sigma_j y^i + \sigma_0 \delta^i_j - \sigma^i y_j, \\ \tilde{C}_j{}^i{}_k = C_j{}^i{}_k. \end{cases}$$

By virtue of $(7.1)_2$, we have

$$(7.2) \qquad \tilde{G}^i = G^i + 2\sigma_0 y^i - L^2 \sigma^i.$$

Of course, here, we put $G^i = G^i{}_0$ and $\tilde{G}^i = \tilde{G}^i{}_0$. Therefore, we have $\tilde{y}_m \tilde{G}^m = e^{2\sigma}(y_m G^m + L^2 \sigma_0)$, from which we have

$$\sigma_0 = \tilde{y}_m \tilde{G}^m / \tilde{L}^2 - y_m G^m / L^2.$$

By putting

$$(7.3) \qquad \alpha = y_m G^m / L^2, \quad \alpha_k = \dot{\partial}_k \alpha, \quad \alpha^k = g^{km} \alpha_m,$$

we have

$$(7.4) \qquad \sigma_0 = \tilde{\alpha} - \alpha, \quad \sigma_k = \tilde{\alpha}_k - \alpha_k.$$

Substituting these into (7.2), we have

$$\tilde{G}^i - 2\tilde{\alpha} y^i + \tilde{L}^2 \tilde{\alpha}^i = G^i - 2\alpha y^i + L^2 \alpha^i.$$

That is to say, if put

(7.5) $$\beta^i = G^i - 2\alpha y^i + L^2\alpha^i,$$

we obtain

(7.6) $$\tilde{\beta}^i = \beta^i.$$

Using this conformally invariant quantity $\beta^i$, we can show

**Theorem 7.1.** *In order that a Finsler manifold $(M, L(x, y))$ is a conformally flat Finsler manifold, it is necessary and sufficient that, for any point $p$ of $M$, there exists a local coordinate neighbourhood $(U, x^i)$ containing $p$ such that the relations*

(7.7) $$\beta^i = 0, \quad \dot{\partial}_j\alpha_k = 0, \quad \partial_j\alpha_k = \partial_k\alpha_j$$

*hold in each $U$.*

Proof. Assume that $(M, L(x, y))$ is a conformally flat Finsler manifold. Then $M$ is covered by a system of local coordinate neighbourhoods $\{(U, x^i)\}$ such that a local scalar $\sigma(x)$ exists in each $U$ satisfying $e^{2\sigma(x)}g_{ij}(x, y) = \tilde{g}_{ij}(y)$. By means of this $\sigma(x)$, we define a conformal change $L \to \tilde{L} = e^{\sigma(x)}L$ in each $U$. Of course, the Finsler metric tensor of $\tilde{L}$ is $\tilde{g}_{ij}(y)$ itself. Now, in each $(U, x^i)$, we have $\tilde{G}^i = 0$, from which we have $\tilde{\alpha} = 0$, $\tilde{\alpha}_k = 0$ and $\tilde{\beta}^i = 0$. Hence (7.6) says that $\beta^i = 0$ holds in each $(U, x^i)$. Moreover (7.4) leads us, in each $U$, to

$$\dot{\partial}_j\alpha_k = -\dot{\partial}_j\sigma_k = 0, \quad \partial_j\alpha_k = -\partial_j\sigma_k = -\partial_k\sigma_j = \partial_k\alpha_j.$$

Conversely, let us assume that $M$ is covered by a system of local coordinate neighbourhoods satisfying (7.7). Here we put $\sigma_k = -\alpha_k$, then (7.7) gives us that, for any point $p$ of $M$, there are a local coordinate neighbourhood $(U, x^i)$ containing $p$ and a local scalar $\sigma(x)$ defined on $U$ such that $\partial_k\sigma = \sigma_k$ and (7.7) hold true. By using this $\sigma(x)$, we consider the conformal change $L \to \tilde{L} = e^{\sigma(x)}L$ defined on each $U$. In this case, from $(7.4)_2$, we have $\tilde{\alpha}_k = 0$ and $\tilde{\alpha} = y^m\tilde{\alpha}_m = 0$. Hence we have, in each $(U, x^i)$, $\tilde{G}^i = \tilde{\beta}^i = \beta^i = 0$. Namely, $\tilde{L}$ is a locally Minkowski metric on $U$. Therefore, $L$ is a conformally flat Finsler metric on $M$. Accordingly the proof is completed.

Now, it is easily seen that

$$y_m G^m = \frac{1}{2}y_m g^{mr}(\partial_t g_{rs} + \partial_s g_{rt} - \partial_r g_{st})y^s y^t = \frac{1}{2}(\partial_r g_{st})y^r y^s y^t.$$

So, we have $\alpha = y^r\partial_r L^2/2L^2$, namely,

(7.8) $$\alpha = y^r\partial_r \log L.$$

It is to be noted that, from the form of (7.8), $\alpha$ is obviously not a scalar field, and also (7.7) is not written in tensor forms.

If an $(L, N)$-structure is conformally flat, then, due to Theorem 5.1, $L$ must be a conformally flat Finsler metric. In addition to the above, if $L$ satisfies $C \neq 0$, the proof of Theorem 5.2 tells us that $\sigma_i = B_i$. So, in this case, $\sigma_i$ must be a vector field defined globally on $M$. Now, we show finally

**Theorem 7.2.** *Let $M$ be a manifold endowed with a conformally flat Finsler metric $L(x, y)$, that is, $M$ is endowed with a Finsler metric $L(x, y)$ and is covered by a system of local coorinate neighbourhoods each of which admits a local scalar $\sigma(x)$ such that $e^{\sigma(x)} L(x, y)$ is a locally Minkowski metric. If $\sigma_i = \partial_i \sigma(x)$ is a vector field defined globally on $M$, then the non-linear connection $N$ which is determined by*

$$N^i{}_j = G^i{}_j + L^2 C_j{}^i{}_m \sigma^m + \sigma_0 \delta^i_j - \sigma^i y_j \tag{7.9}$$

*composes, together with the Finsler metric $L$, a conformally flat $(L, N)$-structure.*

Proof, From the assumption that $\sigma_i(x)$ is a globally defined vector field on $M$, (7.9) gives $M$ a non-linear connection $N$. Then the connection coefficient $F_j{}^i{}_k$ of the $(L, N)$-connection is written in the form

$$\begin{aligned} F_j{}^i{}_k = \Gamma^*_j{}^i{}_k - \sigma_0 C_j{}^i{}_k + \sigma^m (C_m{}^i{}_k y_j + C_m{}^i{}_j y_k - C_{mjk} y^i) \\ - L^2 \sigma^m (C_r{}^i{}_k C_m{}^r{}_j + C_r{}^i{}_j C_m{}^r{}_k - C_m{}^i{}_r C_j{}^r{}_k). \end{aligned} \tag{7.10}$$

In this case we have $P^i{}_{k0} = \sigma_0 \delta^i_k - \sigma^i y_k$. On the other hand, $\tilde{L} = e^{\sigma(x)} L$ is a locally Minkowski metric. And, from $(4.1)_6$, it follows that the $(\tilde{L}, N)$-structure satisfies

$$C_j{}^i{}_m \tilde{P}^m{}_{k0} = C_j{}^i{}_m (P^m{}_{k0} - \sigma_k y^m - \sigma_0 \delta^m_k + \sigma^m y_k) = 0.$$

Consequently, Theorem 5.1 shows us that the $(L, N)$-structure is, surely, conformally flat.

## References


[1] Y. Ichijyō, Conformal curvature tensors of a generalized Finsler metric, TENSOR (N.S.), **47** (1988), 172–179.

[2] M. Matsumoto, Foundations of Finsler geometry and special Finsler spaces, Kaiseisha Press, Saikawa, Ohtsu-shi 520, Japan, (1986).

[3] M. Matsumoto, On *h*-recurrent Finsler connections and conformally Minkowski spaces, TENSOR (N.S.), **49** (1990), 18–23.