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Remarks on the Structure of Power Semigroups

By

Yuji KOBAYASHI

Dedicated to Professor M. Yamada on his 60th birthday

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1. Introduction

Let S be a semigroup. The *power semigroup* $P(S)$ of S is the set of all non-empty subsets of S with the operation defined by

$$XY = \{xy \mid x \in X,\ y \in Y\}$$

for X, Y in $P(S)$. This concept is old as is found in Dubriel [1], Liapin [4] and Tamura [9], but precise studies have begun recently (see, for example, Gould and Iskra [2], Tamura [7]). Even if S has a simple structure, the structure of $P(S)$ can be very complicated. This is especially so if S is infinite. Suppose that S is a commutative semigroup with a non-preiodic element. In this paper we show that (i) $P(S)$ has uncountably many incomparable archimedean components, and (ii) $P(S)$ contains uncountably many free generators. The first result answers to a question posed by Tamura [8]. The second result may be interesting in connection with the embedding problem in power semigroups.

The set of positive integers will be denoted by $\mathbf{P}$.

2. Archimedean components

If S is a commutatie semigroup, then so is $P(S)$. A standard way to investigate a commutative semigroup is to decompose it into a semilattice of archimedean semigroups. Let T be a commutative semigroup. The relation ρ on T defined by

$$x\rho y \text{ if } x^n = yz \text{ and } y^n = xw \text{ for some } n \in \mathbf{P} \text{ and } z,\ w \in T,$$

is a congruence of T. The ρ-classes are archimedean subsemigroups of T and are the *archimedean components* of T. The archimedean component containing $x \in T$

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is denoted by $\mathscr{A}_x$. The quotient T/ρ is a (lower) semilattice and is the greatest semilattice image of T. For $\mathscr{A}_x$, $\mathscr{A}_y \in T/\rho$, $\mathscr{A}_x \leqq \mathscr{A}_y$ if and only if $x^n = yz$ for some $n \in \mathbf{P}$ and $z \in T$.

Now we shall show the semilattice decomposition of $P(S)$ is intricate in general.

Theorem 1. *Let S be a commutatie semigroup with a non-periodic element. Then $P(S)$ has uncountably many incomparable archimedean components.*

To prove the theorem we need the following easy lemma.

Lemma 1. *Let X be a countable infinite set. Then there is an uncountable family $\{X_\alpha\}_{\alpha \in I}$ of subsets X_α of X such that the difference $X_\alpha \backslash X_\beta$ is infinite for any different α and β in I.*

PROOF. We may suppose X is the set of rational numbers and I is the set of real numbers. For $\alpha \in I$ define a subset X_α of X by $X_\alpha = \{x \in X \mid \alpha \leqq x \leqq \alpha + 1\}$. Then the family $\{X_\alpha\}_{\alpha \in I}$ satisfies the desired property.

PROOF of Theorem 1. Let a be a non-periodic element of S. First we choose an infinite sequence $X = \{n(i) \mid i \in \mathbf{P}\}$ of positive integers such that $n(1) = 1$ and $n(i+1) > N(i)^2$ for $i \in \mathbf{P}$. By Lemma 1 we can find an uncountable family $\{X_\alpha\}_{\alpha \in I}$ of subsequences of X such that $X_\alpha \backslash X_\beta$ is infinite for any different α, $\beta \in I$. We may assume that every X_α contains 1. Let A_α be an element of $P(S)$ defined by $A_\alpha = \{a^n \mid n \in X_\alpha\}$ and let $\mathscr{A}_\alpha$ be the archimedean component of A_α in $P(S)$. We shall show that $\mathscr{A}_\alpha$ and $\mathscr{A}_\beta$ are incomparable if $\alpha \neq \beta$.

Assume to the contrary that $\mathscr{A}_\alpha \leqq \mathscr{A}_\beta$, that is,

$$A_\alpha^\ell = A_\beta C \tag{1}$$

for some $m_1,\dots,m_\ell \in X_\alpha$. It follows from (2) and (3) that $a^{n_1+\dots+n_\ell} = a^{m_1+\dots+m_\ell+n-1}$ or

$$a^{n_1} a^{n_2} \cdots a^{n_\ell} = a^n c \tag{2}$$

for some $n_1,\dots, n_\ell \in X_\alpha$. Since $a \in A_\beta$, again by (1) we have

$$a^{m_1} a^{m_2} \cdots a^{m_\ell} = ac \tag{3}$$

for some $m_1,\dots, m_\ell \in X_\alpha$. It follows from (2) and (3) that $a^{n_1+\cdots+n_\ell} = a^{m_1+\cdots+m_\ell+n-1}$ or

$$n_1 + \cdots + n_\ell = m_1 + \cdots + m_\ell + n - 1. \tag{4}$$

Since n is not in X_α, n is different from any of $n_1 \dots, n_\ell$ and $m_1,\dots, m_\ell$.

If $n > \max(n_1,\dots, n_\ell)$ then $\sqrt{n} > \max\{n_1,\dots, n_\ell\}$ by the property of the sequence X. So by (4) we have $n \leqq n_1 + \cdots + n_\ell < \sqrt{n} \cdot \ell < n$, a contradiction. Hence $n < \max\{n_1, \ldots, n_\ell\}$. We may suppose that $n_1 = \max\{n_1, \ldots, n_\ell\}$ and $m_1 = \max\{m_1, \ldots, m_\ell\}$. If $n_1 > m_1$, then $\sqrt{n_1} > m_i$ for $i = 1, \ldots, \ell$. Noting $\sqrt{n_1} > n$, we get the impossible inequalities

$$n_1 \leqq m_1 + \cdots + m_\ell + n - 1 < \sqrt{n_1} \cdot (\ell + 1) \leqq \sqrt{n_1} \cdot n < n_1.$$

In the same way $n_1 < m_1$ is impossible and we have $n_1 = m_1$. Thus we can cancel n_1 and m_1 in (4) and we get

$$n_2 + \cdots + n_\ell = m_2 + \cdots + m_\ell + n - 1.$$

Repeating the above argument, we can cancel all the n_i and m_i in (4) and finially we would have $n = 1$, but this is impossible.

Similarly, $\mathscr{A}_\alpha \geqq \mathscr{A}_\beta$ is impossible either. Therefore $\mathscr{A}_\alpha$ and $\mathscr{A}_\beta$ are incomparable and the proof of the theorem is complete.

What about the cardinality of each archimedean component of $P(S)$? We can show that some of the components are uncountable. In fact, let S be a commutative semigroup with a non-periodic element a. Consider the subsets of $\{a^i | i \in \mathbf{P}\}$ containing a^2 and a^i for all positive odd integer i. There are uncountably many such sets and the square of them are all equal. Therefore they are in the same archimedean component which are uncountable. Thus, $P(S)$ has uncountably many archimedean components some of which are uncountable.

The semilattice decomposition of $P(G)$ for a finite group G was described by Putcha [5]. Tamura [8] studied the archimedean components of $P(G)$ for the infinite cyclic group G and asked how many archimedean components $P(G)$ has. The answer is "uncountable" due to Theorem 1.

3. Free commutative subsemigroups

The embedding problem in power semigroups has been of interest (Gould and Iskra [3], Trnkova [10]). In this section we shall prove a somewhat surprizing result that the power semigroup of a semigroup with a non-periodic element has a very large free commutative subsemigroup.

Theorem 2. *Let C be a infinite cyclic semigroup. Then $P(C)$ contains a subsemigroup isomorphic to a free commutative semigroup on an uncountable set of generators.*

We need the following lemma stronger than Lemma 1. The result is due to Sierpinski [6].

Lemma 2. *Let X be a countable infinite set. Then there is an uncountable* *family* $\{X_\alpha\}_{\alpha\in I}$ *of infinite subsets* X_α *of* X *such that* $X_\alpha \cap X_\beta$ *is finite for any different* α *and* β *in* I.

PROOF. We may suppose $X = \mathbf{P}$ and I is the set of real numbers between 1/2 and 1. For $\alpha\in I$, define $X_\alpha = \{\mathrm{Int}(2^n\alpha) | n\in\mathbf{P}\}$, where Int($t$) for a real number t is the greatest integer not exceeding t. Let α and β be different elements in I. Since $2^{n-1} < 2^n\alpha < 2^n$ and $2^{m-1} < 2^m\beta < 2^m$ for any $n, m \in \mathbf{P}$, we see that $\mathrm{Int}(2^n\alpha) \neq \mathrm{Int}(2^m\beta)$ if $n \neq m$. Moreover, $\mathrm{Int}(2^n\alpha) \neq \mathrm{Int}(2^n\beta)$ if $n \geqq -\log_2|\alpha-\beta|$. It follows that $X_\alpha \cap X_\beta$ is finite.

PROOF of Theorem 2. We may assume that C is the additive semigroup of positive integers. The operation of $P(C)$ is also written additively and nA denotes the sum of n A's for $n\in\mathbf{P}$ and $A\in P(C)$. Let $X = \{n(i) | i\in\mathbf{P}\}$ be an infinite sequence of positive integers such that $n(i+1) > n(i)^2$ for all $i\in\mathbf{P}$. Let $\{X_\alpha\}_{\alpha\in I}$ be an uncountable family of subsequences $X_\alpha = \{n(\alpha, i) | i\in\mathbf{P})$ of X such that $X_\alpha \cap X_\beta$ is finite for any different α and β in I. The existence of such a family is guaranteed by Lemma 2. X_α are considered to be elements of $P(C)$. We claim that the subsemigroup generated by $\{X_\alpha\}_{\alpha\in I}$ is a free commutative semigroup with the free generating set $\{X_\alpha\}_{\alpha\in I}$.

Let $\{m_\alpha\}_{\alpha\in I}$ be a set of non-negative integers indexed by I such that only a finite number of m_α are positive. Let

$$Y = \textstyle\sum_{\alpha\in I} m_\alpha X_\alpha = m_1 X_{\alpha_1} + \cdots + m_r X_{\alpha_r},$$

where $\{m_i = m_{\alpha_i} | i = 1,\dots, r\}$ is the set of all positive integers in $\{m_\alpha\}_{\alpha\in I}$. We have to show that the integer m_α is determined only by Y and α for any $\alpha\in I$. Let $\alpha\in I$ and $i\in\mathbf{P}$, and set

$$Y(\alpha, i) = (n\in Y | n \leqq n(\alpha, i)^2\}.$$

Let n be in $Y(\alpha, i)$, then n is written as

$$n = \sum_{j=1}^{r} \sum_{k=1}^{m_j} n(\alpha_j, i_j(k)). \tag{5}$$

If some of $n(\alpha_j, i_j(k))$ in (5) were greater than $n(\alpha, i)$, then $n > n(\alpha, i)^2$, a contradiction. So every $n(\alpha_j, i_j(k))$ in (5) is not greater than $n(\alpha, i)$. If just m numbers in $n(\alpha_j, i_j(k))$ are equal to $n(\alpha, i)$, then

$$n = m\cdot n(\alpha, i) + p,$$

where $0 \leqq p < M\cdot\sqrt{n(\alpha, i)}$ with $M = \sum_{\alpha\in I} m_\alpha = m_1 + \cdots + m_r$. By the choice of the family $\{X_\alpha\}_{\alpha\in I}$, there exists a positive integer N such that if $i \geqq N$, then $n(\alpha, i) \geqq M^2$ and $n(\alpha, i)$ is not in X_{α_j} for each $j = 1,\dots, r$ with $\alpha_j \neq \alpha$. Therefore, if $i \geqq N$, then the greatest number in $Y(\alpha, i)$ is of the form $m_\alpha\cdot n(\alpha, i) + p$ with $p < n(\alpha, i)$

$< M\cdot\sqrt{n(\alpha, i)}$. This implies $m_\alpha = \mathrm{Int}\left(\frac{\max Y(\alpha, i)}{n(\alpha, i)}\right)$ for $i \geqq N$. Consequently we have

$$m_\alpha = \lim_{i\to\infty} \mathrm{Int}\left(\frac{\max Y(\alpha, i)}{n(\alpha, i)}\right),$$

showing that m_α is determined by Y and α.

Corollary 1. *Any commutative semigroup S whose cardinality is not greater than the cardinality of the real numbers devides the power semigroup $P(C)$ of the infinite cyclic semigroup C, that is, S is a homomorphic image of a subsemigroup of $P(C)$.*

Corollary 2. *If a semigroup S contains a non-periodic element, then $P(S)$ contains an uncountale free commutative semigroup.*

The above results imply that $P(S)$ contains a large cancellative subsemigroup in general. It may be interesting to point out that $P(S)$ itself is not cancellative at all.

Proposition. *If S is a semigroup with at least two elements, then $P(S)$ is not cancellative.*

PROOF. If S is a band of order greater than 1, then S is not cancellative and neither is $P(S)$. If S is not a band, then S has a non-idempotent element a. Let A be the subsemigroup of S gnerated by a. Then we have $A\cdot\{a\} = A\cdot\{a, a^2\}$ in $P(S)$. Since $\{a\} \neq \{a, a^2\}$, cancellation does not hold in $P(S)$.

If S is a monoid with a non-periodic element a, then $P(S)$ contains an uncountable null subsemigroup as well as an uncountable free commutative semigroup. In fact, subsets of $\{a^i | i \in \mathbf{P}\}$ containing 1 and a^i for all positive odd integers i form an uncountable null subsemigroup of $P(S)$.

Depatment of Mathematics and Computer Sciences,
Faculty of Integral Arts and Sciences,
Tokushima University

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