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A Note on Conjectures of the Ideal of s-generic Points in P^4

By

Hiroshi OKUYAMA
Department of Mathematics and Computor Sciences,
Faculty of Integral Arts and Sciences,
The University of Tokushima
Tokushima 770, *JAPAN*
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Abstract

Let $A = k[X_0, \dots, X_n]/I$ be the homogeneous coordinate ring of s points in generic position in P^n. In this note, we formulate natural conjectures for the number of generators of I, and for the Cohen-Macaulay type of A in terms of the minimal free resolution of I in case $n = 4$. We then consider the relationship between these conjectures from a stand point of the minimal resolution conjecture made by E. Ballico and A. V. Geramita.

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Introduction

Let $X = \{P_1, \dots, P_s\}$ be a set of s distinct points in a projective n-space $P^n(k)$, k an algebraically closed field, and let I be the defining ideal of X in the polynomial ring $R = k[X_0, \dots, X_n] = \oplus_{i \geqq 0} R_i$.

We denote by A the homogeneous coordinate ring of X, $A = R/I = \oplus_{i \geqq 0} A_i$. We say, following Geramita and Orecchia [4], that the points $P_1, \dots, P_s$ are in generic position (or are in generic s-position) if the Hilbert function $H_A(t) := \dim_k A_t$ satisfies

$$H_A(t) = \min\left\{s, \binom{n+t}{n}\right\} \qquad \text{for all} \quad t \geqq 0.$$

It is well known that almost every set of s points in $P^n(k)$ (k infinite) is in generic position (see [4]) and that A is an one dimensional Cohen-Macaulay graded k-algebra whose Cohen-Macaulay type $r(A)$ is defined as the k-dimension of the socle of an artinian reduction of A.

In a series of papers [3], [4], [5], [8], the defining ideal I of X has been extensively studied and conjectures made about the minimal number of generators of I and the Cohen-Macaulay type $r(A)$.

These conjectures were verified for P^2 in [3] and in [2] for many infinite families of s in P^n when $n > 2$. Further, in [1], viewing these conjectures from a point of graded free resolution of the R-module I, it was proved that three conjectures (ideal generation conjecture, Cohen-Macaulay type conjecture and minimal resolution conjecture) are valid for $n = 3$.

In this note we shall deal with the case $n = 4$ and consider the relations between these conjectures.

1. The ideal generation, Cohen-Macaulay type and minimal resolution conjectures

We begin by reviewing these conjectures briefly.

In order to fix the notation, we assume that $P_1, \dots, P_s$ are points in generic s-position in $P^n(k)$. Then, there is an integer d such that $\binom{d-1+n}{n} \leqq s < \binom{d+n}{n}$, and it follows from [3], [7] that the ideal $I = \oplus_{t \geqq 0} I_t$ of s points $P_1, \dots, P_s$ satisfies

i) $I = I_d \oplus I_{d+1} \oplus \cdots$,

ii) $\dim_k I_d = \binom{d+n}{n} - s$,

iii) I always generated by forms of degree d and $d+1$: $I = \langle I_d, I_{d+1} \rangle$,

iv) the graded minimal free resolution of I has the form

$$(*)\quad 0 \longrightarrow R(-(d+n-1))^{\alpha_{n-1}} \oplus R(-(d+n))^{\beta_{n-1}} \xrightarrow{d_{n-1}} \cdots \longrightarrow R(-(d+i))^{\alpha_i} \oplus$$

$$R(-(d+i+1))^{\beta_i} \xrightarrow{d_i} \cdots \xrightarrow{d_1} R(-(d))^{\alpha_0} \oplus R(-(d+1))^{\beta_0} \longrightarrow I \longrightarrow 0.$$

Ideal generation conjecture

From iii) above, the minimal number of generators of the ideal I (denoted $v(I)$) is determined completely on knowing how much of I_{d+1} can be obtained from I_d. Thus we have

$$v(I) = \dim_k I_d + \dim_k I_{d+1} - \dim_k R_1 I_d.$$

The ideal generation conjecture of [5] is that the elements of $R_1 I_d$ are 'as independent as possible' in I_{d+1}; explicitely

$$\dim_k R_1 I_d = \min\{(n+1)\dim_k I_d, \dim_k I_{d+1}\}.$$

This gives, as a conjectured generic value for $\nu(I)$, that:

$$\nu(I) = \max\{\dim_k I_{d+1} - n \dim_k I_d, \dim_k I_d\}.$$

Since the Hilbert function for points in generic s-position in P^n determines the dimensions of the graded pieces of the ideal of points, this formula can be described completely in terms of n and s, i.e.

$$\nu(I) = \max\left\{\binom{d+1+n}{n} - n\binom{d+n}{n} + s(n-1), \binom{d+n}{n} - s\right\}.$$

On the other hand, from the formula (*) of graded minimal resolution of I, we have $\alpha_0 = \dim_k I_d$ and $\beta_0 = \dim_k I_{d+1} - \dim_k R_1 I_d$. So the ideal generation conjecture is the one about the value of β_0:

$$\beta_0 = \max\{0, \dim_k I_{d+1} - (n+1)\dim_k I_d\}.$$

Cohen-Macaulay type conjecture

Let $A = R/I$ be, as above, the coordinate ring of s points in P^n and let $L \in R_1$ be a non zero divisor on A. If $B = \oplus_{i \geq 0} B_i$ denote the graded artinian ring $B := A/LA$, then the socle of B is the k-vector space $(I, L)\colon R_1/(I, L)$ which we denote by $s(B)$. Notice that, if k is an infinite field, we may take X_0 as L_1 without loss of generality.

Since $H_B(t) = 0$ for all $t > d$ (see, for instance, [2] (1.3)), the Cohen-Macaulay type of A is given by $r(A) = \dim_k s(B)_d + \dim_k s(B)_{d-1}$. Obviously, $s(B)_d = B_d$, therefore we need only to compute $\dim_k s(B)_{d-1}$.

Since $s(B)_{d-1}$ is the kernel of the linear transformation

$$\varphi\colon B_{d-1} \longrightarrow \mathrm{Hom}_k(B_1, B_d)$$

which is induced by the multiplication of B, it is clear that

$$\dim_k s(B)_{d-1} \geqq \dim_k B_{d-1} - (\dim_k B_1 \cdot \dim_k B_d).$$

The Cohen-Macaulay type conjecture made by L. G. Roberts in [8] is that for a general set of points in generic position in P^n, we have

$$\dim_k s(B)_{d-1} = \max\{0, \dim_k B_{d-1} - n \dim_k B_d\}.$$

This gives, as a conjectured generic value for $r(A)$, that:

$$r(A) = \max\{\dim_k B_d, \dim_k B_{d-1} + (1-n)\dim_k B_d\}$$

or phrased in terms of n and s only (see [2] (1.6)):

$$r(A) = \max\left\{s - \binom{d-1+n}{n}, (1-n)s + n\binom{d-1+n}{n} - \binom{d-2+n}{n}\right\}.$$

It is well known that the another characterization for the Cohen-Macaulay type of A is given by the $(n-1)$-th Betti number of I:

$$r(A) = \dim_k \mathrm{Tor}_{n-1}^R(I, k).$$

On the other hand, from the graded minimal free resolution $(*)$ of I, we get $\dim_k \mathrm{Tor}_{n-1}^R(I, k) = \alpha_{n-1} + \beta_{n-1}$ and $\beta_{n-1} = \dim_k(\mathrm{Tor}_{n-1}^R(I, k))_{d+n} = \dim_k s(B)_d$ $(= \dim_k B_d)$ by using the graded isomorphism (i.e. an isomorphism preserving degrees) $\mathrm{Tor}_{n-1}^R(I, k) \equiv s(B)(-n)$ (see [6, p. 280]). Hence, we have $\alpha_{n-1} = \dim_k s(B)_{d-1}$.

Therefore, the Cohen-Macaulay type conjecture is the conjecture about the value of α_{n-1}:

$$\alpha_{n-1} = \max\{0, \dim_k B_{d-1} - n \dim_k B_d\}$$

Minimal resolution conjecture

Again we consider the graded minimal free resolution $(*)$ of the ideal I of s points in generic position in P^n and let N_i the i-th syzygy module of I: $N_i = \mathrm{Im}\, d_i$ for $i = 1, \dots, n$ and $N_0 = I$. Then N_i has the following properties:

i) $(N_i)_t = 0$ for $t < d + i$,
ii) $N_i = \langle (N_i)_{d+i}, (N_i)_{d+i+1} \rangle$,
iii) $\alpha_i = \dim_k (N_i)_{d+i}$ and $\beta_i = \dim_k (N_i)_{d+i+1} - \dim_k R_1(N_i)_{d+i}$.

The minimal resolution conjecture of [1] is that, for a general set of s points in generic position, $\dim_k R_1(N_i)_{d+i}$ is as large as possible; explicitely

$$\dim_k R_1(N_i)_{d+i} = \min\{(n+1)\dim_k(N_i)_{d+i}, \dim_k(N_i)_{d+i+1}\}$$

for $i = 0, 1, \dots, n-1$.

Notice that, the ideal generation conjecture is the special case of the minimal resolution conjecture and by iii) above, the minimal resolution conjecture is the one about the values of β_i:

$$\beta_i = \max\{0, \dim_k(N_i)_{d+i+1} - (n+1)\dim_k(N_i)_{d+i}\} \qquad \text{for } i = 0, 1, \dots n-1.$$

2. The conjecture for $n = 4$.

We shall deal with the case $n = 4$. Let $P_1 \dots, P_s$ are points in generic position in P^4 and let d be the least degree of a form in R vanishing at $P_1, \dots, P_s$, then $\binom{d+3}{4} \leqq s < \binom{d+4}{4}$ and that the graded minimal free resolution of I has the form

$$(\dagger)\quad 0 \longrightarrow R(-(d+3))^{\alpha_3} \oplus R(-(d+4))^{\beta_3} \xrightarrow{d_3} R(-(d+2))^{\alpha_2} \oplus R(-(d+3))^{\beta_2}$$

$$\xrightarrow{d_2} R(-(d+1))^{\alpha_1} \oplus R(-(d+2))^{\beta_1} \xrightarrow{d_1} R(-d)^{\alpha_0} \oplus R(-(d+1))^{\beta_0} \longrightarrow I \longrightarrow 0.$$

We denote as above the syzygy modules of I by $N_i (i = 1, 2, 3)$, and $N_0 = I$. As we have noted, α_0 and β_3 are determined by the Hilbert function alone; $\alpha_0 = \dim_k I_d = \binom{d+4}{4} - s$, $\beta_3 = \dim_k B_d = s - \binom{d+3}{4}$.

In the following, we calculate the values α_i for $i \geqq 1$.

Lemma 1. *With the notation above, we have*

(i) $\alpha_1 = \dim_k (N_1)_{d+1} = \dfrac{(d+4)(d+3)(d+2)d}{6} - 4s + \beta_0$.

(ii) $\alpha_2 = \dim_k (N_2)_{d+2} = \dfrac{(d+4)(d+3)(d+1)d}{4} - 6s + \beta_1$.

(iii) $\alpha_3 = \dim_k (N_3)_{d+3} = \dfrac{(d+4)(d+2)(d+1)d}{6} - 4s + \beta_2$.

Proof. Consider the graded short exact sequence obtained from $(\dagger)$

$$0 \longrightarrow N_1 \longrightarrow R(-d)^{\alpha_0} \oplus R(-(d+1))^{\beta_0} \longrightarrow I \longrightarrow 0.$$

Then, from the additivity of the Hilbert function, we get

$$\alpha_1 = \dim_k (N_1)_{d+1} = \dim_k (R(-d)^{\alpha_0})_{d+1} + \dim_k (R(-(d+1)^{\beta_0})_{d+1} - \dim_k I_{d+1}$$

$$= 5\alpha_0 + \beta_0 - \binom{d+5}{4} + s.$$

After a simple, direct calculation we get the desired form.

Similary by considering the graded short exact sequence started from N_2 (resp. N_3)

$$0 \longrightarrow N_2 \longrightarrow R(-(d+1))^{\alpha_1} \oplus R(-(d+2))^{\beta_1}$$
$$\longrightarrow R(-d)^{\alpha_0} \oplus R(-(d+1))^{\beta_0} \longrightarrow I \longrightarrow 0,$$

$$0 \longrightarrow N_3 \longrightarrow R(-(d+2))^{\alpha_2} \oplus R(-(d+3))^{\beta_2}$$
$$\longrightarrow R(-(d+1))^{\alpha_1} \oplus R(-(d+2))^{\beta_1}$$
$$\longrightarrow R(-d)^{\alpha_0} \oplus R(-(d+1))^{\beta_0} \longrightarrow I \longrightarrow 0$$

and using the values α_1 (resp. α_2) we get the desired formula for α_2 (resp. α_3).

The conjectures on the ideal of points in generic position mentioned in the preceding section can be write down in terms of d and s.

Lemma 2. *Let I be the ideal of points in generic s-position in P^4 and (†) a graded minimal free resolution of I. Then*

(i) *The Cohen-Macaulay type conjecture is equivalent to the following condition*

$$\alpha_3 = \max\left\{0, \frac{(d+4)(d+2)(d+1)d}{6} - 4s\right\}.$$

(ii) *The minimal resolution conjecture is equivalent to the following conditions*:

(a) $$\beta_0 = \max\left\{0, 4s - \frac{(d+4)(d+3)(d+2)d}{6}\right\},$$

(b) $$\beta_1 = \max\left\{0, 6s - \frac{(d+4)(d+3)(d+1)d}{4}\right\},$$

(c) $$\beta_2 = \max\left\{0, 4s - \frac{(d+4)(d+2)(d+1)d}{6}\right\}, \quad \textit{and}$$

(d) $$\beta_3 = \max\left\{0, s - \binom{d+3}{4}\right\}.$$

Proof of (i). As we have discussed in the preceding section, the Cohen-Macaulay type conjecture is the conjecture about the value of α_3 and is equivalent to the condition

$$\alpha_3 = \max\{0, \dim_k B_{d-1} - 4\dim_k B_d\}$$

where $B = \bigoplus_{i=0}^{d} B_i$ is the graded artinian ring $B := A/X_0A = R/(I, X_0)$.

Now, the Hilbrt function of B is given by

$$H_B(t) = \dim_k B_t = \begin{cases} \binom{t+3}{3} & \text{for} \quad t < d, \\ s - \binom{d+3}{4} & \text{for} \quad t = d, \\ 0 & \text{for} \quad t > d, \end{cases}$$

and hence we get the desired form for α_3.

Proof of (ii). Similary, the minimal resolution conjecture is the conjecture about the values β_i and is equivalent to the following conditions

$$\beta_i = \max\{0, \dim_k(N_i)_{d+i+1} - 5\dim_k(N_i)_{d+i}\} \qquad \text{for} \quad i = 0, 1, 2, 3.$$

(a) In case $i = 0$, we have $N_0 = I$ and $\dim_k I_{d+1} = \binom{d+5}{4} - s$.

Therefore $\dim_k I_{d+1} - 5\dim_k I_d = 4s - \frac{(d+4)(d+3)(d+2)d}{6}$.

(b) In case $i = 1$, from the short exact sequence considered in the proof of Lemma 1, we have

$$\dim_k(N_1)_{d+2} = 15\alpha_0 + 5\beta_0 - \binom{d+6}{4} + s = \frac{(7d+17)(d+4)(d+3)d}{12} + 5\beta_0 - 14s.$$

Therefore, we get $\dim_k(N_1)_{d+2} - 5\dim_k(N_1)_{d+1} = 6s - \frac{(d+4)(d+3)(d+1)d}{4}$ by Lemma 1 (i).

(c) In case $i = 2$, by considering the graded short exact sequence as in the proof of Lemma 1, we obtain

$$\begin{aligned}\dim_k(N_2)_{d+3} &= 15\alpha_1 + 5\beta_1 - 35\alpha_0 - 15\beta_0 + \binom{d+7}{4} - s \\ &= \frac{(13d^2+54d+41)(d+4)d}{12} - 26s + 5\beta_1.\end{aligned}$$

The formula for the second term of the bracket in β_2 now follows after applying (ii) of Lemma 1.

(d) In case $i = 3$, the similar argumentation shows that

$$\begin{aligned}\dim_k(N_3)_{d+4} &= 15\alpha_2 + 5\beta_2 - 35\alpha_1 - 15\beta_1 + 70\alpha_0 + 35\beta_0 - \binom{d+8}{4} + s \\ &= \frac{(19d^2+96d+77)(d+2)d}{24} - 19s + 5\beta_2.\end{aligned}$$

By applying (iii) of Lemma 1, we obtain

$$\dim_k(N_3)_{d+4} - 5\dim_k(N_3)_{d+3} = s - \binom{d+3}{4}.$$

This complets the proof.

Remark 1. The ideal generation conjecture is described as a special case of the minimal resolution conjecture by the following condition:

$$\beta_0 = \max\left\{0, 4s - \frac{(d+4)(d+3)(d+2)d}{6}\right\}.$$

Remark 2. The last condition (d) in Lemma 2 is satisfied always whenever $s \geqq \binom{d+3}{4}$, since it holds $\beta_3 = s - \binom{d+3}{4}$.

3. The relationship between conjectures.

In this section we study the relations between these conjectures.

In particular, we shall show that if the minimal resolution conjecture is true for points in P^4, then the Cohen-Macaulay type conjecture is also true. But conversely, the Cohen-Macaulay type conjecture does not imply, in general, the ideal generation conjecture nor the minimal resolution conjecture.

Proposition. *Let I be the ideal of s points in generic position in P^4 and let* (†) (*as given in* p. 5) *be its graded minimal resolution. Then*

(a) *The minimal resolution conjecture implies the Cohen-Macaulay type conjecture.*

(b) *Conversely, suppose the Cohen-Macaulay type conjecture is true, then the minimal resolution conjecture is also true in the following cases*:

(1) $\binom{d+3}{4} \leqq s \leqq \left[\frac{(d+4)(d+2)(d+1)d}{24}\right]$

(*where* $[n]$ *denotes the greatest integer* $\leqq n$).

(2) $\left[\frac{(d+4)(d+2)(d+1)d}{24}\right] < s \leqq \left[\frac{(d+4)(d+3)(d+1)d}{24}\right]$ *and* $\beta_1 = 0$.

(3) $\left[\frac{(d+4)(d+3)(d+1)d}{24}\right] < s \leqq \left[\frac{(d+4)(d+3)(d+2)d}{24}\right]$ *and* $\beta_0 = \alpha_2 = 0$.

(4) $\left[\frac{(d+4)(d+3)(d+2)d}{24}\right] < s < \binom{d+4}{4}$ *and* $\alpha_1 = 0$.

Proof of (a). We distinguish four cases.

Case 1. $\binom{d+3}{4} \leqq s \leqq \left[\frac{(d+4)(d+2)(d+1)d}{24}\right]$.

In this case $4s - \frac{(d+4)(d+2)(d+1)d}{6} \leqq 0$ and so the minimal resolution conjecture would say that $\beta_2 = 0$ by Lemma 2. Hence we have

$\alpha_3 = \frac{(d+4)(d+2)(d+1)d}{6} - 4s$ in view of Lemma 1, which implies that Cohen-Macaulay type conjecture is true.

Case 2. $$\left[\frac{(d+4)(d+2)(d+1)d}{24}\right] < s \leqq \left[\frac{(d+4)(d+3)(d+1)d}{24}\right].$$

In this case, since $4s - \frac{(d+4)(d+2)(d+1)d}{6} > 0$, it holds $\beta_2 \neq 0$ by Lemma 2. Hence $\dim_k R_1(N_2)_{d+2} \neq \dim_k(N_2)_{d+3}$. From the minimal resolution conjecture we have $\dim_k R_1(N_2)_{d+2} = \min\{\dim_k(N_2)_{d+3}, 5\dim_k(N_2)_{d+2}\}$.

Therefore

$$\dim_k R_1(N_2)_{d+2} = 5\dim_k(N_2)_{d+2} = 5\alpha_2.$$

By the result of Lorenzini [7, Proposition 2.1] we have $\alpha_3 = 0$.

This is what the C-M type conjecture predicts and so the minimal resolution conjecture implies the C-M type conjecture in this case.

Case 3. $$\left[\frac{(d+4)(d+3)(d+1)d}{24}\right] < s \leqq \left[\frac{(d+4)(d+3)(d+2)d}{24}\right].$$

In this case, since $6s - \frac{(d+4)(d+3)(d+1)d}{4} > 0$ it holds $\beta_1 \neq 0$ by Lemma 2. Hence $\dim_k R_1(N_1)_{d+1} \neq \dim_k(N_1)_{d+2}$. From the minimal resolution conjecture we have $\dim_k R_1(N_1)_{d+1} = 5\dim_k(N_1)_{d+1} = 5\alpha_1$ which implies $\alpha_2 = \alpha_3 = 0$ [7, Proposition 2.1]. The C-M type conjecture then follows immediately.

Case 4. $$\left[\frac{(d+4)(d+3)(d+2)d}{24}\right] < s < \binom{d+4}{4}.$$

In this case $4s - \frac{(d+4)(d+3)(d+2)d}{6} > 0$. Hence the minimal resolution conjecture implies $\beta_0 \neq 0$ and $\dim_k R_1 I_d = 5\dim_k I_d = 5\alpha_0$.

By the result of Lorenzini, we have $\alpha_1 = \alpha_2 = \alpha_3 = 0$, which yields

$$\alpha_3 = \max\left\{0, \frac{(d+4)(d+2)(d+1)d}{6} - 4s\right\}.$$

Thus the minimal resolution conjecture implies the Cohen-Macaulay type conjecture in all cases.

Proof of (b). Suppose the Cohen-Macaulay type conjecture is correct.

In case (1), from the C-M type conjecture, we have

$$\alpha_3 = \frac{(d+4)(d+2)(d+1)d}{6} - 4s$$

which yields $\beta_2 = 0$ by virtue of Lemma 1. This implies $\beta_1 = 0$ and $\beta_0 = 0$ by the result of Lorenzini [6, Proposition 3.1].

The minimal resolution conjecture now follows by applying (ii) of Lemma 2 and its remark 2.

In case (2), from the C-M type conjecture, we have $\alpha_3 = 0$ since $\frac{(d+4)(d+2)(d+1)d}{6} - 4s < 0$. Therefore it holds by Lemma 1 that $\beta_2 = 4s - \frac{(d+4)(d+2)(d+1)d}{6}$ which is a positive integer in this case. This means that $\beta_2 = \max\left\{0, 4s - \frac{(d+4)(d+2)(d+1)d}{6}\right\}$ and since $\beta_3 = s - \binom{d+3}{4}$, the last two conditions (c), (d) of Lemma 2 (ii) are satisfied.

Consequently, if $\beta_1 = 0$ (therefore $\beta_0 = 0$) we get the minimal resolution conjecture.

In case (3), the last two conditions (c), (d) of Lemma 2 (ii) are satisfied in the same way as in case (2). Since $4s - \frac{(d+4)(d+3)(d+2)d}{6} \leqq 0$, the first condition (a) of Lemma 2(ii) (i.e. the ideal generation conjecture) is satisfied if and only if $\beta_0 = 0$. On the other hand, since $6s - \frac{(d+4)(d+3)(d+1)}{4} > 0$, the second part (b) of Lemma 2(ii) is satisfied if and only if $\beta_1 = 6s - \frac{(d+4)(d+3)(d+1)d}{4}$. From (ii) of Lemma 1, this is equivalent to $\alpha_2 = 0$. Therefor if $\beta_0 = 0$ and $\alpha_2 = 0$ then the minimal resolution conjecture holds.

In case (4), by the similar argument as in case (2), the last two conditions (c), (d) of Lemma 2(ii) are easily verified. Since $4s - \frac{(d+4)(d+3)(d+2)d}{6} > 0$ in this case, the first condition (a) of Lemma 2(ii) is satisfied if and only if $\beta_0 = 4s - \frac{(d+4)(d+3)(d+2)d}{6}$. From (i) of Lemma 1, this is equivalent to $\alpha_1 = 0$. On the other hand, by [6, Proposition 3.1], $\alpha_1 = 0$ induces $\alpha_2 = 0$ which yields $\beta_1 = 6s - \frac{(d+4)(d+3)(d+1)d}{4}$ according to (ii) of Lemma 1.

Therefore the second part (b) of Lemma 2 (ii) is exactly satisfied if $\alpha_1 = 0$. Consequently, the s points in P^4 which satisfy both C-M type conjecture and ideal generation conjecture also satisfies the minimal resolution conjecture in this case.

We have just completed the proof of Proposition.

Remark. In [2], Geramita et al. investigated the condition when the C-M type conjecture or the ideal generation conjecture can be proved by the lifting of monomials in P^n. It follows from [2, Theorem 4.7], using the method of graph theory, that

(a) The ideal generation conjecture for s points in P^4 can be proved, by the lifting of monomial ideals, for all $s \leqq 70$ except for $s = 10, 24, 25, 26, 27, 28, 29, 53, 54, 55, 56, 57, 58$.

(b) The C-M type conjecture for s points in P^4 can be proved, by the lifting of monomials, for all $s \leqq 70$ except for $s = 6, 16, 17, 18, 19, 37, 38, 39, 40, 41, 42$.

Therefore, for example, s points in P^4 such as $s = 5, 15, 35, 36, 70$ which are in generic position verifies all these three conjectures.

Note added in proof.

The Cohen-Macaulay type conjecture has been solved for any n by N. V. Trung and G. Valla, J. Algebra. 125 (1989), 110–119.

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