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The $D(O(n))$-Structures in Tangent Bundles

By

Yoshihiro ICHIJYŌ

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As for an n-dimensional C^∞-manifold M, the following theorem is very famous: "*M admits an $O(n)$-structure if and only if M is a Riemann manifold.*"

Corresponding to this theorem, the present author has obtained in the paper [11] the following:

"*The tangent bundle $T(M)$ admits an $F(n)$-structure depending on $\mathscr{C}_0$ if and only if M admits a generalized metric and a certain quantity α.*"

"*M is a Finsler manifold if and only if the tangent boundle $T(M)$ admits the above mentioned structure satisfying $d\Omega = 0$.*"

In the above, $F(n)$ is the linear Lie group such that

$$F(n) = \left\{ \begin{pmatrix} A & , & 0 \\ SA & , & A \end{pmatrix} \middle| A \in O(n),\ S \in \mathrm{Symm}(n) \right\},$$

$\mathscr{C}_0$ is the homogeneous standard tangent structure of $T(M)$, and Ω is a differential 2-form associated with an almost symplectic structure induced from the $F(n)$-structure.

Moreover, in the paper [13], the properties of the almost symplectic structure and almost Hamilton vectors, which are derived from Ω in $T(M)$, have been studied in detail. It is, however, to be noted that these arguments do not always depend upon the choice of Finsler connections in $T(M)$.

On the other hand, the following propositions are well-known:

M is a Riemann manifold satisfying $R^h_{ijk} = 0$	$\rightleftarrows$	*M is locally Euclidean*	$\rightleftarrows$	*M admits an integrable $O(n)$-structure*

Corresponding to these, we will consider

M is a Finsler manifold satisfying $R^h_{ijk} = 0$, $C^h_{ij\|k} = 0$	$\rightleftarrows$	*M is locally Minkowskian*	$\rightleftarrows$	*$T(M)$ admits some G-structure*

The left hand side of these propositions is the well-known theorem. However, the right hand side is unknown yet. The main purpose of the present paper is to show the right hand side of these propositions holds true under some condition. Now, the theorem of the left hand side depends upon curvatures. Hence, it is anticipated that we can not help dealing with, at least, a non-linear connection in our argument. If $T(M)$ admits a non-linear connection, $T(M)$ admits an almost product structure. Therefore, instead of $F(n)$, we should be concerned with a $D(O(n))$-structure in $T(M)$, where

$$D(O(n))=\left\{\begin{pmatrix} A & , & 0 \\ 0 & , & A \end{pmatrix} \mid A\in O(n)\right\}.$$

The final result will be shown as Theorem 11.

§1. The $D(GL(n, R))$-structures depending on $\mathscr{C}_0$

Let M be an n-dimensional connected C^∞-manifold and $T(M)$ be the tangent bundle over M. Since we shall be concerned, in the following, with Finsler metrics and non-linear connections, we ought to treat the open subbundle $T(M)$-M of $T(M)$, that is, the bundle consisting of all non-zero vectors tangent to M. From now on, for the sake of brevity, we denote it by the same notation $T(M)$ and simply call it the tangent bundle over M. As is well-known, $T(M)$ admits the homogeneous standard integrable almost tangent structure $\mathscr{C}_0$ ([1], [2], [5], [10], [11], [13]), whose structure group is the so-called tangent group given by

$$T_n=\left\{\begin{pmatrix} A & , & 0 \\ B & , & A \end{pmatrix} \mid A\in GL(n, R),\ B\in gl(n, R)\right\}.$$

Now, let $\pi: T(M)\to M$ be the natural projection, $\{(U, x^i)\}$[1] be a system of coordinate neighbourhoods of M, then, for each (U, x^i), there exists in $T(M)$ the caninical coordinate neighbourhood $(\pi^{-1}(U), (x^i, y^i))$. In each $\pi^{-1}(U)$, the natural frame $\{\partial/\partial x^i, \partial/\partial y^i\}$ (which will be denoted simply by $\{\partial/\partial x^A\}$ hereafter) is an adapted frame of the G-structure $\mathscr{C}_0$. On the other hand, there exists, on $T(M)$, a (1,1) tensor field Q such that $Q^2=0$, which is called the structure tensor of $\mathscr{C}_0$. The components of Q with respect to (x^A) are, as is well-known, given by $Q=\begin{pmatrix} 0 & , & 0 \\ E_n & , & 0 \end{pmatrix}$. The above mentioned tangent group T_n is rewritten as $T_n=\{T \mid T\in GL(2n, R),\ TQ=QT\}$.

1) Throughout the paper, we use the following indices and notation:
$A, B, C,\dots, P, Q, R,\dots$ run over the range $\{1,2,3,\dots, 2n\}$;
$a, b, c,\dots, i, j, k,\dots$ run oer the range $\{1,2,3,\dots, n\}$;
$\bar{a}, \bar{b},\dots, \bar{i}, \bar{j},\dots$ stand for $a+n, b+n,\dots, i+n, j+n,\dots$ respectively;
With respect to any canonical coordinate system in a tangent bundle, $(x^A)=(x^a, x^{\bar{a}})=(x^a, y^a)$, i.e., $x^{\bar{a}}=y^a$, and the natation ∂_i and $\dot{\partial}_i$ stand for $\partial/\partial x^i$ and $\partial/\partial y^i$ respectively.

Now we suppose that G is a Lie subgroup of T_n and $T(M)$ admits the G-structure as a reduction of $\mathscr{C}_0$, which is called the G-structure depending on $\mathscr{C}_0$([3], [11], [13]). It is easy to show that the condition for $T(M)$ to admit a G-structure depending on $\mathscr{C}_0$ is given by

(1) G is a Lie subgroup of T_n.

(2) $T(M)$ admits the G-structure, i.e., M is covered by be a system of local coordinate neighbourhoods $\{(U_\alpha, x^i_\alpha)\}$ such that each $\pi^{-1}(U_\alpha)$ admits a $2n$-frame $\{Z^{(\alpha)}_A\}$ satisfying the relation $Z^{(\alpha)}_A = P^B_A Z^{(\beta)}_B$ where $(P^B_A)\in G$ in $\pi^{-1}(U_\alpha)\cap\pi^{-1}(U_\beta)$ if $U_\alpha\cap U_\beta\neq\phi$. ($\{Z^{(\alpha)}_A\}$ is called an adapted frame of the G-structure in $\pi^{-1}(U_\alpha)$).

(3) The adapted frame $\{Z_A\}$ can be written in $(\pi^{-1}(U), x^A)$ as $Z_a = \gamma^i_a(x, y)\, \partial/\partial x^i + \sigma^i_a(x, y)\, \partial/\partial y^i$, $Z_{\bar a} = \gamma^i_a(x, y)\, \partial/\partial y^i$ where $\det|\gamma^i_a(x, y)|\neq 0$, $\gamma^i_a(x, y)$ is $(0)p$-homogeneous for y and $\sigma^i_a(x, y)$ is $(1)p$-homogeneous for y.

The homogeneity condition in (3) can be rewritten as $d\bar\lambda(Z_a)_{(x,y)} = (Z_a)_{(x,\lambda y)}$ where $d\bar\lambda$ is the differential of the mapping $\bar\lambda: T(M)\to T(M)((x, y)\to(x, \lambda y))$, λ being any positive number.

Now we assume that $T(M)$ admits a G-structure $\mathscr{C}$ which is depending on $\mathscr{C}_0$. If M is covered by a system of local coordinate neighbourhoods $\{(U, x^i)\}$ such that the natural frame $\{\partial/\partial x^A\}$ of the canonical coordinate neighbourhood $(\pi^{-1}(U), x^A)$ for each (U, x^i) is adapted to the structure $\mathscr{C}$, then the G-structure $\mathscr{C}$, which is depending on $\mathscr{C}_0$, is called integrable.

Putting

$$D(GL(n, R)) = \left\{\begin{pmatrix} A & , & 0 \\ 0 & , & A\end{pmatrix} \Big| A\in GL(n, R)\right\},$$

we see $D(GL(n, R))$ is a Lie subgroup of T_n. In this section, we treat the case where $T(M)$ admits a $D(GL(n, R))$-structure depending on $\mathscr{C}_0$, and denote it simply by $\mathscr{C}_1$.

If we put $P_0 = \begin{pmatrix} E_n & , & 0 \\ 0 & , & -E_n\end{pmatrix}$, then we see that $TP_0 = P_0 T$ holds for any $T\in D(GL(n, R))$. Hence, if $T(M)$ admits a structure $\mathscr{C}_1$ then it also admits an almost product structure. Let $\{Z_A\}$ be an adapted frame of $\mathscr{C}_1$ in each $(\pi^{-1}(U), x^A)$, and let us put $Z_A = \gamma^B_A\partial_B$, then $\Gamma = (\gamma^B_A)$ is written as

$$\Gamma = (\gamma^B_A) = \begin{pmatrix} \gamma^i_a & , & 0 \\ \sigma^i_a & , & \gamma^i_a\end{pmatrix}.$$

Putting $\gamma = (\gamma^i_a)$ and $\sigma = (\sigma^i_a)$, we see $\Gamma^{-1} = \begin{pmatrix} \gamma^{-1} & , & 0 \\ -\gamma^{-1}\sigma\gamma^{-1} & , & \gamma^{-1}\end{pmatrix}$. Now $P = \Gamma P_0\Gamma^{-1} = \begin{pmatrix} E_n & , & 0 \\ 2\sigma\gamma^{-1} & , & -E_n\end{pmatrix}$ satisfies $P^2 = E_{2n}$ and becomes a globally defined (1, 1)-tensor field on $T(M)$, i.e., P is the almost product tensor field associated with the given almost product structure [3].

Putting $N=(N^i_j)=-\sigma\gamma^{-1}$, we see, as is well-known, N is a non-linear connection defined on $T(M)$. Of course, N^i_j is $(1)p$-homogeneous for y. Now we show

Theorem 1. *A tangent bundle $T(M)$ admits a structure $\mathscr{C}_1$ (namely, a $D(GL(n, R))$-structure depending on $\mathscr{C}_0$) if and only if the underlying manifold M admits a non-linear connection.*

PROOF. The necessity is shown already. So, we show the condition is sufficient. In each $(\pi^{-1}(U), x^A)$, let us put $X_i=\partial/\partial x^i - N^m_i\,\partial/\partial y^m$ and $X_{\bar{i}}=Y_i=\partial/\partial y^i$, then $\{X_A\}$ is a $2n$-frame in each $\pi^{-1}(U)$, which we call the N-frame hereafter. Let $\{\bar{X}_A\}$ be the N-frame in $(\pi^{-1}(\bar{U}), \bar{x}^A)$. If $U\cap\bar{U}\neq\phi$, it is easy to see that $\bar{Y}_m\frac{\partial\bar{x}^m}{\partial x^i}=Y_i$ and $\bar{X}_m\frac{\partial\bar{x}^m}{\partial x^i}=X_i$ in $\pi^{-1}(U)\cap\pi^{-1}(\bar{U})$. So, if we put $X_A=T^B_A\bar{X}_B$, we have $(T^B_A)=\begin{pmatrix}\frac{\partial\bar{x}^i}{\partial x^j}, & 0\\ 0, & \frac{\partial\bar{x}^i}{\partial x^j}\end{pmatrix}\in D(GL(n, R))$. Since N^i_j is $(1)p$-homogeneous for y, we see that $T(M)$ admits a structrue $\mathscr{C}_1$ and the N-frame is an adapted frame of the structure $\mathscr{C}_1$ in each $(\pi^{-1}(U), x^A)$. Q.E.D.

Next, if we put $J_0=\begin{pmatrix}0, & -E_n\\ E_n, & 0\end{pmatrix}$, then we see directly that $TJ_0=J_0T$ holds for any $T\in D(GL(n, R))$. This means that $D(GL(n, R))\subset GL(n, C)$, namely, $T(M)$ admits an almost complex structure if $T(M)$ admits a structure $\mathscr{C}_1$. The almost complex structure tensor F associated with this structure is given by $F=\Gamma J_0\Gamma^{-1}$. Of course, F is a globally defined (1,1)-tensor field on $T(M)$ satisfying $F^2=-E_{2n}$[3]. The components of F with respect to the canonical coordinate $\{x^A\}$ is given by ([7], [14], [15], [19], [20])

$$F=\Gamma J_0\Gamma^{-1}=\begin{pmatrix}-N, & -E_n\\ E_n+N^2, & N\end{pmatrix}.$$

Moreover, the components of F with respect to the N-frame $\{X_A\}$ are given by J_0 itself and also F satisfies $F(X_i)=Y_i$, $F(Y_i)=-X_i$. According to Matsumoto [16], this almost complex structure is called the almost complex N structure. Now we show

Theorem 2. *In order that a tangent bundle $T(M)$ admits an integrable $D(GL(n, R))$-structure depending on $\mathscr{C}_0$, it is necessary and sufficient that the underlying manifold M is locally affine.*

PROOF. If $T(M)$ admits the structure $\mathscr{C}_1$ which is integrable, then M is covered by a system of local coordinate neighborhoods $\{(U, x^i)\}$ such that the natural frame $\{\partial/\partial x^A\}$ of each $(\pi^{-1}(U), x^A)$ is adapted to the structure $\mathscr{C}_1$. On the other hand, the N-frame is also an adapted frame of the structure $\mathscr{C}_1$ in $(\pi^{-1}(U), x^A)$. So, we have $\partial/\partial x^A = T_A^B X_B$ where $(T_A^B) \in D(GL(n, R))$. That is, $\partial/\partial x^i = T_i^m(\partial/\partial x^m - N_m^r \partial/\partial y^r)$ and $\partial/\partial y^i = T_i^m \partial/\partial y^m$. These yield $T_i^m = \delta_i^m$ and $N_j^i = 0$. Next, let $\{\bar{U}, \bar{x}^i\}$ be another coordinate neighbourhood satisfying the above. If $U \cap \bar{U} \neq \phi$, then, in $\pi^{-1}(U) \cap \pi^{-1}(\bar{U})$, the relations

$$\bar{N}_r^i \frac{\partial \bar{x}^r}{\partial x^j} = \frac{\partial \bar{x}^i}{\partial x^r} N_j^r - \frac{\partial^2 \bar{x}^i}{\partial x^j \partial x^m} y^m, \ N_j^i = 0 \text{ and } \bar{N}_j^i = 0$$

hold, and these lead us to $\dfrac{\partial^2 \bar{x}^i}{\partial x^j \partial x^k} = 0$, that is, M is locally affie.

Conversely, if M is locally affine, there globally exists a flat affine connection $\Gamma_{jk}^i(x)$ on M. Then $T(M)$ is endowed with a non-linear connection such as $N_j^i = \Gamma_{mj}^i(x) y^m$. Owing to Theorem 1, $T(M)$, therefore, admits a $D(GL(n, R))$-structure depending on $\mathscr{C}_0$, i.e., a structure $\mathscr{C}_1$, whose N-frame is adapted to $\mathscr{C}_1$. Since $\Gamma_{jk}^i(x)$ is a global flat affine connection on M, M is covered by a system of local coordinate neighbourhoods $\{(U, x^i)\}$ such that $\Gamma_{jk}^i(x) = 0$ holds in each U. Then, in each U, $N_j^i = 0$ holds, from which we have $X_i = \partial/\partial x^i$ and $Y_i = \partial/\partial y^i$. Namely, the canonical natural frame $\{\partial/\partial x^A\}$ is adapted to $\mathscr{C}_1$ in each $(\pi^{-1}(U), x^A)$.

Q.E.D.

§2. Holonomy mapping associated with a non-linear connection

Let us assume, in this section, that the tangent bundle $T(M)$ admits the structure $\mathscr{C}_1$. That is to say, a non-linear connection N is defined on $T(M)$. In the papers ([8], [9], [12]), the notion of holonomy mappings associated with a non-linear connection has been introduced and some results concerning them have been obtained. For the later use, we shall give a brief sketch of them.

Let p and q be arbitrary two points of M endowed with a non-linear connection N, and C be any piecewise differentiable curve joining the two points p and q. Let $\tilde{C}$ be a horizontal lift of C to the tangent bundle $T(M)$, that is, a curve in $T(M)$ represented as $(x(t), y(t))$ satisfying locally $\dfrac{dy^i}{dt} + N_m^i(x(t), y(t))\dfrac{dx^m}{dt} = 0$, where we denote by $x(t)$ the curve C and by $x(0)$ the point p and by $x(1)$ the point q. Denote by $T_p(M)$ and $T_q(M)$ the fibre spaces over the points p and q respectively. For each point $(x(0), y)$ in $T_p(M)$, there exists one and only one horizontal lift $\tilde{C}$ of C passing through the point $(x(0), y)$. And the curve $\tilde{C}$ passes $T_q(M)$ by one point, which we denote by $(x(1), \bar{y})$. Then the correspondence $y \to \bar{y}$ defines a bijective differentiable mapping $\psi: T_p(M) \to T_q(M)$. We call the mapping

ψ a holonomy mapping from $T_p(M)$ to $T_q(M)$ along the curve C with respect to the non-linear connection N.

Let $S(x, y)$ be a quasi tensor field[1]. That is, if $S(x, y)$ be, for example, of (1,1)-type, S is written as $S = S^i_j(x, y)\frac{\partial}{\partial x^i} \otimes dx^j$ in each coordinate neighbourhood (U, x^i) of M. Of course, the components $S^i_j(x, y)$ are C^∞ functions in $(\pi^{-1}(U), x^A)$ and satisfy $\bar{S}^i_j(\bar{x}, \bar{y}) = S^r_s(x, y)\frac{\partial \bar{x}^i}{\partial x^r}\frac{\partial x^s}{\partial \bar{x}^j}$ in $U \cap \bar{U}(\neq \phi)$ where $\bar{S}^i_j(\bar{x}, \bar{y})$ are the componentes of S in $(\bar{U}, \bar{x}^i)$. Now, for any point $p \in U$, $S^i_j(x, y)\frac{\partial}{\partial y^i} \otimes dy^j$ can be regarded as a tensor field on the space $T_p(M)(\subset \pi^{-1}(U))$, because $\frac{\partial \bar{y}^i}{\partial y^j} = \frac{\partial \bar{x}^i}{\partial x^j}$ holds in $\pi^{-1}(U) \cap \pi^{-1}(\bar{U})$. We shall denote this tensor by $S^*(p, y)$ or $S^*(x, y)$ and call it a lifted tensor on $T_p(M)$. It is, however, to be noted that the lifted tensor S^* is neither the so-called vertical lift of S nor the horizontal lift of S, and moreover it is not a tensor field on $T(M)$. It is only the tensor field on $T_p(M)$.

Next, we denote by $\overset{N}{\nabla}$ the h-covariant derivative with respect to the non-linear connection N, that is, for any quasi tensor field T, for example, of (1,1)-type,

$$\overset{N}{\nabla}_k T^i_j = \partial_k T^i_j - \dot{\partial}_m T^i_j N^m_k + T^m_j \dot{\partial}_m N^i_k - T^i_m \dot{\partial}_j N^m_k.$$

Concerning these notions, we have obtained ([9], [12])

Theorem 3. *Let M be a manifold admitting a non-linear connection N, and let $S(x, y)$ be a quasi tensor field defined on M. Let p and q be arbitrary two points in M. In order that any holonomy mapping from $T_p(M)$ to $T_q(M)$ with respect to N always transfers the lifted tensor field $S^*(p, y)$ on $T_p(M)$ to the lifted tensor field $S^*(q, y)$ on $T_q(M)$ for any curve joining the two points p and q, it is necessary and sufficient that $\overset{N}{\nabla} S = 0$ holds good.*

Let us assume moreover that M is a Finsler manifold and g be the Finsler metric tensor. For any point $p \in M$, the lifted tensor $g^*(p, y)$ gives $T_p(M)$ a Riemann metric. Now, let p and q be arbitrary two points in M. If any holonomy mapping from $T_p(M)$ to $T_q(M)$ which is associated with a non-linear connection N is always an isometry from the Riemann space $\{T_p(M), g^*(p, y)\}$ to the Riemann space $\{T_q(M), g^*(q, y)\}$, then the non-linear connection N is called *metrical*. The condition for N to be metrcal is written as $\overset{N}{\nabla} g = 0$. Let $\overset{*}{\Gamma}{}^i_{jk}$ be the

1) A quasi tensor is a so-called tensor of Finsler type. However, to avoid the confusion with a Finsler metric tensor, we shall adopt the terminology "*quasi tensor*".

Finsler connection defined by E. Cartan, and G_j^i be the non-linear connection given by $G_j^i = \overset{*}{\Gamma}{}^i_{mj} y^m$, which is called the Cartan's non-linear connection hereafter. With respect to these, the following results have been obtained [9].

Theorem 4. *Let M be a Finsler manifold. Any holonomy mapping associated with the Cartan's non-linear connection is always metrical, if and only if the given Finsler metric is a Landsberg metric.*

Theorem 5. *Let M be a Finsler manifold whose metric tensor is g, and let C be a quasi tensor field defined by* $C_{jk}^i = \frac{1}{2} g^{im} \dot{\partial}_m g_{jk}$. *Any holonomy mapping associated with the Cartan's non-linear connection leaves the lifted tensor* C^* *invariant if and only if M is a Berwald space.*

§3. The $D(O(n))$-structure depending on $\mathscr{C}_0$

Putting

$$D(O(n)) = \left\{ \begin{pmatrix} A & , & 0 \\ 0 & , & A \end{pmatrix} \mid A \in O(n) \right\},$$

we see that $D(O(n))$ is a Lie subgroup of T_n. In this section, we assume that the tangent bundle admits a $D(O(n))$-structure depending on $\mathscr{C}_0$, and denote it by $\mathscr{C}_2$.

Since $D(O(n)) \subset D(GL(n, R))$, if $T(M)$ admits a structure $\mathscr{C}_2$, then it admits a structure $\mathscr{C}_1$, that is to say, $T(M)$ admits a non-linear connection N. Moreover, since $D(O(n)) \subset O(2n)$, $T(M)$ admits also a positive definite Riemann metric G. To be precisely, let $(\pi^{-1}(U), x^A)$ be any canonical coordinate neighbourhood attached to the structure $\mathscr{C}_2$ in $T(M)$, and let $\{Z_A\}$ be an adapted frame of $\mathscr{C}_2$ in $\pi^{-1}(U)$, then

$$Z_a = \gamma_a^i \partial/\partial x^i + \sigma_a^i \partial/\partial y^i, \ Z_{\bar{a}} = \gamma_a^i \partial/\partial y^i$$

where $\det|\gamma_a^i| \neq 0$, γ_a^i are (0)p-homogeneous for y and σ_a^i are (1)p-homogeneous for y. The non-linear connection N is also given by $N = -\sigma\gamma^{-1}$. For any point $P \in \pi^{-1}(U)$, we can define an inner product in $T_P(T(M))$ by $\langle Z_A, Z_B \rangle_{\pi^{-1}(U)} = \delta_{AB}$. It is easy to show that this inner product is globally well-defined on $T(M)$. Let $\{X_A\}$ be the N-frame associated with the given non-linear connection N and let us denote $\gamma^{-1} = \beta = (\beta_i^a)$. Then β_i^a are (0)p-homogeneous for y. Now, it is easy to see that

$$X_i = \beta_i^a Z_a, \ Y_i = \beta_i^a Z_{\bar{a}},$$

$$\langle X_i, X_j \rangle = g_{ij}, \ \langle X_i, Y_j \rangle = 0, \ \langle Y_i, Y_j \rangle = g_{ij},$$

where we put $g_{ij}=\sum_{a=1}^{n}\beta_i^a\beta_j^a$. Obviously, g_{ij} are (0)p-homogeneous for y, and g_{ij} give M a global quasi tansor field g of (0, 2)-type and $g_{ij}\xi^i\xi^i$ is a positive definite quadratic form. Thus we obtain that g is a generalized metric in Moór's sense[18], namely, a homogeneous positive definite generalized Finsler metric ([6], [17]). For the sake of brevity, we call g a Moór metric. Now we show

Theorem 6. *A tangent bundle $T(M)$ admits a structure $\mathscr{C}_2$(i.e., a $D(O(n))$-structure depending on $\mathscr{C}_0$), if and only if the underlying manifold M admits a non-linear connection and a Moór metric.*

PROOF. It is shown already that the condition is necessary. Now we show the condition is sufficient. In a local coordinate neighbourhood (U, x^i) in M, let $g_{ij}(x, y)$ and $N_j^i(x, y)$ be the components of the Moór metric g and the non-linear connection N respectively. Since $g_{ij}(x, y)$ is positive definite and symmetric for i and j, there exists n linearly independent local quasi vector fields $\gamma_a^i(x, y)(a=1,2,\dots,n)$ in U such that $g_{ij}(x, y)\gamma_a^i(x, y)\gamma_b^j(x, y)=\delta_{ab}$. Of course, $\gamma_a^i(x, y)$ are (0)p-homogeneous for y. If we put $\gamma=(\gamma_a^i)$ and $\gamma^{-1}=\beta=(\beta_i^a)$, we have $g=(g_{ij})={}^t\beta\beta$. Let $\{X_A\}$ be the N-frame in $\pi^{-1}(U)$, and put $Z_a=\gamma_a^iX_i$, $Z_{\bar{a}}=\gamma_a^iY_i$. Then, $\{Z_A\}$ is a $2n$-frame in $\pi^{-1}\{U\}$. Moreover we can define an inner product $\langle\,,\rangle$ in $\pi^{-1}(U)$ by

$$\langle X_i, X_j\rangle_{\pi^{-1}(U)}=g_{ij}(x, y),\ \langle X_i, Y_j\rangle_{\pi^{-1}(U)}=0,$$
$$\langle Y_i, Y_j\rangle_{\pi^{-1}(U)}=g_{ij}(x, y).$$

Then we have $\langle Z_a, Z_b\rangle_{\pi^{-1}(U)}=\gamma_a^i(x, y)\gamma_b^j(x, y)\langle X_i, X_j\rangle_{\pi^{-1}(U)}=\delta_{ab}$. Similarly, we have $\langle Z_a, Z_{\bar{b}}\rangle_{\pi^{-1}(U)}=0$ and $\langle Z_{\bar{a}}, Z_{\bar{b}}\rangle_{\pi^{-1}(U)}=\delta_{ab}$. That is, we have $\langle Z_A, Z_B\rangle_{\pi^{-1}(U)}=\delta_{AB}$. Let $(\bar{U}, \bar{x}^i)$ be another local coordinate neighbourhood in M. Then we can define similarly the N-frame$\{\bar{X}_A\}$ and $2n$-frame $\{\bar{Z}_A\}$ in $\pi^{-1}(\bar{U})$. And also, by the same procedure, we can define an inner product in $\pi^{-1}(\bar{U})$ and obtain $\langle\bar{Z}_A, \bar{Z}_B\rangle_{\pi^{-1}(\bar{U})}=\delta_{AB}$. Here, we assume $U\cap\bar{U}\neq\phi$, then, in $\pi^{-1}(U)\cap\pi^{-1}(\bar{U})$, we have

$$\langle X_i, X_j\rangle_{\pi^{-1}(\bar{U})}=\langle\frac{\partial\bar{x}^m}{\partial x^i}\bar{X}_m, \frac{\partial\bar{x}^r}{\partial x^j}\bar{X}_r\rangle_{\pi^{-1}(\bar{U})}=\frac{\partial\bar{x}^m}{\partial x^i}\frac{\partial\bar{x}^r}{\partial x^j}\bar{g}_{mr}(\bar{x}, \bar{y})$$
$$=g_{ij}(x, y)=\langle X_i, X_j\rangle_{\pi^{-1}(U)}.$$

Similarly we have $\langle X_i, Y_j\rangle_{\pi^{-1}(\bar{U})}=\langle X_i, Y_j\rangle_{\pi^{-1}(U)}$ and $\langle Y_i, Y_j\rangle_{\pi^{-1}(\bar{U})}=\langle Y_i, Y_j\rangle_{\pi^{-1}(U)}$. Namely, thus defined local inner product in each $\pi^{-1}(U)$ is the inner product which is defined globally on $T(M)$. Next, it is evident, in $\pi^{-1}(U)\cap\pi^{-1}(\bar{U})$, that $\bar{Z}_A=P_A^CZ_c$ where $(P_A^C)\in GL(2n, R)$ holds. Since the inner product is the global one on $T(M)$, it follows that

$$\delta_{AB} = \langle \bar{Z}_A, \bar{Z}_B \rangle = \langle P_A^C Z_C, P_B^D Z_D \rangle = \sum_{C=1}^{2n} P_A^C P_B^C,$$

i.e., $(P_A^C) \in O(2n)$. On the other hand, it is seen

$$\bar{Z}_a = \bar{\gamma}_a^m \bar{X}_m = \bar{\gamma}_a^m \frac{\partial x^r}{\partial \bar{x}^m} X_r = \bar{\gamma}_a^m \frac{\partial x^r}{\partial \bar{x}^m} \beta_r^b Z_b,$$

$$\bar{Z}_{\bar{a}} = \bar{\gamma}_a^m \bar{Y}_m = \bar{\gamma}_a^m \frac{\partial x^r}{\partial \bar{x}^m} Y_r = \bar{\gamma}_a^m \frac{\partial x^r}{\partial \bar{x}^m} \beta_r^b Z_{\bar{b}},$$

i.e., $(P_A^C) \in D(GL(n, R))$. Thus we have $(P_A^C) \in D(O(n))$. Of course, we have

$$Z_a = \gamma_a^i \partial/\partial y^i - N_m^i \gamma_a^m \partial/\partial y^i, \ Z_{\bar{a}} = \gamma_a^i \partial/\partial y^i,$$

where γ_a^i are (0)p-homogeneous for y and $-N_m^i \gamma_a^m$ are (1)p-homogeneous for y. Summerizing up all the above, we obtain that $T(M)$ admits a structure $\mathscr{C}_2$.
Q.E.D.

If $T(M)$ admits a structure $\mathscr{C}_2$, it also admits a structure $\mathscr{C}_1$. So, $T(M)$ admits an almost product structure. Let P be the almost product tensor and G be the Riemann metric tensor on $T(M)$ which is derived from the structure $\mathscr{C}_2$.

With respect to the N-frame, P and G have the components as

$$P = \begin{pmatrix} E_n & , & 0 \\ 0 & , & -E_n \end{pmatrix}, \ G = \begin{pmatrix} g & , & 0 \\ 0 & , & g \end{pmatrix}$$

respectively. Then, it is easy to verify that ${}^tPGP = G$ holds. This tells us that P and G construct an almost product metric structure.

As is already seen, the components of the almost complex N structure F with respect to the N-frame is written as $F = J_0$. So, it is seen directly that ${}^tFGF = G$ holds good. This tells us that F and G construct an almost Hermit structure.

As is well-known, on putting $\omega = GF$, ω is a skew-symmetric non-degenerate tensor field of (0, 2)-type. So, we can consider the differential 2-form Ω which is determined by ω. This is the differential 2-form associated with the almost symplectic structure which is induced from the almost Hermit structure $\{F, G\}$ ([4], [20]).

Now we write down the components of ω with respect to the canonical coordinate (x^i, y^i). First, direct calculation gives us that

$$\langle \partial/\partial y^i, \partial/\partial y^j \rangle = g_{ij}, \ \langle \partial/\partial x^i, \partial/\partial y^j \rangle = g_{jm} N_i^m,$$

$$\langle \partial/\partial x^i, \partial/\partial x^j \rangle = g_{ij} + N_i^m N_j^r g_{mr}.$$

Hence we get $$G = \begin{pmatrix} g + {}^tNgN & , & {}^tNg \\ gN & , & g \end{pmatrix}.$$

On the other hand, in §1, we have

$$F=\begin{pmatrix} -N & , & -E_n \\ E_n+N^2 & , & N \end{pmatrix}.$$

From these, we get

$$\omega=(\omega_{AB})=\begin{pmatrix} -gN+{}^tNg & , & -g \\ g & , & 0 \end{pmatrix}.$$

Therefore we obtain

$$\Omega=-(g_{im}N_j^m-g_{jm}N_i^m)dx^i{}_\wedge dx^j-2g_{ij}dx^i{}_\wedge dy^j.$$

Remark. This Ω essentially coincides with the differential 2-form associated with the almost Finsler structure ([11], [13]).

If the differential 2-form associated with an almost Hermit structure $\{F, G\}$ is closed, the structure $\{F, G\}$ is called an almost Kähler structure. Now we show

Theorem 7. *Let $T(M)$ be a tangent bundle admitting a structure $\mathscr{C}_2$. In order that the induced almost Hermit structure $\{F, G\}$ from the structure $\mathscr{C}_2$ is almost Kählerian, it is necessary and sufficient that the following hold good:*

(1) *The Moór metric g is a Finsler metric,*

(2) *The non-linear connection N satisfies*

$$y^m(\overset{N}{\nabla}_i g_{mj}-\overset{N}{\nabla}_j g_{mi})=y^m g_{mr}(\dot{\partial}_i N_j^r-\dot{\partial}_j N_i^r).$$

PROOF. Since Ω can be written as

$$\Omega=2(g_{jr}N_i^r dx^i_\wedge dx^j-g_{ij}dx^i_\wedge dy^j),$$

we have

$$\begin{aligned} d\Omega=2\{&\partial_k(g_{jr}N_i^r)dx^k_\wedge dx^i_\wedge dx^j \\ &+(\dot{\partial}_k(g_{jr}N_i^r)+\partial_j g_{ik})dy^k_\wedge dx^i_\wedge dx^j-\dot{\partial}_k g_{ij}dy^k_\wedge dx^i_\wedge dy^j\}. \end{aligned}$$

Therefore, the condition $d\Omega=0$ is written as

$$\begin{cases} \text{(a)} & \partial_k(g_{jr}N_i^r)+\partial_j(g_{ir}N_k^r)+\partial_i(g_{kr}N_j^r) \\ & \qquad -\partial_k(g_{ir}N_j^r)-\partial_j(g_{kr}N_i^r)-\partial_i(g_{jr}N_k^r)=0, \\ \text{(b)} & \dot{\partial}_k(g_{jr}N_i^r)+\partial_j g_{ik}-\dot{\partial}_k(g_{ir}N_j^r)-\partial_i g_{jk}=0, \\ \text{(c)} & \dot{\partial}_k g_{ij}-\dot{\partial}_j g_{ik}=0. \end{cases}$$

First, it is well-known that (c) is the condition for g_{ij} to be a Finsler metric. Contracting y^k to the equation (b), we have

$$(*)\qquad y^m(\partial_i g_{jm} - \partial_j g_{im}) = g_{jr} N^r_i - g_{ir} N^r_j.$$

Using this and the fact that g_{ij} is a Finsler metric, we can easily verify that the condition (2) is satisfied.

Conversely, if the condition (1) and (2) are satisfied, the condition (c) is satisfied evidently and the condition (2) is rewritten as the equation (*). Differentiating the equation (*) by y^k, we can see that the condition (b) is satisfied. Finally, using the equation (*), it follows that

the left hand side of (a)

$$= \partial_k(g_{jr}N^r_i - g_{ir}N^r_j) + \partial_j(g_{ir}N^r_k - g_{kr}N^r_i) + \partial_i(g_{kr}N^r_j - g_{jr}N^r_k)$$
$$= \partial_k\{y^m(\partial_i g_{jm} - \partial_j g_{im})\} + \partial_j\{y^m(\partial_k g_{im} - \partial_i g_{km})\} + \partial_i\{y^m(\partial_i g_{km} - \partial_k g_{jm})\}$$
$$= 0.$$

Thus the condition (a), (b) and (c) are all satisfied, namely, $d\Omega = 0$ holds good. Q.E.D.

In the case of Theorem 7, Ω can be rewritten as $\Omega = d(2y^m g_{mj} dx^j)$. This is the well-known differential 2-form defined on a tangent bundle over a Finsler manifold ([14], [20]).

Theorem 8. *In order that a manifold M is a Finsler manifold, it is necessary and sufficient that the tangent bundle $T(M)$ admits a structure $\mathscr{C}_2$ whose induced almost Hermit structure $\{F, G\}$ is almost Kählerian.*

Proof. In order to prove this, according to Theorem 6 and 7, it is sufficient to show that a Finsler manifold M admits a non-linear connection such that the relation

$$y^m(\overset{N}{\nabla}_i g_{mj} - \overset{N}{\nabla}_j g_{im}) = y^m g_{mr}(\dot{\partial}_i N^r_j - \dot{\partial}_j N^r_i)$$

holds good. To do this, we show that the Cartan's non-linear connection G^i_j satisfies it. With respect to the Cartan's Finsler connection, the relations

$$\partial_i g_{jk} = \dot{\partial}_r g_{jk} G^r_i + g_{kr} \overset{*}{\Gamma}{}^r_{ji} + g_{jr} \overset{*}{\Gamma}{}^r_{ki},\ \dot{\partial}_j G^i_k = \dot{\partial}_k G^i_j$$

hold true. Then, it is easily seen that

$$y^m(\overset{G}{\nabla}_i g_{mj} - \overset{G}{\nabla}_j g_{im}) = y^m(\partial_i g_{mj} - \partial_j g_{mi} - g_{rj}\dot{\partial}_m G^r_i + g_{ri}\dot{\partial}_m G^r_j)$$
$$= y^m(\dot{\partial}_r g_{mj} G^r_i + g_{rj} \overset{*}{\Gamma}{}^r_{mi} + g_{mr} \overset{*}{\Gamma}{}^r_{ji} - \dot{\partial}_r g_{mi} G^r_j - g_{ri} \overset{*}{\Gamma}{}^r_{mj} - g_{mr} \overset{*}{\Gamma}{}^r_{ij})$$

$$- g_{rj}G^r_i + g_{ri}G^r_j$$
$$= 0.$$

And the relation $y^m g_{mr}(\dot{\partial}_i G^r_j - \dot{\partial}_j G^r_i) = 0$ is evident. Thus the proof is completed.

§4. Integrability conditions

We have considered, in Theorem 2, the condition for the structure $\mathscr{C}_1$ to be integrable. On the other hand, if $T(M)$ admits a structure $\mathscr{C}_1$, then $T(M)$ also admits an almost product structure and, at the same time, an almost complex structure. To consider the integrability conditions of them, we calculate the Nijenhuis tensor.

Let S be a tensor field of (1,1)-type. The Nijenhuis tensor N_S of S is given by

$$N_S(U, V) = [SU, SV] + S^2[U, V] - S[SU, V] - S[U, SV],$$

U, V being any vector fields. For the N-frame $\{X_A\}$, we have

$$[X_i, X_j] = R^h_{ij} Y_h,\ [X_i, Y_j] = \dot{\partial}_j N^h_i Y_h,\ [Y_i, Y_j] = 0,$$

where $R^h_{ij} = -\partial_i N^h_j + \partial_j N^h_i + \dot{\partial}_m N^h_j N^m_i - \dot{\partial}_m N^h_i N^m_j$.

First, we consider the almost product structure P which is induced from the structure $\mathscr{C}_1$ in $T(M)$. P satisfies $P(X_i) = X_i$ and $P(Y_i) = -Y_i$. Therefore the Nijenhuis tensor N_P has the relations

$$N_P(X_i, X_j) = 4R^h_{ij} Y_h,\ N_P(X_i, Y_j) = 0,\ N_P(Y_i, Y_j) = 0.$$

Hence, in a tangent bundle admitting a structure $\mathscr{C}_1$, the Nijenhuis tensor with respect to the almost product structure induced from $\mathscr{C}_1$ vanishes when and only when the non-linear connection derived from $\mathscr{C}_1$ satisfies $R^h_{ij} = 0$.

Next, we consider the almost complex N structure F. F satisfies $F(X_i) = Y_i$, $F(Y_i) = -X_i$. Hence the Nijenhuis tensor N_F has the relations

$$N_F(X_i, X_j) = -R^h_{ij} Y_h - (\dot{\partial}_i N^h_j - \dot{\partial}_j N^h_i) X_h,$$
$$N_F(X_i, Y_j) = (\dot{\partial}_i N^h_j - \dot{\partial}_j N^h_i) Y_h - R^h_{ij} X_h,$$
$$N_F(Y_i, Y_j) = R^h_{ij} Y_h + (\dot{\partial}_i N^h_j - \dot{\partial}_j N^h_i) X_h,$$

Therefore, in a tangent bundle admitting a structure $\mathscr{C}_1$, the Nijenhuis tensor with respect to the almost complex N structure derived from $\mathscr{C}_1$ vanishes when and only when the non-linear connection derived from $\mathscr{C}_1$ satisfies $R^h_{ij} = 0$ and $\dot{\partial}_i N^h_j - \dot{\partial}_j N^h_i = 0$. This is the well-known result ([7], [14], [16]).

Of course, if $N_F = 0$, $T(M)$ is a complex manifold. However, with respect to the complex structure, it is not always true that $T(M)$ is covered by a system of the suitable canonical coordinate neighbourhoods $\{(\pi^{-1}(U), x^A)\}$ such that $(z^i) = (x^i + \sqrt{-1}\, y^i)$ becomes the local complex coordinate of $\pi^{-1}(U)$.

As is shown in Theorem 2, if the structure $\mathscr{C}_1$ is integrable, M is locally affine. In this case, M is covered by a system of local coordinate neighbourhoods $\{(U, x^i)\}$ such that $N^i_j = 0$ holds in each $\pi^{-1}(U)$. Hence, both N_P and N_F vanish. Moreover the natural $2n$-frame $\{\partial/\partial x^A\}$ in each $\pi^{-1}(U)$ is adapted to the almost complex N structure F.

Conversely, in a tangent bundle $T(M)$ admitting a structure $\mathscr{C}_1$, if $T(M)$ is covered by a system of the local canonical coordinate neighbourhoods $\{(\pi^{-1}(U), x^A)\}$ such that the canonical natural frame $\{\partial/\partial x^A\}$ is adapted to the induced almost complex N structure in each $\pi^{-1}(U)$, then it follows, in each $(\pi^{-1}(U), x^A)$, that $F = J_0$ and $F(\partial/\partial x^i) = \partial/\partial y^i$, $F(\partial/\partial y^i) = -\partial/\partial x^i$ hold. On the other hand, $F(X_i) = Y_i$ and $F(Y_i) = -X_i$ hold. Hence we get $N^i_j = 0$ in each $\pi^{-1}(U)$, that is, M is locally affine. Thus we obtain

Theorem 9. *Let $T(M)$ be a tangent bundle admitting a structure $\mathscr{C}_1$, and let us consider the almost complex N structure induced from the structure $\mathscr{C}_1$. The tangent bundle $T(M)$ is covered by a system of local canonical coordinate neighbourhoods $\{(\pi^{-1}(U), x^A)\}$ such that the natural frame $\{\partial/\partial x^A\}$ in each $\pi^{-1}(U)$ is adapted to the almost complex N structure, if and only if the underlying manifold M is locally affine, that is, the structure $\mathscr{C}_1$ is integrable.*

Next we show

Theorem 10. *In order that a tangent bundle $T(M)$ admits an integrable $D(O(n))$-structure depending on $\mathscr{C}_0$, it is necessary and sufficient that the underlying manifold M admits a flat Riemann metric.*

Proof. First, let us consider a tangent bundle $T(M)$ admitting an integrable $\mathscr{C}_2$ structure. Due to the definition, $T(M)$ is covered by a system of canonical coordinate neighbourhoods $\{(\pi^{-1}(U), x^A)\}$ such that the natural frame $\{\partial/\partial x^A\}$ is adapted to the structure $\mathscr{C}_2$. Then $\langle \partial/\partial x^A, \partial/\partial x^B\rangle = \delta_{AB}$ holds. That is, $g_{ij} = \delta_{ij}$ holds true with respect to each (U, x^i). Of course, $\{(U, x^i)\}$ covers M. Hence M is locally Euclidean.

Conversely, if M admits a flat Riemann metric g_{ij}, then M is covered by a system of local coordinate neighbourhoods $\{(U, x^i)\}$ with respect to which $g_{ij} = \delta_{ij}$ holds always. Then $\left\{ {i \atop jk} \right\} = 0$ holds. Now, the system of the canonical coordinate neighbourhoods $\{(\pi^{-1}(U), x^A)\}$ covers $T(M)$. With respect to these coordinate neighbourhoods, the non-linear connection $N^i_j = \left\{ {i \atop mj} \right\} y^m$ vanishes. Hence the N-frame $\{X_i, Y_i\}$ for the non-linear connection coincides with the canonical natural frame $\{\partial/\partial x^A\}$. On the other hand, according to Theorem 6, g_{ij}

and N_j^i determine a structure $\mathscr{C}_2$ in $T(M)$ and the relations $\langle X_i, X_j\rangle = g_{ij}$, $\langle Y_i, Y_j\rangle = g_{ij}$ and $\langle X_i, Y_j\rangle = 0$ hold. Hence we have $\langle \partial/\partial x^A, \partial/\partial x^B\rangle = \delta_{AB}$. Thus the natural frame $\{\partial/\partial x^A\}$ is adapted to the structure $\mathscr{C}_2$. That is, the sructure $\mathscr{C}_2$ is integrable.

Finally we show

Theorem 11. *In order that a manifold M is a locally Minowski manifold, it is necessary and sufficient that the tangent bundle $T(M)$ admits a structure $\mathscr{C}_2$ satisfying*

(1) *The structure $\mathscr{C}_1$ induced from $\mathscr{C}_2$ is integrable,*
(2) *The almost Hermit structure $\{F, G\}$ induced from $\mathscr{C}_2$ is almost Kählerian,*
(3) *The non-linear connection derived from $\mathscr{C}_2$ is metrical with respect to the Moór metric derived from $\mathscr{C}_2$.*

Proof. Let M be locally Mikowskian and g_{ij} be the metric tensor. Then M is covered by a system of local coordinate neighbourhoods $\{(U, x^i)\}$ such that $\partial_k g_{ij} = 0$ holds good in each U. Then, in these coordinate neighbourhoods, the Cartan's Finsler connection $\overset{*}{\Gamma}{}^i_{jk}$ and the Cartan's non-linear connection G^i_j vanish. Now, g_{ij} and G^i_j induce a structure $\mathscr{C}_2$ in $T(M)$. We consider this structure $\mathscr{C}_2$. The N-frame associated with G^i_j is an adapted frame of the structure $\mathscr{C}_1$ determined by G^i_j, and $G^i_j = 0$ holds in each $\pi^{-1}(U)$. So, the natural frame $\{\partial/\partial x^A\}$ is adapted to the structure $\mathscr{C}_1$. Of course, by definition, $\mathscr{C}_1$ coincides with the $D(GL(n, R))$-structure induced from $\mathscr{C}_2$. Namely, the structure $\mathscr{C}_1$ satisfies the condition (1). On the other hand, it is obvious in each U that $\overset{G}{\nabla}_k g_{ij} = \partial_k g_{ij} - \dot{\partial}_m g_{ij} G^m_k - g_{mj}\dot{\partial}_i G^m_k - g_{im}\dot{\partial}_j G^m_k = 0$. Hence the condition (3) is satisfied. Moreover, $\dot{\partial}_i G^k_j - \dot{\partial}_j G^k_i = 0$ is evident. And, of course, g_{ij} is a Finsler metric. Hence, owing to Theorem 7, the structure $\mathscr{C}_2$ satisfies the condition (2).

Conversely, let us assume that $T(M)$ admits a structure $\mathscr{C}_2$ satisfying (1), (2) and (3). Then there given a non-linear connection N and a Moór metric g. Now, according to Theorem 7, the condition (2) tells us that g is a Finsler metric. And the condition (1) implies that $T(M)$ is covered by a system of local coordinate neighbourhoods $\{(\pi^{-1}(U), x^A)\}$ such that $\{(U, x^i)\}$ covers M and $N^i_j = 0$ holds in each $\pi^{-1}(U)$. On the other hand, the condition (3) means $\overset{N}{\nabla} g = 0$. Then, with respect to these coordinates, $\partial_k g_{ij} = 0$ holds. Namely, M is covered by a system of local coordinate neighbourhoods $\{(U, x^i)\}$ such that $\partial_k g_{ij} = 0$ holds good in each U. Therefore M is locally Minkowskian.

Remark. According to Theorem 9, the almost Hermit structure $\{F, G\}$ in

Theorem 11 is Kählerian.

Department of Mathematics,
College of General Education,
Tokushima University

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