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Theory of (Vector Valued) Fourier Hyperfunctions. Their Realization as Boundary Values of (Vector Valued) Slowly Increasing Holomorphic Functions, (III)

By

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Introduction

This paper is the third part of this series of papers, which includes Chapter 9. For the outline of this paper, see "Contents" in the first part of this series of papers [37]. Here we note that "isomorphisms" usually mean topological ones without explicit mention for the contrary. For References we refer to the lists of references at the end of papers [37], [38] and this one.

Here I wish to express my hearty thanks to Professors H. Komatsu, A. Kaneko and M. Morimoto and Drs. S. Nagamachi and H. Kaneta for many valuable advices and discussions during the preparartion of this work.

Chapter 9. Cases of sheaves $\mathcal{O}^{\natural}$, $\mathcal{A}^{\natural}$, $\mathcal{O}_{\flat}$ and $\mathcal{A}_{\flat}$

9.1. The Oka-Cartan-Kawai Theorem B

In this section we will prove the Oka-Cartan-Kawai Theorem B for the sheaves $\mathcal{O}^{\natural}$ and $\mathcal{O}_{\flat}$.

For a pair $n = (n_1, n_2)$ of nonnegative integers with $|n| = n_1 + n_2 \neq 0$, we denote by $\boldsymbol{H}^n$ the product space $\boldsymbol{C}^{n_1} \times \boldsymbol{E}^{n_2}$ and by $\tilde{\boldsymbol{D}}^n$ the product space $\boldsymbol{R}^{n_1} \times \boldsymbol{D}^{n_2}$ and by $\boldsymbol{C}^{|n|}$ the space $\boldsymbol{C}^{n_1+n_2} = \boldsymbol{C}^{n_1} \times \boldsymbol{C}^{n_2}$. We denote $z = (z', z'') \in \boldsymbol{C}^{|n|}$ so that $z' = (z_1, \dots, z_{n_1})$ and $z'' = (z_{n_1+1}, \dots, z_{|n|})$.

Definition 9.1.1 (The sheaf $\mathcal{O}^{\natural}$ of germs of partially slowly increasing holomorphic functions). *We define $\mathcal{O}^{\natural}$ to be the sheafification of the presheaf $\{\mathcal{O}^{\natural}(\Omega); \Omega \subset \boldsymbol{H}^n$ open$\}$, where the section module $\mathcal{O}^{\natural}(\Omega)$ on an open set Ω in $\boldsymbol{H}^n$ is the space of all holomorphic functions $f(z)$ on $\Omega \cap \boldsymbol{C}^{|n|}$ such that, for any positie number ε and for any compact set K in Ω, the estimate $\sup\{|f(z)|e(-\varepsilon|z|); z \in K \cap \boldsymbol{C}^{|n|}\} < \infty$ holds.*

Definition 9.1.2 (The sheaf $\mathcal{O}_{\flat}$ of germs of partially rapidly decreasing holomorphic functions). *We define $\mathcal{O}_{\flat}$ to be the sheafification of the presheaf*

$\{\mathcal{O}_\natural(\Omega); \Omega \subset \boldsymbol{H}^n$ open$\}$, where the section module $\mathcal{O}_\natural(\Omega)$ on an open set Ω in $\boldsymbol{H}^n$ is the space of all holomorphic functions $f(z)$ on $\Omega \cap \boldsymbol{C}^{|n|}$ such that, for any compact set K in Ω, there exists some positive constant δ so that the estimate $\sup\{|f(z)|e(\delta|z|); z \in K \cap \boldsymbol{C}^{|n|}\} < \infty$ holds.

Difinition 9.1.3. *An open set V in $\boldsymbol{H}^n$ is said to be an $\mathcal{O}^\natural$-pseudoconvex open set if it satisfies the conditions:*

(1) $\sup\{|\mathrm{Im}\, z''| - |\mathrm{Re}\, z''| : z = (z', z'') \in V \cap \boldsymbol{C}^{|n|}\} < \infty$.

(2) *There exists a C^∞-plurisubharmonic function $\varphi(z)$ on $V \cap \boldsymbol{C}^{|n|}$ having the following two properties:*

(i) *The closure of $V_t = \{z \in V \cap \boldsymbol{C}^{|n|};\ \varphi(z) < t\}$ in $\boldsymbol{H}^n$ is a compact subset of V for any real t.*

(ii) *$\varphi(z)$ is bounded on $L \cap \boldsymbol{C}^{|n|}$ for any compact subset L of V.*

Then we can prove the Oka-Cartan-Kawai Theorem B by a similar way to that in section 1.1.

Theorem 9.1.4(The Oka-Cartan-Kawai Theorem B). *For every $\mathcal{O}^\natural$-pseudoconvex open set V in $\boldsymbol{H}^n$, we have $H^s(V, \mathcal{O}^{\natural,p}) = 0$, ($p \geqq 0$ and $s \geqq 1$).*

Proof. Since V is paracompact, $H^s(V, \mathcal{O}^{\natural,p})$ coincides with the Čech cohomology group. So we have only to prove $\varinjlim_{\mathfrak{U}} H^s(\mathfrak{U}, \mathcal{O}^{\natural,p}) = 0$, where $\mathfrak{U} = \{V_j\}_{j \geqq 1}$ is a locally finite open covering of V so that $U_j = V_j \cap \boldsymbol{C}^{|n|}$ is pseudoconvex.

Now we definie $C^s(Z^{\mathrm{loc},\natural}_{(p,q)}(\{U_j\}))$ to be the set of all cochains $c = \{c_J; J = (j_0, j_1, \dots, j_s) \in \boldsymbol{N}^{s+1}\}$ of forms of type (p, q) satisfying the three conditions:

(i) The coefficients of c_J are locally square integrable functions on $U_J = U_{j_0} \cap U_{j_1} \cap \cdots \cap U_{j_s}$.

(ii) $\bar{\partial} c_J = 0$ in U_J.

(iii) For every positive ε and every finite subset M of $\boldsymbol{N}^{s+1}$, the estimate

$$\sum_{J \in M} \int_{U_J} |c_J|^2 e(-\varepsilon \|z\|) d\lambda < \infty$$

holds, where $d\lambda$ is the Lebesgue measure on $C^{|n|}$ and $\|z\|$ denote the modification of $\sum_{j=1}^{|n|} |z_j|$ so as to become C^∞ and convex.

Here we need the following

Lemma 9.1.5. *If $c \in C^s(Z^{\mathrm{loc},\natural}_{(p,q)}(\{U_j\}))$ satisfies the condition $\delta c = 0$, then we can find some $c' \in C^{s-1}(Z^{\mathrm{loc},\natural}_{(p,q)}(\{U_j\}))$ such that $\delta c' = c$. Here δ means the coboundary operator.*

Postponing the proof of Lemma 9.1.5 at the end of the proof of this theorem, the conclusion will follow from Lemma 9.1.5 as the special case where $q=0$ because we can use Cauchy's integral formula to change the L_2-norm to the sup-norm for holomorphic functions. This completes the proof of the theorem.

Q.E.D.

Now we will prove Lemma 9.1.5.

PROOF OF LEMMA 9.1.5. We denote by $\{\chi_j\}$ the partition of unity subordinate to $\{U_j\}$ such that $\sup_j(\sup\{|\bar{\partial}\chi_j|; z\in U_j\})<\infty$ holds, and define $b_I=\sum_j \chi_j c_{jI}$ for $I\in \mathbf{N}^s$. Since $\delta c=0$, we have $\delta b=c$. So $\delta\bar{\partial}b=0$ because $\bar{\partial}c=0$. Since $\sum\chi_j=1$ and $\chi_j\geqq 0$, we have

$$\int_{U_I}|b_I|^2 e(-\varepsilon\|z\|)d\lambda \leqq \sum_j\int_{U_I}\chi_j|c_{jI}|^2 e(-\varepsilon||z||)d\lambda$$

for any positive number ε by virtue of Cauchy-Schwarz' inequality.

By the assumption of the existence of a C^∞-plurisubharmonic function $\varphi(z)$ in Definition 9.1.3, we can find some plurisubharmonic function $\psi(z)$ on $U=V\cap \boldsymbol{C}^{|n|}$ which satisfies the following three conditions:

(1) $\sum|\bar{\partial}\chi_j|\leqq e(\psi(z))$.

(2) The closure of $V_t=\{z\in V\cap\boldsymbol{C}^{|n|}; \psi(z)<t\}$ in $\boldsymbol{H}^n$ is a compact subset of V for every real t.

(3) $\sup\{\psi(z); z\in K\cap \boldsymbol{C}^{|n|}\}\leqq C_K$ for every $K\Subset V$.

Thus it follows from the condition on c that

$$\sum_{I\in N}\int_{U_I}|\bar{\partial}b_I|^2 e(-\varepsilon\|z\|-\psi(z))d\lambda<\infty$$

for every positive number ε and every finite subset N of $\mathbf{N}^s$.

Now we consider the case $s=1$. By the fact that $\delta(\bar{\partial}b)=0$, $\bar{\partial}b$ defines a global section f on $U=V\cap\boldsymbol{C}^{|n|}$. Then, by Hörmander[4], Theorem 4.4.2, p. 94, we can prove the existence of u such that $\bar{\partial}u=f$ and the estimate

$$\int_{K\cap C^{|n|}}|u|^2 e(-\varepsilon\|z\|)(1+|z|^2)^{-2}d\lambda<\infty$$

holds for every positive number ε and every $K\Subset V$.

If we define $c'_I=b_I-u|U_I$, then $\bar{\partial}c'_I=0$ and $\delta c'=\bar{\partial}b=c$. Clearly $c'\in C^{s-1}(Z^{\mathrm{loc},\natural}_{(p,q)}(\{U_j\}))$.

Now we go on to the case $s>1$. In this case we use the induction on s. By the induction hypotheses there exists $b'\in C^{s-2}(Z^{\mathrm{loc},\natural}_{(p,q+1)}(\{U_j\}))$ such that $\delta b'=\bar{\partial}b$. By virtue of Hörmander [4], Theorem 4.4.2, p.94, we can also find $b''=\{b''_H\}_{H\in\mathbf{N}^{s-1}}$ such that $b'_H=\bar{\partial}b''_H$ and the estimate

$$\sum_{H\in L}\int_{U_H} |b''_H|^2 e(-\varepsilon\|z\|-\psi(z))(1+|z|^2)^{-2}d\lambda<\infty$$

holds for every positive number ε and every finite subset L of N^{s-1}. Therefore $c' = b-\delta b''$ satisfies all the required conditions. Q.E.D.

Now we will prove the Malgrange theorem for the sheaf $\mathscr{A}^\natural$ of germs of partially slowly increasing real analytic functions. Here we define the sheaf $\mathscr{A}^\natural$ to be the restriction of $\mathcal{O}^\natural$ to $\tilde{\boldsymbol{D}}^n$: $\mathscr{A}^\natural=\mathcal{O}^\natural|\tilde{\boldsymbol{D}}^n$. Then we have the following

Theorem 9.1.6 (Malgrange). *For an arbitrary set Ω in $\tilde{\boldsymbol{D}}^n$, we have $H^s(\Omega, \mathscr{A}^{\natural,p})=0$, ($p\geqq 0$ and $s\geqq 1$).*

Proof. We know, by virtue of Ito[11], Theorem 2.1.13, that Ω has a fundamental system $\{\tilde{\Omega}\}$ of $\mathcal{O}^\natural$-pseudoconvex open neighborhoods. Then, it follows from the Oka-Cartan-Kawai Theorem B and Schapira [34], Theorem B42, that, for $p\geqq 0$ and $s\geqq 1$, we have

$$H^s(\Omega, \mathscr{A}^{\natural,p})=\operatorname*{ind\,lim}_{\tilde{\Omega}\cap\tilde{D}^n=\Omega} H^s(\tilde{\Omega}, \mathcal{O}^{\natural,p})=0. \qquad \text{Q.E.D.}$$

Next we will prove the Oka-Cartan-Kawai Theorem B for the sheaf $\mathcal{O}_\natural$. This can be proved by a similar way to Theorem 9.1.4. Thus we have the following

Theorem 9.1.7 (The Oka-Cartan-Kawai Theorem B). *For some δ, $(0<\delta<1)$ and $A>0$, put $X=\operatorname{int}\{z=(z',z'')\in \boldsymbol{C}^{|n|}; |\operatorname{Im} z''|^2\leqq\delta^2|\operatorname{Re} z''|^2+A^2\}^a$ in $\boldsymbol{H}^n$. For every $\mathcal{O}^\natural$-pseudoconvex open set V in X, we have $H^s(V, \mathcal{O}^p_\natural)=0$ for $p\geqq 0$ and $s\geqq 1$.*

Proof. Since V is paracompact, $H^s(V, \mathcal{O}^p_\natural)$ coincides with the Čech cohomology group. So we have only to prove $\operatorname{ind\,lim}_{\mathfrak{U}} H^s(\mathfrak{U}, \mathcal{O}^p_\natural)=0$, where $\mathfrak{U}=\{V_j\}_{j\geqq 1}$ is a locally finite open covering of V so that $U_j=V_j\cap \boldsymbol{C}^{|n|}$ is pseudoconvex.

Here we use the notations in the proof of Theorem 9.1.4.

For any cocycle $d=\{d_J\}$ representing an element in $H^s(\mathfrak{U}, \mathcal{O}^p_\natural)$, we can define an element $c=\{c_J\}$ in $C^s(Z^{\mathrm{loc},\natural}_{(p,0)}(\{U_j\}))$ such as $\delta c=0$ by putting $c_J=d_J\cdot h_\varepsilon(z)$ with $h_\varepsilon(z)=\cosh(\varepsilon\sqrt{(z'')^2/2})$ for some positive ε, where δ denotes the coboundary operator. Then we can find some $c'\in C^{s-1}(Z^{\mathrm{loc},\natural}_{(p,0)}(\{U_j\}))$, such that $\delta c'=c$. If we put $d'_I=c'_I\cdot(h_\varepsilon(z))^{-1}$, then $d'=\{d'_I\}$ is a cochain with values in $\mathcal{O}_\natural$ such that $\delta d'=d$. Thus the element in $H^s(\mathfrak{U}, \mathcal{O}^p_\natural)$ represented by d is zero. Since a class $[d]$ with a representative d is an arbitrary element in $H^s(\mathfrak{U}, \mathcal{O}^p_\natural)$, we have $H^s(\mathfrak{U}, \mathcal{O}^p_\natural)=0$. This completes the proof. Q.E.D.

At last we will prove the Malgrange theorem for the sheaf $\mathscr{A}_\natural$ of germs of partially rapidly decreasing real analytic functions. Here we define the sheaf $\mathscr{A}_\natural$ to be the restriction of $\mathcal{O}_\natural$ to $\tilde{\boldsymbol{D}}^n$: $\mathscr{A}_\natural=\mathcal{O}_\natural|\tilde{\boldsymbol{D}}^n$. Then we have the following

Theorem 9.1.8 (Malgrange). *For an arbitrary set Ω in $\tilde{\boldsymbol{D}}^n$, we have* $H^s(\Omega, \mathscr{A}^p_\natural) = 0$ for $p \geqq 0$ and $s \geqq 1$.

PROOF. We can prove this by a way similar to that of Theorem 9.1.6.
Q.E.D.

9.2. The Dolbeault-Grothendieck resolution $\mathscr{O}^\natural$ and $\mathscr{O}_\natural$

In this section we will construct soft resolutions of $\mathscr{O}^\natural$ and $\mathscr{O}_\natural$ and prove some of their consequences.

At first we will define the sheaf $L^\natural = L^\natural_{2,\mathrm{loc}}$ of germs of partially slowly increasing locally L_2-functions over $\boldsymbol{H}^n$.

Definition 9.2.1. *We define the sheaf $L^\natural$ to be the sheafification of the presheaf* $\{L^\natural(\Omega); \Omega \subset \boldsymbol{H}^n$ open$\}$, *where, for an open set Ω in $\boldsymbol{H}^n$, the section module $L^\natural(\Omega)$ is the space of all $f \in L_{2,\mathrm{loc}}(\Omega \cap \boldsymbol{C}^{|n|})$ such as, for any $\varepsilon > 0$ and any relatively compact open subset ω of Ω, $e(-\varepsilon\|z\|)f(z)|\omega$ belongs to $L_2(\omega \cap \boldsymbol{C}^{|n|})$. Here $(e(-\varepsilon\|z\|)f(z))|\omega$ denotes the restriction of $e(-\varepsilon\|z\|)f(z)$ to ω and $\|z\|$ denotes the modification of $\sum_{j=1}^{|n|}|z_j|$ so as to become C^∞ and convex.*

Then it is easy to see that $L^\natural$ is a soft FS*-sheaf. Then we give

Definition 9.2.2. *We define the sheaf $\mathscr{L}^{\natural,p,q} = \mathscr{L}^{\natural,p,q}_{2,\mathrm{loc}}$ to be the sheafification of the presheaf $\{\mathscr{L}^{\natural,p,q}(\Omega); \Omega \subset \boldsymbol{H}^n$ open$\}$, where, for an open set Ω in $\boldsymbol{H}^n$, the section module $\mathscr{L}^{\natural,p,q}(\Omega)$ is the space of all $f \in L^{\natural,p,q}(\Omega)$ such that $\bar\partial f \in L^{\natural,p,q+1}(\Omega)$. We put $\mathscr{L}^\natural = \mathscr{L}^{\natural,0,0}$.*

Then $\mathscr{L}^{\natural,p,q}$ is a soft FS*-sheaf with respct to the graph topology. Then we have the following

Theorem 9.2.3 (The Dolbeault-Grothendieck resolution). *For some $d > 0$, we put* $X = \mathrm{int}\{z \in \boldsymbol{C}^{|n|}; |\mathrm{Im}\ z''| - |\mathrm{Re}\ z''| < d\}^a$, *where* $\mathrm{int}\{\ \}^a$ *denotes the interior of the closure in $\boldsymbol{H}^n$ of a set $\{\ \}$. Then the sequence of sheaves over X*

$$0 \longrightarrow \mathscr{O}^{\natural,p}|X \longrightarrow \mathscr{L}^{\natural,p,0}|X \xrightarrow{\bar\partial} \mathscr{L}^{\natural,p,1}|X \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{L}^{\natural,p,|n|}|X \longrightarrow 0$$

is exact. Here p is a nonnegative integer.

Remark. We note that, in Theorem 9.2.3, all $\bar\partial$'s are continuous linear operators with respect to the graph topology.

PROOF. The exactness of the sequence

$$0 \longrightarrow \mathscr{O}^{\natural,p}|X \longrightarrow \mathscr{L}^{\natural,p,0}|X \xrightarrow{\bar\partial} \mathscr{L}^{\natural,p,1}|X$$

is evident. In fact, let Ω be a relatively compact open set in X. Let $u \in \mathscr{L}^{\natural,p,0}(\Omega)$ such that $\bar{\partial}u = 0$. Then, if we write u in the form

$$u = \sum_{|I|=p} u_I dz^I,$$

we have

$$\partial u_I / \partial \bar{z}_j = 0, \; j = 1, 2, \ldots, |n|,$$

if follows from Weyl's Lemma that u_I's are analytic on Ω. So that we can conclude that u_I's are holomorphic. The fact that $u_I \in \mathcal{O}^\natural(\Omega)$ follows from the exchangeability of L_2-norm and sup-norm for holomorphic functions. Thus the exactness of the above sequence was proved.

Next we have to prove the exactness of the sequence

$$\mathscr{L}^{\natural,p,0}|X \xrightarrow{\bar{\partial}} \mathscr{L}^{\natural,p,1}|X \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathscr{L}^{\natural,p,|n|}|X \longrightarrow 0.$$

For this purpose, we have only to prove the exactness of the sequence of stalks

$$\mathscr{L}^{\natural,p,0}_z \xrightarrow{\bar{\partial}} \mathscr{L}^{\natural,p,1}_z \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathscr{L}^{\natural,p,|n|}_z \longrightarrow 0$$

for every $z \in X$. But this is an easy consequence of Hörmander [4], Theorem 4.4.2 because every $z \in X$ has a fundamental system of $\mathcal{O}^\natural$-pseudoconvex open neighborhoods. Q.E.D.

Corollary 1. *Let X be as in Theorem 9.2.3. For an open set Ω in X, we have the following isomorphism*:

$$H^q(\Omega, \mathcal{O}^{\natural,p}) \cong \{f \in \mathscr{L}^{\natural,p,q}(\Omega); \bar{\partial}f = 0\}/\{\bar{\partial}g; g \in \mathscr{L}^{\natural,p,q-1}(\Omega)\}, \; (p \geqq 0 \text{ and } q \geqq 1).$$

Corollary 2. *Let X be as in Theorem 9.2.3. Let Ω be an $\mathcal{O}^\natural$-pseudoconvex open set in X. Then the equation $\bar{\partial}u = f$ has a solution $u \in \mathscr{L}^{\natural,p,q}(\Omega)$ for every $f \in \mathscr{L}^{\natural,p,q+1}(\Omega)$ such that $\bar{\partial}f = 0$. Here p and q are nonnegative integers.*

Proof. It follows from Theorem 9.1.4 and Corollary 1 to Theorem 9.2.3. Q.E.D.

We will now define the sheaf $L_\natural = L_{\natural,2,\mathrm{loc}}$ of germs of rapidly decreasing locally L_2-functions.

Definition 9.2.4. *We define the sheaf $L_\natural$ to be the sheafification of the presheaf $\{L_\natural(\Omega); \Omega \subset \boldsymbol{H}^n \text{ open}\}$, where, for an open set Ω in $\boldsymbol{H}^n$, the section module $L_\natural(\Omega)$ is the space of all $f \in L_{\natural,2,\mathrm{loc}}(\Omega \cap \boldsymbol{C}^{|n|})$ such as, for any relatively compact open subset ω of Ω, there exists some positive δ such that $e(\delta \| z \|) f(z) | \omega \in L_2(\omega \cap \boldsymbol{C}^{|n|})$.*

Then it is easy to see that $L_{\natural}$ is a soft FS*-sheaf.

Definition 9.2.5 (The sheaf $\mathcal{L}_{\natural}^{p,q}$). *We define the sheaf $\mathcal{L}_{\natural}^{p,q} = \mathcal{L}_{\natural,2,\mathrm{loc}}^{p,q}$ to be the sheafification of the presheaf $\{\mathcal{L}_{\natural}^{p,q}(\Omega); \Omega \subset \boldsymbol{H}^n \text{ open}\}$, where, for an open set Ω in $\boldsymbol{H}^n$, the section module $\mathcal{L}_{\natural}^{p,q}(\Omega)$ is the space of all $f \in L_{\natural}^{p,q}(\Omega)$ such that $\bar{\partial} f \in L_{\natural}^{p,q+1}(\Omega)$. We put $\mathcal{L}_{\natural} = \mathcal{L}_{\natural}^{0,0}$.*

Then $\mathcal{L}_{\natural}^{p,q}$ is a soft FS*-sheaf with respect to the graph topology. Then we have the following

Theorem 9.2.6 (The Dolbeault-Grothendieck resolution). *For some $d > 0$, put $X = \operatorname{int}\{z \in \boldsymbol{C}^{|n|}; |\operatorname{Im} z''| - |\operatorname{Re} z''| < d\}^a$. Then the sequence of sheaves over X*

$$0 \longrightarrow \mathcal{O}_{\natural}^{p}|X \longrightarrow \mathcal{L}_{\natural}^{p,0}|X \xrightarrow{\bar{\partial}} \mathcal{L}_{\natural}^{p,1}|X \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{L}_{\natural}^{p,|n|}|X \longrightarrow 0$$

is exact.

Remark. We note that, in Theorem 9.2.6, all $\bar{\partial}$'s are continuous linear operators with respect to the graph topology.

PROOF. The exactness of the sequence

$$0 \longrightarrow \mathcal{O}_{\natural}^{p}|X \longrightarrow \mathcal{L}_{\natural}^{p,0}|X \xrightarrow{\bar{\partial}} \mathcal{L}_{\natural}^{p,1}|X$$

can be proved by a similar way to that of Theorem 9.2.3.

Next we have to prove the exactness of the sequence

$$\mathcal{L}_{\natural}^{p,0}|X \xrightarrow{\bar{\partial}} \mathcal{L}_{\natural}^{p,1}|X \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{L}_{\natural}^{p,|n|}|X \longrightarrow 0.$$

Let $z = (z', z'') \in X$ and Ω an open neighborhood in X of z of the form $\Omega' \times \Omega''$ where Ω' is an open neighborhood of z' in $\boldsymbol{C}^{n_1}$ and Ω'' is an open neighborhood of z'' in $\boldsymbol{E}^{n_2}$ of the form $V_{\delta,A}$ in Lemma 3.2.7 for some δ and A such as $0 < \delta < 1$ and $A > 0$. Let f be an element in $\mathcal{L}_{\natural}^{p,q+1}(\Omega)$ such that $\bar{\partial} f = 0$. Then, for some $\varepsilon > 0$, we have $f \cdot h_\varepsilon(z) \in \mathcal{L}^{\natural,p,q+1}(\Omega)$, where we put $h_\varepsilon(z) = \cosh(\varepsilon\sqrt{(z'')^2}/2)$. Since $\bar{\partial}(f \cdot h_\varepsilon(z)) = 0$, we can find some $v \in \mathcal{L}^{\natural,p,q}(\omega)$ for some open neighborhood $\omega = \omega' \times \omega''$ of z with $z' \in \omega' \subset \Omega'$ and $z'' \in \omega'' \subset \Omega''$ such that $\bar{\partial} v = f \cdot h_\varepsilon(z)$. Here we may assume that $h_\varepsilon(z) \neq 0$ on $\omega \cap \boldsymbol{C}^{|n|}$. Then $u = v/h_\varepsilon(z)$ belongs to $\mathcal{L}_{\natural}^{p,q}(\omega)$ and $\bar{\partial} u = f$ holds. This completes the proof. Q.E.D.

Corollary 1. *Let X be as in Theorem 9.2.6. For an open set Ω in X, we have the following isomorphism:*

$$H^q(\Omega, \mathcal{O}_{\natural}^{p}) \cong \{f \in \mathcal{L}_{\natural}^{p,q}(\Omega); \bar{\partial} f = 0\}/\{\bar{\partial} g; g \in \mathcal{L}_{\natural}^{p,q-1}(\Omega)\}, \ (p \geqq 0 \text{ and } q \geqq 1).$$

Corollary 2. *Let X be as in Theorem* 9.2.6. *Let Ω be an $\mathcal{O}^{\natural}$-pseudoconvex open set in X. Then the equation $\bar{\partial}u = f$ has a solution $u \in \mathcal{L}^{p,q}_{\natural}(\Omega)$ for every $f \in \mathcal{L}^{p,q+1}_{\natural}(\Omega)$ such that $\bar{\partial}f = 0$. Here p and q are nonnegative integers.*

Proof. It follows from Theorem 9.1.7 and Corollary 1 to Theorem 9.2.6. Q.E.D.

Now, for later applications, we will construct another soft resolutions of $\mathcal{O}^{\natural}$ and $\mathcal{O}_{\natural}$.

At first we will define the sheaf $\mathcal{E}^{\natural}$ of germs of partially slowly increasing C^{∞}-functions over $\boldsymbol{H}^n$.

Definition 9.2.7. *We define the sheaf $\mathcal{E}^{\natural}$ to be the sheafification of the presheaf $\{\mathcal{E}^{\natural}(\Omega); \Omega \subset \boldsymbol{H}^n \text{ open}\}$, where, for an open set Ω in $\boldsymbol{H}^n$, the section module $\mathcal{E}^{\natural}(\Omega)$ is defined as follows:*

$\mathcal{E}^{\natural}(\Omega) = \{f \in \mathcal{E}(\Omega \cap \boldsymbol{C}^{|n|});$ *for any positive ε and any compact set K in Ω and any $\alpha \in \boldsymbol{N}_0^{2|n|}$, the estimate* $\sup\{|f^{(\alpha)}(z)| \cdot e(-\varepsilon|z|); z \in K \cap \boldsymbol{C}^{|n|}\} < \infty$ *holds*$\}$.

Here $f^{(\alpha)}(z)$ means the derivative of the form

$$f^{(\alpha)}(z) = \partial^{|\alpha|} f(z)/\partial x_1^{\alpha_1}\,\partial y_1^{\alpha_2} \cdots \partial x_n^{\alpha_{2|n|-1}}\,\partial y_n^{\alpha_{2|n|}},$$

and $\boldsymbol{N}_0$ means the set of all nonnegative integers.

Then it is easy to see that $\mathcal{E}^{\natural}$ is a soft nuclear Fréchet sheaf. Then we have the following

Theorem 9.2.8 (The Dolbeault-Grothendieck resolution). *Put $X = \operatorname{int}\{z = (z', z'') \in \boldsymbol{C}^{|n|}; |\operatorname{Im} z''| < 1 + |\operatorname{Re} z''|/\sqrt{3}\}^a$. Then the sequence of sheaves over X*

$$0 \longrightarrow \mathcal{O}^{\natural,p}|X \longrightarrow \mathcal{E}^{\natural,p,0}|X \xrightarrow{\bar{\partial}} \mathcal{E}^{\natural,p,1}|X \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{E}^{\natural,p,|n|}|X \longrightarrow 0$$

is exact. Here p is a nonnegative integer.

Proof. It goes in a similar way to Ito[10], Corollary to Theorem 3.1 using weight functions similar to $J(z)$ in Kaneko [17], Theorem 8.6.6, p. 175. Q.E.D.

Corollary 1. *Let X be as in Theorem* 9.2.8. *For an open set Ω in X, we have the following isomorphism:*

$$H^q(\Omega, \mathcal{O}^{\natural,p}) \cong \{f \in \mathcal{E}^{\natural,p,q}(\Omega); \bar{\partial}f = 0\}/\{\bar{\partial}g; g \in \mathcal{E}^{\natural,p,q-1}(\Omega)\}, \ (p \geqq 0 \text{ and } q \geqq 1).$$

Corollary 2. *Let X be as in Theorem* 9.2.8 *and Ω an $\mathcal{O}^{\natural}$-pseudoconvex open set in X. Then the equation $\bar{\partial}u = f$ has a solution $u \in \mathcal{E}^{\natural,p,q}(\Omega)$ for every $f \in \mathcal{E}^{\natural,p,q+1}(\Omega)$*

such that $\bar{\partial} f = 0$. *Every solution of the equation* $\bar{\partial} u = f$ *has this property when* $q = 0$.

Now we will define the sheaf $\mathcal{E}_{\natural}$ of germs of partially rapidly decreasing C^{∞}-functions over $\boldsymbol{H}^n$.

Definition 9.2.9. *We define the sheaf* $\mathcal{E}_{\natural}$ *to be the sheafification of the presheaf* $\{\mathcal{E}_{\natural}(\Omega); \Omega \subset \boldsymbol{H}^n$ *open*$\}$, *where the section module* $\mathcal{E}_{\natural}(\Omega)$ *on an open set* Ω *in* $\boldsymbol{H}^n$ *is the space of all* C^{∞}*-functions on* $\Omega \cap \boldsymbol{C}^{|n|}$ *such that, for any compact set* K *in* Ω *and any* $\alpha \in \boldsymbol{N}_0^{2|n|}$, *there exists some positive constant* δ *so that the estimate* $\sup\{|f^{(\alpha)}(z)| e(\delta|z|); z \in K \cap \boldsymbol{C}^{|n|}\} < \infty$ *holds.*

Then $\mathcal{E}_{\natural}$ becomes a soft nuclear Fréchet sheaf. Then we have the following

Theorem 9.2.10 (The Dolbeault-Grothendieck resolution). *Put* $X = \operatorname{int}\{z = (z', z'') \in \boldsymbol{C}^{|n|}; |\operatorname{Im} z''| - |\operatorname{Re} z''| < d,\ |\operatorname{Im} z''| < 1 + |\operatorname{Re} z''|/\sqrt{3}\}^a$ *for some* $d > 0$. *Then the sequence of sheaves over* X

$$0 \longrightarrow \mathcal{O}_{\natural}^p | X \longrightarrow \mathcal{E}_{\natural}^{p,0} | X \xrightarrow{\bar{\partial}} \mathcal{E}_{\natural}^{p,1} | X \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{E}_{\natural}^{p,|n|} | X \longrightarrow 0$$

is exact.

Proof. Let $z = (z', z'') \in X$ and Ω an open neighborhood of z in X of the form $\Omega' \times \Omega''$, where Ω' is an open neighborhood of z' in $\boldsymbol{C}^{n_1}$ and Ω'' is an open neighborhood of z'' in $\boldsymbol{E}^{n_2}$ of the form $V_{\delta,A}$ in Lemma 3.2.7 for some δ and A such as $0 < \delta < 1$ and $A > 0$. Let f be an element in $\mathcal{E}_{\natural}^{p,q+1}(\Omega)$ such that $\bar{\partial} f = 0$. Then, for some $\varepsilon > 0$, we have $f \cdot h_{\varepsilon}(z) \in \mathcal{E}^{\natural,p,q+1}(\Omega)$, where we put $h_{\varepsilon}(z) = \cosh(\varepsilon\sqrt{(z'')^2}/2)$. Since $\bar{\partial}(f \cdot h_{\varepsilon}(z)) = 0$, we can find some $v \in \mathcal{E}^{\natural,p,q}(\omega)$ for some open neighborhood $\omega = \omega' \times \omega''$ of z with $z' \in \omega' \subset \Omega'$ and $z'' \in \omega'' \subset \Omega''$ such that $\bar{\partial} v = f \cdot h_{\varepsilon}(z)$. Here we may assume that $h_{\varepsilon}(z) \neq 0$ on $\omega \cap \boldsymbol{C}^{|n|}$. Then $u = v/h_{\varepsilon}(z)$ belongs to $\mathcal{E}_{\natural}^{p,q}(\omega)$ and $\bar{\partial} u = f$ holds. This completes the proof. Q.E.D.

Corollary 1. *Let* X *be as in Theorem* 9.2.10 *and* Ω *an open set in* X. *Then we have the following isomorphism*:

$$H^q(\Omega, \mathcal{O}_{\natural}^p) \cong \{f \in \mathcal{E}_{\natural}^{p,q}(\Omega); \bar{\partial} f = 0\}/\{\bar{\partial} g; g \in \mathcal{E}_{\natural}^{p,q-1}(\Omega)\}, \ (p \geqq 0 \text{ and } q \geqq 1).$$

Corollary 2. *Put* $X = \operatorname{int}\{z = (z', z'') \in \boldsymbol{C}^{|n|}; |\operatorname{Im} z''| < 1 + |\operatorname{Re} z''|/\sqrt{3},\ |\operatorname{Im} z''|^2 < \delta^2 |\operatorname{Re} z''|^2 + A^2\}^a$ *for some* δ, $(0 < \delta < 1)$ *and* $A > 0$. *Let* Ω *be an* $\mathcal{O}^{\natural}$*-pseudoconvex open set in* X. *Then the equation* $\bar{\partial} u = f$ *has a solution* $u \in \mathcal{E}_{\natural}^{p,q}(\Omega)$ *for every* $f \in \mathcal{E}_{\natural}^{p,q+1}(\Omega)$ *such that* $\bar{\partial} f = 0$. *Here* p *and* q *are nonnegative integers.*

9.3. The Malgrange Theorem

In this section we will prove the following

Theorem 9.3.1 (The Malgrange Theorem). *For some constants $d > 0$, $0 < \delta < 1$ and $A > 0$, put $X = \mathrm{int}\{z = (z', z'') \in \boldsymbol{C}^{|n|}; |\mathrm{Im}\, z''| - |\mathrm{Re}\, z''| < d, |\mathrm{Im}\, z''|^2 \leqq \delta^2 |\mathrm{Re}\, z''|^2 + A^2\}^a$ in $\boldsymbol{H}^n$. Let Ω be an open set in X. Then we have $H^{|n|}(\Omega, \mathcal{O}^{\natural}) = 0$.*

PROOF. By virtue of Corollary 1 to Theorem 9.2.3, we have only to prove the exactness of the sequence

$$\mathscr{L}^{\natural,0,|n|-1}(\Omega) \xrightarrow{\bar{\partial}} \mathscr{L}^{\natural,0,|n|}(\Omega) \longrightarrow 0$$

in the notations of Theorem 9.2.3. Here we consider its dual sequence

$$\mathscr{L}^{0,1}_{\natural,c}(\Omega) \xleftarrow{-\bar{\partial}^{\Omega}} \mathscr{L}^{0,0}_{\natural,c}(\Omega) \longleftarrow 0.$$

Then, by virtue of the Serre-Komatsu duality theorem for FS*-spaces, it suffices to show the injectiveness and closedrangeness of $-\bar{\partial}^{\Omega} = (\bar{\partial}^{\Omega})'$. Since $\bar{\partial}^{\Omega}$ is elliptic, its injectivity is an immediate consequence of the unique continuation property. Now we will prove its closedrangeness. This is surely true if Ω is replaced by a large $\mathcal{O}^{\natural}$- pseudoconvex open set $\tilde{\Omega}$ in X containing Ω because $H^p(\tilde{\Omega}, \mathcal{O}^{\natural}) = 0$ for $p \geqq 1$. Then we consider the commutative diagram:

$$\begin{array}{ccc} \mathscr{L}^{0,1}_{\natural,c}(\Omega) & \xleftarrow{-\bar{\partial}^{\Omega}} & \mathscr{L}^{0,0}_{\natural,c}(\Omega) \longleftarrow 0 \\ \downarrow i & & \downarrow \\ \mathscr{L}^{0,1}_{\natural,c}(\tilde{\Omega}) & \xleftarrow{-\bar{\partial}^{\tilde{\Omega}}} & \mathscr{L}^{0,0}_{\natural,c}(\tilde{\Omega}) \longleftarrow 0, \end{array}$$

where the map i is the natural injection. By the above remark, $-\bar{\partial}^{\tilde{\Omega}}$ is of closed range. Then $\mathrm{Im}(-\bar{\partial}^{\Omega}) = i^{-1}(\mathrm{Im}(-\bar{\partial}^{\tilde{\Omega}}))$ is closed. Therefore $-\bar{\partial}^{\Omega}$ is of closed range. This completes the proof. Q.E.D.

Corollary. Flabby dim $\mathcal{O}^{\natural} \leqq |n|$.

9.4. The Serre Duality Theorem

In this section we will prove the Serre Duality Theorem.

Theorem 9.4.1. *Let X be as in Theorem 9.3.1. Let Ω be an open set in X and assume that $\dim H^p(\Omega, \mathcal{O}^{\natural}) < \infty$ holds for each $p \geqq 1$. Then we have the isomorphism $[H^p(\Omega, \mathcal{O}^{\natural})]' \cong H^{|n|-p}_c(\Omega, \mathcal{O}_{\natural})$, $(0 \leqq p \leqq |n|)$.*

PROOF. By virtue of Corollary 1 to Theorem 9.2.3 and Corollary 1 to Theorem 9.2.6, cohomology groups $H^p(\Omega, \mathcal{O}^{\natural})$ and $H^{|n|-p}_c(\Omega, \mathcal{O}_{\natural})$ are cohomology groups respectively of the dual complexes

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathcal{L}^{\natural,0,0}(\Omega) & \xrightarrow{\bar\partial} & \mathcal{L}^{\natural,0,1}(\Omega) & \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} & \mathcal{L}^{\natural,0,|n|}(\Omega) & \longrightarrow 0 \\ & \updownarrow & & \updownarrow & & \updownarrow & \\ 0 \longleftarrow & \mathcal{L}^{0,|n|}_{\natural,c}(\Omega) & \xleftarrow{-\bar\partial} & \mathcal{L}^{0,|n|-1}_{\natural,c}(\Omega) & \xleftarrow{-\bar\partial} \cdots \xleftarrow{-\bar\partial} & \mathcal{L}^{0,0}_{\natural,c}(\Omega) & \longleftarrow 0. \end{array}$$

Here the upper complex is composed of FS*-spaces and the lower complex is composed of DFS*-spaces. Since the ranges of operators $\bar\partial$ in the upper complex are all closed by virtue of Schwartz' Lemma (cf. Komatsu [20]), the ranges of operators $-\bar\partial = (\bar\partial)'$ in the lower complex are also closed. Hence we have the isomorphism

$$[H^p(\Omega, \mathcal{O}^{\natural})]' \cong H_c^{|n|-p}(\Omega, \mathcal{O}_{\natural})$$

by virtue of Serre's Lemma (cf. Komatsu [20]). Q.E.D.

Remark. The above Theorem is also true for such an open set Ω as, in the above dual complexes, each $\bar\partial$ is of closed range.

9.5. The Martineau-Harvey Theorem

In this section we will prove the Martineau-Harvey Theorem.

Theorem 9.5.1. *Let X be as in Theorem 9.3.1. Let K be a compact set in X such that it has an $\mathcal{O}^{\natural}$-pseudoconvex open neighborhood $\Omega \subset X$ and satisfies the condition $H^p(K, \mathcal{O}_{\natural}) = 0$ for each $p \geqq 1$. Then we have $H_K^p(\Omega, \mathcal{O}^{\natural}) = 0$ for $p \neq |n|$, and algebraic isomorphisms $H_K^{|n|}(\Omega, \mathcal{O}^{\natural}) \cong H^{|n|-1}(\Omega\backslash K, \mathcal{O}^{\natural}) \cong \mathcal{O}_{\natural}(K)'$, $(|n| \geqq 2)$, and $H_K^1(\Omega, \mathcal{O}^{\natural}) \cong \mathcal{O}^{\natural}(\Omega\backslash K)/\mathcal{O}^{\natural}(\Omega)$, $(|n| = 1)$.*

Remark. If a compact set K in X has a fundamental system of $\mathcal{O}^{\natural}$-pseudoconvex open neighborhoods in X, it satisfies the assumptions in Theorem 9.5.1.

Proof. At first assume $|n| \geqq 2$. By the excision theorem, $H_K^p(\Omega, \mathcal{O}^{\natural})$ is independent of an open neighborhood Ω of K. So, we may assume that Ω is the $\mathcal{O}^{\natural}$-pseudoconvex open neighborhood in the assumptions in this theorem. Then in the long exact sequence of cohomology groups (cf. Komatsu [10], Theorem II. 3.2):

$$\begin{aligned} 0 &\longrightarrow H_K^0(\Omega, \mathcal{O}^{\natural}) \longrightarrow H^0(\Omega, \mathcal{O}^{\natural}) \longrightarrow H^0(\Omega\backslash K, \mathcal{O}^{\natural}) \\ &\longrightarrow H_K^1(\Omega, \mathcal{O}^{\natural}) \longrightarrow \cdots \\ &\longrightarrow H_K^{|n|}(\Omega, \mathcal{O}^{\natural}) \longrightarrow H^{|n|}(\Omega, \mathcal{O}^{\natural}) \longrightarrow H^{|n|}(\Omega\backslash K, \mathcal{O}^{\natural}) \longrightarrow \cdots, \end{aligned}$$

we have $H^p(\Omega, \mathcal{O}^{\natural}) = 0$ for each $p \geqq 1$, and $H_K^0(\Omega, \mathcal{O}^{\natural}) = 0$ by the unique continuation theorem. Hence we have algebraic isomorphisms

$$H_K^1(\Omega, \mathcal{O}^\natural) \cong \mathcal{O}^\natural(\Omega\backslash K)/\mathcal{O}^\natural(\Omega),$$

$$H_K^p(\Omega, \mathcal{O}^\natural) \cong H^{p-1}(\Omega\backslash K, \mathcal{O}^\natural),\ p \geqq 2.$$

We also have the long exact sequence of cohomology groups with compact support (cf. Komatsu [10], Theorem II. 3.15):

$$\begin{aligned}0 &\longrightarrow H_c^0(\Omega\backslash K, \mathcal{O}_\natural) \longrightarrow H_c^0(\Omega, \mathcal{O}_\natural) \longrightarrow H^0(K, \mathcal{O}_\natural)\\ &\longrightarrow H_c^1(\Omega\backslash K, \mathcal{O}_\natural) \longrightarrow \cdots\\ &\longrightarrow H_c^p(\Omega\backslash K, \mathcal{O}_\natural) \longrightarrow H_c^p(\Omega, \mathcal{O}_\natural) \longrightarrow H^p(K, \mathcal{O}_\natural) \longrightarrow \cdots.\end{aligned}$$

Here $H^p(K, \mathcal{O}_\natural) = 0$ for each $p \geqq 1$ by the assumption on K. By the theorem 9.4.1, we also have $H_c^p(\Omega, \mathcal{O}_\natural) = 0$ for each $p \neq |n|$. Therefore we obtain the isomorphisms

$$H_c^1(\Omega\backslash K, \mathcal{O}_\natural) \cong \mathcal{O}_\natural(K),$$

$$H_c^p(\Omega\backslash K, \mathcal{O}_\natural) \cong H_c^p(\Omega, \mathcal{O}_\natural) = 0, \text{ for each } p \neq 1, |n|,$$

$$H_c^{|n|}(\Omega\backslash K, \mathcal{O}_\natural) \cong H_c^{|n|}(\Omega, \mathcal{O}_\natural) \cong \mathcal{O}^\natural(\Omega)'.$$

Now we consider the following dual complexes:

$$\begin{array}{l} 0 \longrightarrow \mathcal{L}^{\natural,0,0}(\Omega\backslash K) \xrightarrow{\bar\partial_0} \mathcal{L}^{\natural,0,1}(\Omega\backslash K) \xrightarrow{\bar\partial_1} \cdots \xrightarrow{\bar\partial_{|n|-2}} \mathcal{L}^{\natural,0,|n|-1}(\Omega\backslash K) \longrightarrow (*)\\ \qquad\quad \updownarrow \qquad\qquad\qquad\quad \updownarrow \qquad\qquad\qquad\qquad\qquad \updownarrow \\ 0 \longleftarrow \mathcal{L}_{\natural,c}^{0,|n|}(\Omega\backslash K) \xleftarrow{-\bar\partial_{|n|-1}} \mathcal{L}_{\natural,c}^{0,|n|-1}(\Omega\backslash K) \xleftarrow{-\bar\partial_{|n|-2}} \cdots \xleftarrow{-\bar\partial_1} \mathcal{L}_{\natural,c}^{0,1}(\Omega\backslash K) \longleftarrow (**)\\ (*) \xrightarrow{\bar\partial_{|n|-1}} \mathcal{L}^{\natural,0,|n|}(\Omega\backslash K) \longrightarrow 0\\ \qquad\qquad\quad \updownarrow \\ (**) \xleftarrow{-\bar\partial_0} \mathcal{L}_{\natural,c}^{0,0}(\Omega\backslash K) \longleftarrow 0. \end{array}$$

Here the upper complex is composed of FS*-spaces and the lower complex is composed of DFS*-spaces. Then, since $H_c^p(\Omega\backslash K, \mathcal{O}_\natural) = 0$ for each $p \neq 1, |n|$, the range of $-\bar\partial_j = (\bar\partial_{|n|-j-1})'$ is closed for $j \neq 0, |n|-1$. However $\bar\partial_{|n|-1}$ is of closed range by the Malgrange Theorem. Hence, by the closed range theorem, $-\bar\partial_0$ is of closed range(cf. Komatsu [9], Theorem 19, p. 381).

In order to prove the closedness of the range of $-\bar\partial_{|n|-1}$, we consider the following diagram:

$$\begin{array}{ccc} 0 \longleftarrow \mathcal{L}_{\natural,c}^{0,|n|}(\Omega\backslash K) & \xleftarrow{-\bar\partial_{|n|-1}^{\Omega\backslash K}} & \mathcal{L}_{\natural,c}^{0,|n|-1}(\Omega\backslash K)\\ \big\downarrow i & & \big\downarrow \\ 0 \longleftarrow \mathcal{L}_{\natural,c}^{0,|n|}(\Omega) & \xleftarrow{-\bar\partial_{|n|-1}^{\Omega}} & \mathcal{L}_{\natural,c}^{0,|n|-1}(\Omega), \end{array}$$

where the map i is the natural injection. However, in the dual complexes for Ω, $\bar{\partial}_0^{\Omega}$ is of closed range since $H^1(\Omega, \mathcal{O}^{\natural}) = 0$. Thus, since $-\bar{\partial}_{|n|-1}^{\Omega}$ is of closed range by the closed range theorem, $\operatorname{Im}(-\bar{\partial}_{|n|-1}^{\Omega\backslash K}) = i^{-1}(\operatorname{Im}(-\bar{\partial}_{|n|-1}^{\Omega}))$ is closed. Therefore all $-\bar{\partial}_j^{\Omega\backslash K}$ are of closed range. Hence by the Serre-Komatsu duality theorem, we have isomorphisms $[H^p(\Omega\backslash K, \mathcal{O}^{\natural})]' \cong H_c^{|n|-p}(\Omega\backslash K, \mathcal{O}_{\natural})$, for $0 \leqq p \leqq |n|$. Hence, we have $\mathcal{O}^{\natural}(\Omega\backslash K)' \cong H_c^{|n|}(\Omega\backslash K, \mathcal{O}_{\natural}) \cong H_c^{|n|}(\Omega, \mathcal{O}_{\natural}) \cong \mathcal{O}^{\natural}(\Omega)'$. Here $\mathcal{O}^{\natural}(\Omega\backslash K)$ and $\mathcal{O}^{\natural}(\Omega)$ are both FS-spaces, a posteriori, reflexive. Hence we have the isomorphism $\mathcal{O}^{\natural}(\Omega\backslash K) \cong \mathcal{O}^{\natural}(\Omega)$. Thus we have $H_K^1(\Omega, \mathcal{O}^{\natural}) \cong \mathcal{O}^{\natural}(\Omega\backslash K)/\mathcal{O}^{\natural}(\Omega) = 0$. Hence, for each $p \geqq 2$, $p \neq |n|$, we have $0 = [H_c^{|n|-p+1}(\Omega, \mathcal{O}_{\natural})]' \cong [H_c^{|n|-p+1}(\Omega\backslash K, \mathcal{O}_{\natural})]' \cong H^{p-1}(\Omega\backslash K, \mathcal{O}^{\natural}) \cong H_K^p(\Omega, \mathcal{O}^{\natural})$. Thus $H_K^p(\Omega, \mathcal{O}^{\natural}) = 0$ for each $p \neq |n|$. In the case $p = |n|$, we have the algebraic isomorphisms $H_K^{|n|}(\Omega, \mathcal{O}^{\natural}) \cong H^{|n|-1}(\Omega\backslash K, \mathcal{O}^{\natural}) \cong [H_c^1(\Omega\backslash K, \mathcal{O}_{\natural})]' \cong [H^0(K, \mathcal{O}_{\natural})]' \cong \mathcal{O}_{\natural}(K)'$.

At last we will prove the case $|n| = 1$. In this case, we have the conclusion by virtue of the long exact sequence of relative cohomology groups, the Oka-Cartan-Kawai Theorem B and the Malgrange Theorem. Q.E.D.

Here we mention the important facts used in the proof of Theorem 9.5.1 in the following

Proposition 9.5.2. *Let K and Ω be as in Theorem* 9.5.1. *Then we have* (*topological*) *isomorphisms* $H_c^1(\Omega\backslash K, \mathcal{O}_{\natural}) \cong H^0(K, \mathcal{O}_{\natural})$ *and* $H_c^p(\Omega\backslash K, \mathcal{O}_{\natural}) \cong H_c^p(\Omega, \mathcal{O}_{\natural})$ *for each* $p \geqq 2$.

9.6. The Sato Theorem

In this section we will prove the pure-codimensionality of $\tilde{\boldsymbol{D}}^n$ with respect to $\mathcal{O}^{\natural}$. Then we will realize partial modified Fourier hyperfunctons as "boundary values" of partially slowly increasing holomorphic functions or as (relative) cohomology classes of partially slowly increasing holomorphic functions.

Theorem 9.6.1 (The Sato Theorem). *Assume* $|n| \geqq 2$. *Then we have the following*

(1) $\tilde{\boldsymbol{D}}^n$ *is purely* $|n|$-*codimensional with respect to* $\mathcal{O}^{\natural}$.

(2) *The presheaf over* $\tilde{\boldsymbol{D}}^n$, $\Omega \to H_{\Omega}^{|n|}(V, \mathcal{O}^{\natural})$, *is a flabby sheaf, where* Ω *is an open set in* $\tilde{\boldsymbol{D}}^n$ *and* V *an open set in* X *which contains* Ω *as its closed subset. Here* X *is as in Theorem* 9.5.1.

(2) *This sheaf* (2) *is isomorphic to the sheaf* $\mathcal{BQ}$ *of partial modified Fourier hyperfunctions.*

Proof. (1) We have to prove the vanishing of the derived sheaf $\mathcal{H}_{\tilde{D}^n}^p(\mathcal{O}^{\natural})$ for $p \neq |n|$. This is local in nature. Thus it is sufficient to prove $H_{\Omega}^p(V, \mathcal{O}^{\natural}) = 0$, $(p \neq |n|)$, for a relatively compact open set Ω in $\tilde{\boldsymbol{D}}^n$. Thus it goes in a similar way to Kawai [19], p. 482.

(2) By (1) and Komatsu [21], Theorem II. 3.18, we have the conclusion.

(3) We have only to prove this isomorphism stalkswise. This is local in nature. Consider the following exact sequence of relative cohomology groups for relatively compact open set Ω in $\tilde{\boldsymbol{D}}^n$

$$\begin{aligned} 0 &\longrightarrow H^0_{\partial\Omega}(V, \mathcal{O}^{\natural}) \longrightarrow H^0_{\Omega^a}(V, \mathcal{O}^{\natural}) \longrightarrow H^0_{\Omega}(V, \mathcal{O}^{\natural}) \\ &\longrightarrow H^1_{\partial\Omega}(V, \mathcal{O}^{\natural}) \longrightarrow \cdots \longrightarrow H^{|n|-1}_{\Omega}(V, \mathcal{O}^{\natural}) \\ &\longrightarrow H^{|n|}_{\partial\Omega}(V, \mathcal{O}^{\natural}) \longrightarrow H^{|n|}_{\Omega^a}(V, \mathcal{O}^{\natural}) \longrightarrow H^{|n|}_{\Omega}(V, \mathcal{O}^{\natural}) \\ &\longrightarrow H^{|n|+1}_{\partial\Omega}(V, \mathcal{O}^{\natural}) \longrightarrow \cdots. \end{aligned}$$

Then, by (1) and by the Martineau-Harvey Theorem, we have $H^{|n|-1}_{\Omega}(V, \mathcal{O}^{\natural}) = 0$ and $H^{|n|+1}_{\partial\Omega}(V, \mathcal{O}^{\natural}) = 0$. Thus we have the exact sequence

$$0 \longrightarrow H^{|n|}_{\partial\Omega}(V, \mathcal{O}^{\natural}) \longrightarrow H^{|n|}_{\Omega^a}(V, \mathcal{O}^{\natural}) \longrightarrow H^{|n|}_{\Omega}(V, \mathcal{O}^{\natural}) \longrightarrow 0.$$

Since, by the Martineau-Harvey Theorem, we have algebraic isomorphisms

$$H^{|n|}_{\partial\Omega}(V, \mathcal{O}^{\natural}) \cong \mathscr{A}_{\natural}(\partial\Omega)', \ H^{|n|}_{\Omega^a}(V, \mathcal{O}^{\natural}) \cong \mathscr{A}_{\natural}(\Omega^a)',$$

we obtain the algeraic isomorphism

$$H^{|n|}_{\Omega}(V, \mathcal{O}^{\natural}) \cong \mathscr{A}_{\natural}(\Omega^a)'/\mathscr{A}_{\natural}(\partial\Omega)' = \mathscr{B}\mathscr{Q}(\Omega).$$

Thus the sheaf $\Omega \to H^{|n|}_{\Omega}(V, \mathcal{O}^{\natural})$ is isomorphic to the sheaf $\mathscr{B}\mathscr{Q}$ of partial modified Fourier hyperfunctions over $\tilde{\boldsymbol{D}}^n$. Q.E.D.

Corollary. *Assume $|n| \geqq 2$. Let Ω be an open set in $\tilde{\boldsymbol{D}}^n$ and V an $\mathcal{O}^{\natural}$-pseudoconvex open neighborhood of Ω. Then we have the algebraic isomorphism $H^{|n|}_{\Omega}(V, \mathcal{O}^{\natural}) \cong H^{|n|-1}(V\backslash\Omega, \mathcal{O}^{\natural})$.*

Next we will consider the case $|n| = 1$. This case is either $n = (1, 0)$ or $n = (0, 1)$. The case $n = (1, 0)$ results in the case of Sato hyperfunctions and the case $n = (0, 1)$ results in the case of modified Fourier hyperfunctions. In these cases, the Sato Theorem is also true except that, in the case $n = (0, 1)$, the assertion (3) is not yet proved.

Let Ω be an open set in $\tilde{\boldsymbol{D}}^n$. Then there exists an $\mathcal{O}^{\natural}$-pseudoconvex open neighborhood V of Ω such that $V \cap \tilde{\boldsymbol{D}}^n = \Omega$ (cf. Ito [11], Theorem 2.1.13). We put $V_0 = V$ and $V_j = V\backslash\{z \in V; \operatorname{Im} z_j = 0\}^a$, $j = 1, 2, \ldots, |n|$. Then $\mathfrak{U} = \{V_0, V_1, \ldots, V_{|n|}\}$ and $\mathfrak{U}' = \{V_1, \ldots, V_{|n|}\}$ cover V and $V\backslash\Omega$ respectively. Since V_j and their intersections are also $\mathcal{O}^{\natural}$-pseudoconvex open sets, the covering $(\mathfrak{U}, \mathfrak{U}')$ satisfies the conditions of Leray's Theorem (cf. Komatsu [21]). Thus, by Leray's Theorem, we obtain the isomorphism $H^{|n|}_{\Omega}(V, \mathcal{O}^{\natural}) \cong H^{|n|}(\mathfrak{U}, \mathfrak{U}', \mathcal{O}^{\natural})$. Since the covering $\mathfrak{U}$ is composed of only $|n| + 1$ open sets V_j, $(j = 0, 1, \ldots, |n|)$, we easily obtain the algebraic isomorphisms

$$Z^{|n|}(\mathfrak{U}, \mathfrak{U}', \mathcal{O}^{\natural}) \cong \mathcal{O}^{\natural}(\bigcap_j V_j),$$

$$C^{|n|-1}(\mathfrak{U}, \mathfrak{U}', \mathcal{O}^{\natural}) \cong \bigoplus_{j=1}^{|n|} \mathcal{O}^{\natural}(\bigcap_{i \neq j} V_i).$$

Hence we have the algebraic isomorphism

$$\delta C^{|n|-1}(\mathfrak{U}, \mathfrak{U}', \mathcal{O}^{\natural}) \cong \sum_{j=1}^{|n|} \mathcal{O}^{\natural}(\bigcap_{i \neq j} V_i) | V_1 \cap \cdots \cap V_{|n|}.$$

Thus we have the algebraic isomorphisms

$$\begin{aligned} H^{|n|}_{\Omega}(V, \mathcal{O}^{\natural}) &\cong H^{|n|}(\mathfrak{U}, \mathfrak{U}', \mathcal{O}^{\natural}) \\ &\cong Z^{|n|}(\mathfrak{U}, \mathfrak{U}', \mathcal{O}^{\natural})/\delta C^{|n|-1}(\mathfrak{U}, \mathfrak{U}', \mathcal{O}^{\natural}) \\ &\cong \mathcal{O}^{\natural}(\bigcap_j V_j)/\sum_{j=1}^{|n|} \mathcal{O}^{\natural}(\bigcap_{i \neq j} V_i). \end{aligned}$$

Thus we have the following

Theorem 9.6.2. *We use notations as above. Then we have the algebraic isomorphisms*

$$H^{|n|}_{\Omega}(V, \mathcal{O}^{\natural}) \cong H^{|n|}(\mathfrak{U}, \mathfrak{U}', \mathcal{O}^{\natural}) \cong \mathcal{O}^{\natural}(\bigcap_j V_j)/\sum_{j=1}^{|n|} \mathcal{O}^{\natural}(\bigcap_{i \neq j} V_i).$$

At last we will realize partial modified Fourier analytic functionals with certain compact carrier as (relative) cohomology classes with coefficients in $\mathcal{O}^{\natural}$.

Assume $|n| \geqq 2$. Let X be as in Theorem 9.5.1 and K a compact set in X of the form $K = K_1 \times \cdots \times K_{|n|}$ with compact sets K_j in $\boldsymbol{C}$ for $j = 1, 2, \ldots, n_1$ and in $\boldsymbol{E}$ for $j = n_1 + 1, \ldots, |n|$. Assume that K admits a fundamental system of $\mathcal{O}^{\natural}$-pseudoconvex open neighborhoods. Then we have

$$H^p(K, \mathcal{O}_{\natural}) = 0 \text{ for each } p > 0.$$

By virtue of the Martineau-Harvey Theorem, there exists the algebraic isomorphism

$$\mathcal{O}_{\natural}(K)' \cong H^{|n|}_K(\Omega, \mathcal{O}^{\natural}).$$

Here Ω denotes an open neighborhood of K. Further assume that there exists an $\mathcal{O}^{\natural}$-pseudoconvex open neighborhood Ω of K such that

$$\Omega_j = \Omega \backslash \{z \in \boldsymbol{C}^{|n|}; z_j \in K_j \cap \boldsymbol{C}\}^a$$

is also an $\mathcal{O}^{\natural}$-pseudoconvex open set for $j = 1, 2, \ldots, |n|$. Put $\Omega_0 = \Omega$. Then $\mathfrak{U} = \{\Omega_0, \Omega_1, \ldots, \Omega_{|n|}\}$ and $\mathfrak{U}' = \{\Omega_1, \Omega_2, \ldots, \Omega_{|n|}\}$ form acyclic coverings of Ω and $\Omega \backslash K$. Set

$$\Omega \# K = \bigcap_{j=1}^{|n|} \Omega_j, \quad \Omega^j = \bigcap_{i \neq j} \Omega_i.$$

Let $\sum_j \mathcal{O}^{\natural}(\Omega^j)$ be the image in $\mathcal{O}^{\natural}(\Omega \# K)$ of $\prod_{j=1}^{|n|} \mathcal{O}^{\natural}(\Omega^j)$ by the mapping

$$(f_j)_{j=1}^{|n|} \longrightarrow \sum_{j=1}^{|n|} (-1)^{j+1} f'_j,$$

where f'_j denotes the restriction of f_j to $\Omega \# K$.

Then, by a similar way to that of Theorem 9.6.2, we have the following

Theorem 9.6.3. *Assume $|n| \geqq 2$. We use the notations as above. Then we have the algebraic isomorphisms*

$$\mathcal{O}_{\natural}(K)' \cong H_K^{|n|}(\Omega, \mathcal{O}^{\natural}) \cong H^{|n|}(\mathfrak{U}, \mathfrak{U}', \mathcal{O}^{\natural})$$

$$\cong \mathcal{O}^{\natural}(\Omega \# K) / \sum_j \mathcal{O}^{\natural}(\Omega^j).$$

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