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On the Inverse Scattering Problem for 1-dimensional Schrödinger Operator with Integrable Potential

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In this paper we investigate the inverse scattering problem for 1-dimensional Schrödinger operator

$$H_u = -\partial_x^2 + u(x), \quad \partial_x = d/dx, \ -\infty < x < \infty,$$

with certain real potential $u(x)$ in L_0^1, where

$$L_\lambda^1 = \{u(x) \mid \text{real valued continuous, } \lim_{|x|\to\infty} u(x) = 0 \text{ and } \int_{-\infty}^{\infty} |x|^\lambda |u(x)| dx < \infty\}, \quad \lambda \geqq 0.$$

We consider the unique selfadjoint extension of H_u considered in the space of twice continuously differentiable functions on $(-\infty, \infty)$ with compact support; we denote it again by H_u itself. Throughout the paper, we assume that the unique selfadjoint extension H_u has no bound states, i.e. $\sigma_p(H_u) = \phi$, where $\sigma_p(*)$ denotes the point spectrum.

Let $f_\pm(x, \xi; u)$, $\xi \in \boldsymbol{R}\backslash\{0\}$, be the solutions of the differential equation

$$H_u f_\pm = -f_\pm'' + u(x) f_\pm = \xi^2 f_\pm, \quad ' = d/dx$$

with the asymptotic conditions $f_\pm(x, \xi; u) \sim e^{\pm i\xi x}$ as $x \to \pm\infty$ respectively, which are called the Jost solutions. If we assume $u(x) \in L_0^1$ then the Jost solutions exist uniquely for $\xi \in \boldsymbol{R}\backslash\{0\}$. Since the solutions $f_+(x, \pm\xi; u)$ are linearly independent for $\xi \in \boldsymbol{R}\backslash\{0\}$, there exist $a(\xi; u)$ and $b(\xi; u)$ such that

$$f_-(x, \xi; u) = a(\xi; u) f_+(x, -\xi; u) + b(\xi; u) f_+(x, \xi; u).$$

We have immediately

$$|a(\xi; u)|^2 = 1 + |b(\xi; u)|^2.$$

Therefore $a(\xi; u)$ does not vanish for $\xi \in \boldsymbol{R}\backslash\{0\}$. Put

$$r_\pm(\xi; u) = \pm b(\pm\xi; u)/a(\xi; u).$$

The functions $r_+(\xi; u)$ and $r_-(\xi; u)$ are called the right and left reflection coefficient respectively. Moreover the function

$$t(\xi;\, u) = 1/a(\xi;\, u)$$

is called the transmission coefficient. The 2×2 matrix

$$S(\xi;\, u) = \begin{pmatrix} t(\xi;\, u) & r_-(\xi;\, u) \\ r_+(\xi;\, u) & t(\xi;\, u) \end{pmatrix}$$

is called the scattering matrix for the potential $u(x)$. We refer to [1], [5] and [6] for detail of the scattering theory.

If $u(x) \in L_1^1$ and $\sigma_p(H_u) = \phi$, then, by Levinson's theorem [5; Corollary, p208], the potential $u(x)$ is uniquely determined by the right reflection coefficient $r_+(\xi;\, u)$ only. On the contrary, the present author showed in [8] that if $u(x) \in L_0^1 \backslash L_1^1$ then the right reflection coefficient $r_+(\xi;\, u)$ does not determine the potential $u(x)$ uniquely even without bound states in general, i.e., there exist $u_j(x) \in L_0^1 \backslash L_1^1$ $(j=1, 2)$ such that $u_1(x) \neq u_2(x)$, $\sigma_p(H_{u_j}) = \phi$ and $r_+(\xi;\, u_1) = r_+(\xi;\, u_2)$. The main goal of the present work is to give an algorithm for recovering all potentials $u(x) \in L_0^1$ such that $\sigma_p(H_u) = \phi$ and the right reflection coefficient $r_+(\xi;\, u)$ coincides with the given function $r(\xi)$ with certain additional conditions.

The contents of this paper are as follows. In §1 we explain the classical Darboux's lemma and define the Darboux transformation. In §2 we prove the existence of positive solutions for the equation $H_u f = 0$. In §3 we investigate Tanaka's lemma. In §4 we study the property of the Darboux transformation $\hat{H}_{u,f}$ in the case of $u \in L_0^1$. Finally, in §5, we give a solution of the reconstruction problem. §6 is devoted to the concluding remarks.

1. Darboux's lemma

In [2], G. Darboux obtained the following (see also [3] and [9: pp 88–91]).

Lemma 1 (Darboux's lemma). *Let $f_\lambda(x)$ be the nontrivial solution of the equation $H_u f_\lambda = \lambda f_\lambda$. If $\mu \neq \nu$ then*

$$g(x) = W(f_\mu, f_\nu)/f_\nu$$

solves the equation

$$H_{u_\nu} g = \mu g,$$

where

$$u_\nu(x) = u(x) - 2(d/dx)^2 \log f_\nu(x)$$

and $W(\phi, \psi) = \phi\psi' - \phi'\psi$ is Wronskian. In particular, $h(x) = 1/f_\nu(x)$ solves the equation $H_{u_\nu} h = \nu h$.

PROOF. Put $A=f_\nu^{-1}\partial_x f_\nu$ and $A^*=-f_\nu\partial_x f_\nu^{-1}$ then $H_u=AA^*+\nu$, $H_{u_\nu}=A^*A+\nu$ and $g=A^*f_\mu$ follow. Hence we have

$$\begin{aligned}H_{u_\nu}g&=(A^*A+\nu)A^*f_\mu\\&=A^*(H_u-\nu)f_\mu+\nu A^*f_\mu\\&=A^*H_uf_\mu=\mu A^*f_\mu=\mu g.\end{aligned}$$

The proof for h is quite similar. Q.E.D.

On the other hand, the Darboux transformation is defined as follows: Let $g(x)$ be a nontrivial real valued solution of the differential equation

(1) $$H_ug=-g''+u(x)g=0.$$

Put $A_g=g^{-1}\partial_x g$ and let $A_g^*=-g\partial_x g^{-1}$ be its formal adjoint. Then

(2) $$H_u=A_gA_g^*$$

follows. By exchanging the roles of A_g and A_g^* in (2), we obtain the Darboux transformation

$$\hat{H}_{u,g}=A_g^*A_g$$

of H_u by the solution $g(x)$. The spectral property of $\hat{H}_{u,g}$ was studied precisely in [8] in the case of $u(x)\in L_2^1$ and $\sigma_p(H_u)=\phi$. On the other hand put

$$q(x)=(d/dx)\log g(x)$$

then we have

$$u(x)=q'(x)+q(x)^2.$$

Moreover define $\hat{u}_g(x)$ corresponding to $u(x)$ and $g(x)$ by

$$\hat{u}_g(x)=-q'(x)+q(x)^2=u(x)-2q'(x),$$

then one easily verifies

$$\hat{H}_{u,g}=-\partial_x^2+\hat{u}_g(x).$$

If we require $\hat{u}_g(x)$ to be continuous together with $u(x)$ then it suffices to consider the Darboux transformation $\hat{H}_{u,g}$ only by the positive solution $g(x)$. Consequently we must investigate first of all whether the differential equation (1) has a positive solution or not. Hence we do this in the next section.

2. Positive solution of (1)

In [5], Deift and Trubowitz proved the following: *If $u\in L_1^1$ then the Jost solution*

$f_{\pm}(x, \xi; u)$ are defined even at $\xi=0$. Moreover $\sigma_p(H_u)\neq\phi$ if and only if $f_{\pm}(x, 0; u)$ vanish for some x.

This implies that if $u(x)$ is in L_1^1 and $\sigma_p(H_u)=\phi$ then $S_+(u)\neq\phi$ and $f_{\pm}(x, 0; u)\in S_+(u)$, where $S_+(u)$ is the set of all positive solutions of the differential equation (1). We want to generalize the resuslt of Deift and Trubowitz mentioned above for the potential $u(x)$ in L_0^1. However note that the Jost solutions $f_{\pm}(x, \xi; u)$ are not necessarily defined for $\xi=0$ when $u(x)$ is in L_0^1. Therefore we can not obtain any information about solutions of (1) from $f_{\pm}(x, \xi; u)$.

Let $S^{(\pm)}(u; v)$ $(v=0, \infty)$ be the sets of all solutions $f(x)$ of (1) such that $f(x)$ tend to v as $x\to\pm\infty$ respectively. Similarly let $S^{(\pm)}(u; \boldsymbol{R}_+)$ be the sets of all solution $f(x)$ of (1) such that $\lim_{x\to\pm\infty} f(x)$ exist and belong to $\boldsymbol{R}_+=(0, \infty)$ respectively. Then we have the following.

Theorem 2. *Suppose that $u(x)\in L_0^1$, $\sigma_p(H_u)=\phi$ and $S^{(\pm)}(u; \boldsymbol{R}_+)\neq\phi$. Then $S^{(\pm)}(u; \boldsymbol{R}_+)\subset S_+(u)$ is valid.*

PROOF. It obviously suffices to show that if the solution $f(x)$ of (1) tends to 1 as $x\to\infty$ then $f(x)$ does not vanish for arbitrary x. The following argument is quite similar to [5; pp 163–165]. Assume that $f(x)$ has a zero. Then, without loss of generality, we can assume that $f(0)=0$ and $f(x)\neq 0$ for any $x>0$. Then $f'(0)>0$ follows. Let $\tilde{H}$ be the Friedrichs extension of H_u restricted to $C_0^\infty(-a, \infty)$, where a is an arbitrary positive real number. Note that functions $g(x)$ in the domain $D(\tilde{H})$ vanish at $x=-a$, where $D(*)$ denotes the domain of the operator. Suppose $g(x)\in D(\tilde{H})$ then, by putting $g(x)=0$ for $x<-a$, one can deem that $g(x)\in D(H_u)$ and $(g, H_u g)_{L^2(-\infty,\infty)}=(g, \tilde{H}g)_{L^2(-a,\infty)}$ hold, where $(*, *)_E$ denotes the inner product in the Hilbert space E. Hence, by the minimax principle, $\sharp\sigma_p(H_u)\geqq\sharp\sigma_p(\tilde{H})$ is valid, where $\sharp$ denotes cardinal. Therefore it suffices to prove $\sigma_p(\tilde{H})\neq\phi$. Let

$$\chi_n(x)=\begin{cases}1 & \text{for } x\leqq n,\\ 2-x/n & \text{for } n\leqq x\leqq 2n,\\ 0 & \text{for } x\geqq 2n,\end{cases}$$

and $\phi(x)\in C_0^\infty(-1, 1)$ with $\phi(x)\geqq 0$ and $\phi(0)=1$. Now, for $\varepsilon>0$ and $n\geqq 1$, define

$$f_{n,\varepsilon}(x)=\begin{cases}0 & \text{for } x<-a\\ \varepsilon(x/a+1) & \text{for } -a\leqq x<0,\\ \chi_n(x)f(x)+\varepsilon\phi(x) & \text{for } x\geqq 0.\end{cases}$$

Then $f_{n,\varepsilon}\in D(\tilde{H})$ follows. By direct calculation (see [5; p 164] for more detail.), we have

$$\tilde{H}[f_{n,\varepsilon}, f_{n,\varepsilon}]=\int_{-\infty}^{\infty}\{f'_{n,\varepsilon}(x)^2+u(x)f_{n,\varepsilon}(x)^2\}dx$$

$$=c_1\varepsilon^2-2f'(0)\varepsilon+o(1)+\int_n^{2n}\{(\chi_n'(x)f(x)+\chi_n(x)f'(x))^2$$
$$+u(x)\chi_n(x)^2f(x)^2\}dx.$$

One verifies

$$\int_n^{2n}\{(\chi_n'(x)f(x)+\chi_n(x)f'(x))^2+u(x)\chi_n(x)^2f(x)^2\}dx$$
$$\leqq 2\int_n^{2n}(n^{-2}f(x)^2+f'(x)^2)dx+\int_n^{2n}f(x)^2|u(x)|dx$$
$$\leqq c_2n^{-1}+2\int_n^{2n}f'(x)^2dx+c_3\int_n^{2n}|u(x)|dx$$
$$=o(1)+2\int_n^{2n}f'(x)^2dx,\quad n\to\infty.$$

On the other hand one can show

$$f'(x)=-\int_x^\infty u(y)f(y)dy+C.$$

Assume $C\neq 0$ then there exists δ and a positive constant K such that $(f(x)/f'(x))^2\leqq K$ for all $x\geqq\delta$. Since

$$(f'(x)/f(x))'+(f'(x)/f(x))^2=u(x),$$

by straightforward calculation, we have

$$f(x)/f'(x)=x+\int_x^\infty u(y)f(y)^2/f'(y)^2dy+C'$$

for all $x\geqq\delta$. This is contradiction, i.e., $C=0$ follows. Hence $f'(x)$ tends to 0 as $x\to\infty$. Therefore we have

$$\int_n^{2n}f'(x)^2dx\leqq\int_n^\infty f'(x)^2dx=[f(x)f'(x)]_n^\infty-\int_n^\infty f(x)f''(x)dx$$
$$=-f(n)f'(n)-\int_n^\infty u(x)f(x)^2dx=o(1),\ n\to\infty.$$

Thus, for arbitrary $\varepsilon>0$, we have

$$\tilde{H}[f_{n,\varepsilon},f_{n,\varepsilon}]=c_1\varepsilon^2-2f'(0)\varepsilon+o(1),\ n\to\infty.$$

Since $f'(0)>0$ by hypothesis, we can choose $\varepsilon>0$ such as $c_1\varepsilon^2-2f'(0)\varepsilon$, which is independent of n, is negative. Hence $\tilde{H}[f_{n,\varepsilon},f_{n,\varepsilon}]$is negative for sufficiently large n. This implies $\sigma_p(\tilde{H})\neq\phi$. Q.E.D.

Now we classify H_u such that $u(x)\in L_0^1$ and $\sigma_p(H_u)=\phi$ as follows.

Definition. (cf. [8; p 16])

(I) *H_u is of type* I *if and only if $u(x)\in L_2^1$, $\sigma_p(H_u)=\phi$ and $|r_+(\xi; u)|<1$ for all $\xi\in\boldsymbol{R}$.*

(II) *H_u is of type* II *if and only if $u(x)\in L_2^1$, $\sigma_p(H_u)=\phi$ and $r_\pm(0; u)=-1$.*

(III) *H_u is of type* III$\pm$ *if and only if $u(x)\in L_0^1$, $\sigma_p(H_u)=\phi$ and H_u satisfies the following* (A) *and* (B$\pm$) *respectively:*

(A) $$r_\pm(\xi; u)-1=i\rho_\pm\xi+o(\xi),\ \xi\to 0,$$

(B$\pm$) $$S^{(\pm)}(u; \boldsymbol{R}_+)\neq\phi,$$

where $\rho_\pm$ are some real numbers.

Remark. Note that the operators of type III$_\pm$ actually exist. In fact, let H_v be of type II then, by lemma 1, $f(x)=1/g_+(x)$ (resp. $1/g_-(x)$) solves the equation $\hat{H}_{v,g+}f=0$ (resp. $\hat{H}_{v,g\_}f=0$) and $1/g_\pm(x)$ belong to $S^{(\pm)}(\hat{v}_{g\pm}; \boldsymbol{R}_+)$ respectively, where $g_\pm(x)=f_\pm(x, 0; v)$. Moreover, by Tanaka's lemma [8], the condition (A) turns out to be fulfilled by both $\hat{v}_{g\pm}(x)$. Hence $\hat{H}_{v,g\pm}$ are of type III$_\pm$ respectively.

The Darboux transformation of H_u of type I and type II are studied in [7] and [8] respectively. The main purpose of the following sections is to study the Darboux transformation of H_u of type III$\pm$.

3. Tanaka's lemma

In [10], S. Tanaka proved the following: *Suppose that $u(x)\in L_2^1$, $\sigma_p(H_u)=\phi$ and there exists an absolutely continuous $q(x)$ in L_1^1 such that $u(x)=q'(x)+q(x)^2$. Put $v(x)=-q'(x)+q(x)^2$ then $v(x)$ is in L_2^1, $\sigma_p(H_v)=\phi$ and*

$$r_\pm(\xi; u)=-r_\pm(\xi; v)$$

are valid. See [7] for detail.

In [8], this resuslt is slightly generalized. In this section we give further generalization of Tanaka's lemma.

If $f(x)$ is a solution of (1) then

(3) $$f(x)=\int_0^x (x-y)u(y)f(y)dy+f(0)+f'(0)x$$

and

(4) $$f'(x)=\int_0^x u(y)f(y)dy+f'(0)$$

are valid. Suppose that $u(x)\geqq 0$ for all x and $S_+(u)\neq\phi$. Then, by (3),

$$S_+(u)\subset S^{(\pm)}(u;\ 0)\cup S^{(\pm)}(u;\ \boldsymbol{R}_+)\cup S^{(\pm)}(u;\ \infty)$$

follows, i.e., any positive solutions have non-negative limiting values including $+\infty$ as $x\to\pm\infty$.

Taking into account the above consideration, we have

Lemma 3. *Suppose that* $u(x)\in L^1_0$, $u(x)\geqq 0$ *for all* x *and* $S_+(u)\neq\phi$. *If* $f(x)\in S_+(u)$ *then* $q(x)=(d/dx)\log f(x)$ *tends to* 0 *as* $x\to\pm\infty$.

PROOF. First suppose that $f(x)$ is in $S^{(+)}(u;\ 0)\cup S^{(+)}(u;\ \boldsymbol{R}_+)$. Then, by (4), we have

$$f'(x)=C-\int_x^\infty u(y)f(y)dy,$$

where $C=f'(0)+\int_0^\infty u(y)f(y)dy\in\boldsymbol{R}$. Hence $f'(x)$ tends to C as $x\to\infty$. Assume $C\neq 0$ then there exist $D\in\boldsymbol{R}$ and $K\in\boldsymbol{R}_+$ such that $q(x)^{-2}\leqq K$ holds for all $x\geqq D$. Since $q'(x)+q(x)^2=u(x)$, we have immediately

$$-q(x)^{-1}+x=-\int_x^\infty u(y)q(y)^{-2}dy+C'$$

for all $x\geqq D$. This is contradiction. Therefore $C=0$ follows, i.e.,

$$f'(x)=-\int_x^\infty u(y)f(y)dy.$$

Hence it immediately follows that if $f(x)\in S^{(+)}(u;\boldsymbol{R}_+)$ then $q(x)$ tends to 0 as $x\to\infty$. On the other hand let $f(x)$ be in $S^{(+)}(u;\ 0)$ then, by l'Hospital's theorem, we have

$$\lim_{x\to\infty}q(x)^2=\lim_{x\to\infty}\frac{2f''(x)f'(x)}{2f(x)f'(x)}=\lim_{x\to\infty}u(x)=0.$$

Next let $f(x)$ be in $S^{(+)}(u;\ \infty)$. By (4), $f'(x)$ either diverges to ∞ or converges to a finite value as $x\to\infty$. If $f'(x)\to\infty$ as $x\to\infty$ then, similarly to the above, we can show

$$\lim_{x\to\infty}q(x)^2=\lim_{x\to\infty}u(x)=0.$$

If $f'(x)$ converges to a finite value then $q(x)$ obviously tends to 0 as $x\to\infty$. Q.E.D.

Next we have

Proposition 4. (Tanaka's lemma) *Suppose that* $u(x)\in L_0^1$, $\sigma_p(H_u)=\phi$ *and* $S_+(u)\neq\phi$. *Then*

$$r_\pm(\xi;u)=-r_\pm(\xi;\hat{u}_g)$$

and

$$t(\xi;u)=t(\xi;\hat{u}_g)$$

are valid for all $g(x)\in S_+(u)$.

PROOF. Let $\xi\in\boldsymbol{R}\backslash\{0\}$ and $g(x)\in S_+(u)$. Then, by lemma 1, $h_\pm(x,\xi)=A_g^*f_\pm(x,\xi;u)$ solve the equation

$$\hat{H}_{u,g}h_\pm=\xi^2h_\pm.$$

Note that $(\phi,H_u\phi)=(A_g^*\phi,A_g^*\phi)\geqq 0$ for $g\in S_+(u)$ and for $\phi\in C_0^\infty$. On the other hand, if $u(x)$ is negative on some open interval then one easily verifies that there exists $\psi\in D(H_u)$ such that $(\psi,H_u\psi)<0$. Hence it follows that $u(x)$ is non-negative.Hence, by lemma 3, $q(x)=(d/dx)\log g(x)$ tends to 0 as $x\to\infty$. Therefore $h_\pm(x,\xi)$ turn out to behave like $\pm i\xi e^{+i\xi x}$ as $x\to\pm\infty$ respectively. Thus we have

$$f_\pm(x,\xi;\hat{u}_g)=\pm i\xi^{-1}A_g^*f_\pm(x,\xi;u),\ \xi\in\boldsymbol{R}\backslash\{0\}.$$

By straightforward calculation, one verifies

$$\begin{aligned}a(\xi;\hat{u}_g)&=(2i\xi)^{-1})W(-i\xi^{-1}A_g^*f_-(x,\xi;u),\ i\xi^{-1}A_g^*f_+(x,\xi;u))\\&=(2i\xi)^{-1}W(f_-(x,\xi;u),f_+(x,\xi;u))=a(\xi;u)\end{aligned}$$

and

$$\begin{aligned}b(\xi;\hat{u}_g)&=(2i\xi)^{-1}W(-i\xi^{-1}A_g^*f_+(x,-\xi;u),\ -i\xi^{-1}A_g^*f_-(x,\xi;u))\\&=-(2i\xi)^{-1}W(f_+(x,-\xi;u),f_-(x,\xi;u))=-b(\xi;u).\end{aligned}$$

Therefore

$$r_+(\xi;\hat{u}_g)=b(\xi;\hat{u}_g)/a(\xi;\hat{u}_g)=-b(\xi;u)/a(\xi;u)=-r_+(\xi;u)$$

and

$$t(\xi;\hat{u}_g)=1/a(\xi;\hat{u}_g)=1/a(\xi;u)=t(\xi;u)$$

follows. The proof for $r_-(\xi;\hat{u}_g)$ is completely parallel to the above. Q.E.D.

Corollary. *Suppose that* H_u *is of type* III_+ *(resp.* III_-*) and let* $f(x)\in S^{(+)}(u;\boldsymbol{R}_+)$ *(resp.* $S^{(-)}(u;\boldsymbol{R}_+)$*) then*

$$r_\pm(\xi;\hat{u}_f)=-r_\pm(\xi;u) \tag{5}$$

and

(6) $$t(\xi;\hat{u}_f)=t(\xi;u)$$

hold.

PROOF. By Theorem 2, $S^{(+)}(u;\boldsymbol{R}_+)$ is included in $S_+(u)$. Hence, from Proposition 4, (5) and (6) follow. Q.E.D.

4. Properties of $\hat{H}_{u,f}$

In [5], P. Deift and E. Trubowitz solved the characterization problem in the class L_2^1 by giving necessary and sufficient conditions for a given 2×2 matrix $(s_{ij}(\xi))_{i,j=1,2},\xi\in\boldsymbol{R}\backslash\{0\}$, to be the scattering matrix of a potential in L_2^1. In this section, on the basis of this result, we show that if H_u is of type III$\pm$ and satisfies some additional conditions then $\hat{H}_{u,f}$ is of type II. More precisely we have

Theorem 5. *Suppose that H_u is of type* III$_+$ *and $f(x)$ is in $S^{(+)}(u;\boldsymbol{R}_+)$. Moreover assume that*

(7) $$F(x)=\pi^{-1}\int_{-\infty}^{\infty}r_+(\xi;u)e^{2i\xi x}d\xi$$

is absolutely continuous with

(8) $$\int_{\alpha}^{\infty}(1+x^2)|F'(x)|dx<\infty$$

for all $\alpha\in\boldsymbol{R}$. Then, $\hat{H}_{u,f}$ is of type II. *In addition*

$$f_+(x,0;\hat{u}_f)=K/f(x)$$

holds, where $K=\lim_{x\to\infty}f(x)$ is the positive real number. The completely parallel assertion is valid also for H_u of type III$_-$.

PROOF. We show that the scattering matrix of $\hat{H}_{u,f}$ satisfies the conditions (cf. [5; pp 210–212]) for a given 2×2 matrix to be the scattering matrix of the 1-dimensional Schrödinger operator with a potential in L_2^1 without bound states. First, by Proposition 4, we have $r_\pm(\xi;\hat{u}_f)=-r_\pm(\xi;u)$ and $t(\xi;\hat{u}_f)=t(\xi;u)$. Since $f_-(x,\xi;u)$ $(\xi\in\boldsymbol{R}\backslash\{0\})$ solves the integral equation

$$f_-(x,\xi;u)=e^{-i\xi x}-\int_{-\infty}^{x}\xi^{-1}\sin\xi(y-x)u(y)f_-(y,\xi;u)dy,$$

we have

$$f_-(x, \xi; u)=e^{-i\xi x}(1-(2i\xi)^{-1}\int_{-\infty}^{\infty} e^{i\xi y}u(y)f_-(y, \xi; u)dy)$$
$$+e^{i\xi x}((2i\xi)^{-1}\int_{-\infty}^{\infty} e^{-i\xi y}u(y)f_-(y, \xi; u)dy)+o(1)$$

as $x \to \infty$. This implies

$$a(\xi; u)=1-(2i\xi)^{-1}\int_{-\infty}^{\infty} e^{i\xi y}u(y)f_-(y, \xi; u)dy$$

and

$$b(\xi; u)=(2i\xi)^{-1}\int_{-\infty}^{\infty} e^{-i\xi y}u(y)f_-(y, \xi; u)dy.$$

Hence we have

$$r_+(\xi; u)=b(\xi; u)/a(\xi; u)=O(1/\xi), \quad |\xi| \to \infty.$$

Moreover note that, by (A) in Definition (III) and Corollary, $r_\pm(\xi; \hat{u}_f)$ is continuous even at $\xi=0$ and $r_+(0; \hat{u}_f)=-1$ hold. Therefore, by [5; Theorem 3, P 212], it suffices to prove that $t(\xi; \hat{u}_f)$ is the boundary value of the function analytic in the upper half plane. While the Jost solutions $f_\pm(x, \xi; u)$ themselves are not defined for $\operatorname{Im}\xi>0$ in general when $u(x)$ is in L_0^1, we show that $t(\xi; u)$ can be extended analytically into the upper half plane in our case. To do this, put

$$T(\zeta)=\exp\{(2\pi i)^{-1}\int_{-\infty}^{\infty} (\xi-\zeta)^{-1} \log(1-|r_+(\xi; u)|^2)d\xi\}$$

for $\operatorname{Im}\zeta>0$. The function $T(\zeta)$ is defined and analytic in $\operatorname{Im}\zeta>0$ because log $(1-|r_+(\xi; u)|^2)$ is locally integrable by (A) in Definition (III) and $\log(1-|r_+(\xi; u)|^2)$ $=O(1/|\xi|)$ as $|\xi| \to \infty$. Then, by quite similar argument to [5; pp 154–156], one can verify that $t(\xi; u)=t(\xi; \hat{u}_f)$ is the boundary value of the function $T(\zeta)$ analytic in Im $\zeta>0$. Consequently, we have proved that the 2×2 matrix

$$\begin{pmatrix} t(\xi; \hat{u}_f) & r_-(\xi; \hat{u}_f) \\ r_+(\xi; \hat{u}_f) & t(\xi; \hat{u}_f) \end{pmatrix}$$

satisfies the conditions to be the scattering matrix of the operator with the potential in L_2^1. Thus $\hat{u}_f \in L_2^1$ follows. On the other hand, by the commutation formula [4],

$$\sigma(PP^*)\backslash\{0\}=\sigma(P^*P)\backslash\{0\},$$

where P is a densely defined linear operator on the Hilbert space and $\sigma(*)$ denotes spectrum, we have $\sigma_p(\hat{H}_{u,f})\backslash\{0\}=\phi$. Moreover it is well known that $\sigma_c(H_v)=[0, \infty)$ if

$v(x) \in L_1^1$, where $\sigma_c(*)$ denotes the continuous spectrum. Thus we have proved that $\hat{H}_{u,f}$ is of type II. Hence, by [5; Remark 9, p 125], $f_\pm(x, 0; \hat{u}_f)$ are linearly independent. On the other hand, by theorem of Deift-Trubowitz about the virtual levels [5; Theorem 3, p 163], $f_\pm(x, 0; \hat{u}_f)$ turn out to be in $S_+(\hat{u}_f)$. Since $1/f(x) \in S_+(\hat{u}_f)$ by lemma 1, there exist α, $\beta \in \boldsymbol{R}$ such that

$$1/f(x) = \alpha f_+(x, 0; \hat{u}_f) + \beta f_-(x, 0; \hat{u}_f).$$

H_u is realized as the Darboux transformation of $\hat{H}_{u,f}$ by $1/f(x)$ and $\sigma_p(H_u) = \phi$. Hence, by [8; pp 23–24], $\alpha = 0$ or $\beta = 0$ is valid. On the other hand, $\alpha = 1/K$ follows, because $1/f(x)$ tends to $1/K$ and $f_-(x, 0; \hat{u}_f) = O(x)$ as $x \to \infty$. Q.E.D.

5. Solution of reconstruction problem

The reconstruction problem in scattering theory is understood in general to give an algorithm for recovering all potentials from given scatterisng data (cf. [5; p 122]). In this section we give a solution to this problem in the following restricted sene: Let $r(\xi)$, $\xi \in \boldsymbol{R}$, have the following properties.

(R_1) $r(\xi)$ is continuous for all real ξ.

(R_2) $|r(\xi)| < 1$ holds for all $\xi \in \boldsymbol{R} \setminus \{0\}$.

(R_3) $r(\xi) = O(1/\xi)$ as $\xi \to \pm\infty$.

(R_4) $r(\xi) - 1 = i\rho\xi + o(\xi)$ as $\xi \to 0$, where ρ is a real number.

(R_5) $\overline{r(\xi)} = r(-\xi)$.

(R_6) $F(x) = \pi^{-1} \int_{-\infty}^{\infty} r(\xi) e^{2i\xi x} d\xi$ is absolutely continuous with

$$\int_\alpha^\infty (1 + x^2)|F'(x)| dx < \infty \quad \text{for all} \quad \alpha \in \boldsymbol{R}.$$

The reconstruction problem considered here is to construct the set $\Sigma(r(\xi))$ of all potentials $u(x) \in L_0^1$ such that $r_+(\xi; u) = r(\xi)$, $\sigma_p(H_u) = \phi$ and $S^{(+)}(u; \boldsymbol{R}_+) \cup S^{(-)}(u; \boldsymbol{R}_+) \neq \phi$.

We solve the above problem in the following. Roughly speaking, by the solution of the characterization problem [5; Theorem 3, p 212], there exists the unique potential without bound states in L_2^1 such that its reflection coefficient coincides with $-r(\xi)$. More precisely, put

$$F_\pm(x) = \pi^{-1} \int_{-\infty}^{\infty} r_\pm(\xi) e^{\pm 2i\xi x} d\xi,$$

where $r_+(\xi) = -r(\xi)$,

$$a(\xi)=\lim_{\eta\to+0}\exp\{(2\pi i)^{-1}\int_{-\infty}^{\infty}(\zeta-\xi-i\eta)^{-1}\log(1-|r(\zeta)|^2)d\zeta\}$$

and $r_-(\xi)=-r_+(-\xi)a(-\xi)/a(\xi)$. Then, the Gelfand-Levitan-Marchenko equation

$$B_\pm(x, y)\pm\int_0^{\pm\infty}F_\pm(x+y+z)B_\pm(x, z)dz+F_\pm(x+y)=0$$

are uniquely solvable for each x. By the assumption (R_4), $\mp B_\pm(x, 0)$ coincide with each other; we denote it by $v(x)$. Then it follows that $H_v=-\partial_x^2+v(x)$ is of type II and $r_+(\xi; v)=-r(\xi)$. Next put

$$u_\pm(x)=v(x)-2(d/dx)^2\log f_\pm(x, 0; v).$$

Then, by [8; Corollary, p 25], $H_{u\pm}$ turn out to be of type III$\pm$, respectively and $r_+(\xi; u_\pm)=r(\xi)$ follow. Therefore $u_\pm(x)$ belong to $\Sigma(r(\xi))$. Moreover we have

Theorem 6. *If the conditons* (R_1)-(R_6) *are fulfilled, then*

$$\Sigma(r(\xi))=\{u_+(x), u_-(x)\}$$

holds.

PROOF. Suppose that $w(x)$ is in $\Sigma(r(\xi))$. First we assume H_w to be of type III$_+$. Let $h(x)\in S^{(+)}(w; \boldsymbol{R}_+)$. Then, by Theorem 2, $h(x)$ is in $S_+(u)$. Hence $\hat{H}_{w,h}$ is of type II by Theorem 5, and $r_+(\xi; \hat{w}_h)=-r(\xi)$ holds by Corollary. On the other hand, by Levinson's theorem [5; Corollary, p 208]), this implies $\hat{w}_h(x)=v(x)$. Therefore we have $w(x)=u_+(x)$ by Theorem 5. The proof in the case of type III$_-$ is completely parallel to the above.

Q.E.D.

6. Concluding remark

It is well known that if $u(x)$ is in L_1^1 then the Jost solutions $f_\pm(x, \zeta; u)$ are analytic in Im $\zeta>0$ whereas if $u(x)$ is only assumed to be in L_0^1, such analyticity is not necessarily expected. Moreover if $u(x)$ is in L_1^1 then $e^{\mp i\zeta x}f_\pm(x, \zeta; u)-1$ are known to belong to the Hardy space H^{2+} of functions $\phi(\zeta)$ analytic in Im$\zeta>0$ with

$$\sup_{\eta>0}\int_{-\infty}^{\infty}|\phi(\xi+i\eta)|^2d\xi<\infty.$$

This property plays crucial role in the inverse problem of scattering theory. In the following, we show that while the Jost solutions $f_\pm(x, \zeta; u)$ for H_u of type III$\pm$ are extended analytically to Im$\zeta>0$, $e^{\mp i\zeta x}f(x, \zeta; u)-1$ do not belong to H^{2+}.

Suppose that H_u is of type III$_+$. Then, there exists uniquely the potential $v(x)$ such that H_v is of type II and $H_u=\hat{H}_{v,f_+}$ holds, where $f_+=f_+(x,0;v)$. By the argument in the proof of Proposition 4, one verifies

(9) $$f_{\pm}(x,\zeta;u)=\pm i\zeta^{-1}A^*_{f_+}f_{\pm}(x,\zeta;v).$$

Since H_v is of type II, especially $v(x)$ is in L^1_2, the right hand side of (9) is analytic in $\mathrm{Im}\,\zeta>0$. Moreover, because $e^{\mp i\zeta x}f_{\pm}(x,\zeta;v)-1$ is in H^{2+}, the integral representations

$$f_{\pm}(x,\zeta;v)=e^{\pm i\zeta x}\{1\pm\int_0^{+\infty}B_{\pm}(x,y)e^{\pm 2i\zeta y}dy\}$$

are valid (see [5] for more detail.). Hence we have

(10) $$f_{\pm}(x,\zeta;u)=e^{\pm i\zeta x}\{1\pm\int_0^{\pm\infty}B_{\pm}(x,y)e^{\pm 2i\zeta y}dy$$
$$\pm i\zeta^{-1}(q(x)\pm\int_0^{\pm\infty}C_{\pm}(x,y)e^{\pm 2i\zeta y}dy)\},$$

where $C_{\pm}(x,y)=A^*_{f_+}B_{\pm}(x,y)=-B_{\pm x}(x,y)+q(x)B_{\pm}(x,y)$ and $q(x)=(d/dx)\log f_+(x,0;v)$. Obviously, the right hand side of (10) is analytic in $\mathrm{Im}\,\zeta>0$. On the other hand, by (10), $f_{\pm}(x,\zeta;u)$ may be singular at $\zeta=0$. First suppose H_u to be of type III$_+$. Then, by direct calculation, we have

$$\lim_{\zeta\to 0} i\zeta f_+(x,\zeta;u)=0,$$

and

$$\lim_{\zeta\to 0} -iv^{-1}\zeta f_-(x,\zeta;u)=1/f_+(x,0;v),$$

where $v=W(f_+(x,0;v),f_-(x,0;v))>0$ (cf. [8; Corollary 2.2, p 18]). Since $f_+(x,0;v)$ do not vanish by [5; Theorem 3, p 163], $f_-(x,\zeta;u)$ is singular at $\zeta=0$. Similarly, if H_u is of type III$_-$,

$$\lim_{\zeta\to 0} -iv^{-1}\zeta f_+(x,\zeta;u)=1/f_-(x;0;v),$$

and

$$\lim_{\zeta\to 0} i\zeta f_-(x,\zeta;u)=0$$

follow and $f_+(x,\zeta;u)$ turns out to be singular at $\zeta=0$.

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