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On the Darboux Transformation of the **1**-*dimensional Schrödinger Operator and Levinson's Theorem*

By

Mayumi OHMIYA

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In the preceding paper [6], the present author studied the Darboux transformation of the 2-nd order ordinary differential operator of Fuchsian type. In this paper, we investigate the Darboux transformation of the 1-dimensional Schrödinger operator

$$H_u = -\partial_x^2 + u(x), \quad \partial_x = d/dx, \quad -\infty < x < \infty.$$

Throughout the paper, we assume that potential $u(x)$ belongs to L_2^1 unless explicitly stated otherwise, where

$$L_\lambda^1 = \left\{ w(x) \mid \text{real valued, continuous and} \int_{-\infty}^{\infty} (1+|x|^\lambda)|w(x)|dx < \infty \right\}, \ \lambda \geqq 0.$$

We consider the unique selfadjoint extension of H_u defined in the space of twice continuously differentiable functions on $(-\infty, \infty)$ with compact support and denote it again by H_u. In this paper, we study the problem only in case of $\sigma_p(H_u) = \emptyset$, where $\sigma_p(*)$ denotes the set of point spectrum. The *Darboux transformation* of H_u is defined as follows: Let $g = g(x)$ be a non-trivial real valued solution of the homogeneous equation

$$H_u y = -y'' + u(x)y = 0, \quad ' = d/dx. \tag{1}$$

Put

$$A_g = g(x)^{-1}\partial_x g(x) = \partial_x + g'(x)/g(x),$$

then

$$H_u = A_g A_g^* \tag{2}$$

follows, where $A_g^* = -g(x)\partial_x g(x)^{-1}$ is the adjoint operator of A_g. By exchanging the role of A_g and A_g^* in (2), we obtain the operator

$$\hat{H}_{u,g} = A_g^* A_g. \tag{3}$$

We call $\hat{H}_{u,g}$ the *Darboux transformation* of H_u by the solution $g(x)$. Put

$$v(x)=g'(x)/g(x),$$

then we have

(4) $$u(x)=v'(x)+v(x)^2.$$

Moreover, put

(5) $$\hat{u}_g(x)=-v'(x)+v(x)^2,$$

then one verifies easily

$$\hat{H}_{u,g}=-\partial_x^2+\hat{u}_g(x).$$

In this paper, we require that $\hat{u}_g(x)$ is continuous together with $u(x)$. This implies that it suffices to investigate the Darboux transformation $\hat{H}_{u,g}$ by the positive solution $g(x)$.

On the other hand, our result is deeply related to the *commutation relation*

(6) $$\sigma(PP^*)\backslash\{0\}=\sigma(P^*P)\backslash\{0\},$$

where P is a densely defined linear operator on the Hilbert space and $\sigma(*)$ denotes spectrum. As for the commutation relation (6), we refer to [2] and [3].

Some preliminary consideration related to the commutation relation has been already done in case of $\hat{u}_g \in L_2^1$ by the present author in [5]. If $\hat{u}_g \in L_2^1$ then, according to [5], spectral property of $\hat{H}_{u,g}$ turns out to be very similar to that of H_u;

$$\sigma(H_u)=\sigma(\hat{H}_{u,g}).$$

However, the new potential $\hat{u}_g$ does not necessarily belong to L_2^1. The main purpose of the present work is to clarify spectral property of $\hat{H}_{u,g}$ in case of $\hat{u}_g \notin L_2^1$ in connection with the scattering theory.

On the other hand, by classical Levinson's theorem, it is well known that the potential in L_1^1 without bound states is uniquely determined by its reflection coefficient. On the contrary, by applying the method of the Darboux transformation, we can show that the uniqueness of the potential corresponding to the reflection coefficient does not hold in general for the potential in $L_0^1\backslash L_1^1$ even without bound states. Thus, it turns out that the assumption of Levinson's theorem such that the potential $u(x)$ is in L_1^1 and $\sigma_p(H_u)=\emptyset$ is the best possible one. As for Levinson's theorem, we refer to [3; Corollary, p. 208].

In section 1, we briefly state about the reflection coefficients of H_u. In section 2, we investigate the positive solutions of the equation (1). In section 3, we study the integrability of $\hat{u}_g$. In section 4, the spectrum of $\hat{H}_{u,g}$ is investigated and the related matter to Levinson's theorem is discussed.

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1. Reflection coefficients

In this section, we briefly sketch the definition and properties of the reflection coefficients of H_u. We refer to [1], [3], [4] and [7] for detail.

First, we assume that the potential $u(x)$ is in L_0^1. Let us consider the integral equation

$$g(x, \xi)=\exp(i\xi x)+\int_x^\infty \xi^{-1}\sin\xi(y-x)u(y)g(y, \xi)dy \tag{1.1}$$

for $\xi\in\boldsymbol{R}\setminus\{0\}$. Put $g_0(x, \xi)=\exp(i\xi x)$ and

$$g_n(x, \xi)=\int_x^\infty \xi^{-1}\sin\xi(y-x)u(y)g_{n-1}(y, \xi)dy.$$

Then we have

$$|g_n(x, \xi)|\leqq|\xi|^{-n}\left(\int_x^\infty |u(y)|dy\right)^n/n!. \tag{1.2}$$

Hence, one verifies immediately that the series $\sum_{n=0}^\infty g_n(x, \xi)$ converges to the solution $f_+(x, \xi)$ of

$$H_uf=\xi^2 f, \quad \xi\in\boldsymbol{R}\setminus\{0\}. \tag{1.3}$$

By (1.2), we have

$$|f_+(x, \xi)-\exp(i\xi x)|\leqq|\xi|^{-1}\eta(x)\exp(|\xi|^{-1}\eta(x)),$$

where

$$\eta(x)=\int_x^\infty |u(y)|dy.$$

Hence $f_+(x, \xi)$ behaves like $\exp(i\xi x)$ as $x\to\infty$. Similarly to the above, we can construct the solution $f_-(x, \xi)$ of (1.3) which behaves like $\exp(-i\xi x)$ as $x\to-\infty$. In view of (1.1), it turns out that the complex conjugate $\overline{f_+(x, \xi)}$ also solves the equation (1.3) and

$$\overline{f_+(x, \xi)}=f_+(x, -\xi)$$

holds. For $\xi\in\boldsymbol{R}\setminus\{0\}$, $f_+(x, \xi)$ and $f_+(x, -\xi)$ are linearly independent, so there exist $a(\xi; u)$ and $b(\xi; u)$ such that

$$f_-(x, \xi)=a(\xi; u)f_+(x, -\xi)+b(\xi; u)f_+(x, \xi).$$

We have

$$a(\xi; u) = (2i\xi)^{-1} W(f_-(x, \xi), f_+(x, \xi)),$$
$$b(\xi; u) = (2i\xi)^{-1} W(f_+(x, -\xi), f_-(x, \xi))$$

and

$$|a(\xi; u)|^2 = 1 + |b(\xi; u)|^2,$$

where $W(f, g) = fg' - f'g$ is the Wronskian. The functions

$$r_\pm(\xi; u) = \pm b(\pm\xi; u)/a(\xi; u), \quad \xi \in \boldsymbol{R}\backslash\{0\}$$

are called right and left reflection coefficients. We have

$$|r_\pm(\xi; u)| < 1, \quad \xi \in \boldsymbol{R}\backslash\{0\}.$$

If the potential $u(x)$ is in L_1^1 then $f_\pm(x, \xi)$ are meaningful even at $\xi = 0$. Moreover, the functions $f'_\pm(x, 0)$ obey the estimates

$$|f'_\pm(x, 0)| \leqq \pm K \int_x^{\pm\infty} |u(y)| dy \tag{1.4}$$

for $\pm x \geqq 0$ respectively, where K is a positive constant (cf. [3; lemma 1, p. 103].). For convenience, we will sometimes adopt the following conventions; $f_{\pm 0}$ stand for $f_\pm(x, 0)$ respectively.

On the other hand, while $r_\pm(\xi; u)$ are defined only for $\xi \neq 0$, if the potential $u(x)$ is in L_2^1 then $r_\pm(\xi; u)$ are continuous even at $\xi = 0$. Moreover, provided $u(x) \in L_2^1$, the following is shown by P. Deift and E. Trubowitz [3; Theorem 1, p. 147]: *If $f_+(x, 0)$ and $f_-(x, 0)$ are linearly dependent, then,*

$$|r_\pm(\xi; u)| \leqq \delta < 1 \tag{1.5}$$

hold for some $\delta > 0$, otherwise, i.e., if $f_+(x, 0)$ and $f_-(x, 0)$ are linearly independent then

$$r_\pm(0; u) = -1 \tag{1.6}$$

hold. This fact plays crucial role in what follows.

If the potential $u(x)$ is in L_2^1 and $\sigma_p(H_u) = \emptyset$, $u(x)$ can be reconstructed uniquely from $r_+(\xi; u)$ by the Gelfand-Levitan-Marchenko procedure (cf. [1], [3] and [4].). Now, according to the classification of the reflection coefficient stated above, we classify H_u itself such as $u(x) \in L_2^1$ and $\sigma_p(H_u) = \emptyset$.

Definition. (1) *We say that H_u is of type* I *if and only if $u(x) \in L_2^1$, $\sigma_p(H_u) = \emptyset$ and the condition* (1.5) *is valid.*

(2) *We say that H_u is of type* II *if and only if $u(x) \in L_2^1$, $\sigma_p(H_u) = \emptyset$ and the condition* (1.6) *is valid.*

2. Positive solutions of (1).

In this section, we investigate the positive solutions of the equation (1).

The following is shown by Deift-Trubowitz [3; Theorem 3, p. 163]: *Let* $u(x) \in L_2^1$. *Then,* $f_\pm(x, 0) > 0$ *for any* $x \in \boldsymbol{R}$ *if and only if* $\sigma_p(H_u) = \emptyset$. Thus, provided $\sigma_p(H_u) = \emptyset$, the equation (1) possesses actually positive solutions $Kf_\pm(x, 0)$ $(K > 0)$. Let $S_+(u)$ be the totality of positive solutions of the equation (1). We introduce the equivalent relation $\sim$ to $S_+(u)$ as follows: For $f(x)$, $g(x) \in S_+(u)$,

$$f(x) \sim g(x)$$

if and only if the ratio $f(x)/g(x)$ is a positive constant. We investigate the quotient set $\tilde{S}_+(u) = S_+(u)/\sim$.

The following lemma plays important role to investigate the asymptotic behavior of $f_\pm(x, 0)$ as $x \to \mp\infty$ respectively.

Lemma 2.1. *Let* $u(x) \in L_2^1$ *and* $\sigma_p(H_u) = \emptyset$. *Then*

$$f_\mp(x, 0) = \pm v f_\pm(x, 0) \int_0^x f_\pm(y, 0)^{-2} dy + c_\pm f_\pm(x, 0) \tag{2.1}$$

are valid, where $v = W(f_+(x, 0), f_-(x, 0))$ *and* $c_\pm = f_\mp(0, 0)/f_\pm(0, 0)$.

Proof. By the assumption, $f_\pm(x, 0)$ do not vanish for any x. Hence, if $v = 0$, i.e., $f_+(x, 0)$ and $f_-(x, 0)$ are linearly dependent, (2.1) is obviously valid. Next suppose $v \neq 0$. Put

$$\phi(x) = f_+(x, 0) \int_0^x f_+(y, 0)^{-2} dy$$

then $f_+(x, 0)$ and $\phi(x)$ are linearly independent. Hence there exist c_j $(j = 1, 2)$ such that

$$f_-(x, 0) = c_1 \phi(x) + c_2 f_+(x, 0).$$

We have

$$v = W(f_+(x, 0), f_-(x, 0)) = c_1 W(f_+(x, 0), \phi(x)) = c_1.$$

Thus we have

$$f_-(x, 0) = v f_+(x, 0) \int_0^x f_+(y, 0)^{-2} dy + c_2 f_+(x, 0).$$

Hence $f_-(0, 0) = c_2 f_+(0, 0)$, i.e., $c_2 = c_+$ follows. Similarly to the above, we can show

$$f_+(x, 0) = -v f_-(x, 0) \int_0^x f_-(y, 0)^{-2} dy + c_- f_-(x, 0). \qquad \text{Q.E.D.}$$

Corollary 2.2. $v=W(f_+(x,0),f_-(x,0))\geqq 0$.

PROOF. Assume $v<0$. Since

$$f_+(x,0)\int_0^x f_+(y,0)^{-2}dy \longrightarrow \infty$$

and $f_+(x,0)\to 1$ as $x\to\infty$, from (2.1), $f_-(x,0)\to -\infty$ follows. This is contradiction, because $f_-(x,0)>0$ for any x. Q. E. D.

From Deift-Trubowitz's theorem mentioned above, $\#\tilde{S}_+(u)\geqq 1$ follows, where $\#$ denotes cardinal. More precisely, we have

Theorem 2.3. (1) *H_u is of type* I *if and only if $u(x)\in L_2^1$, $\sigma_p(H_u)=\emptyset$ and $\#\tilde{S}_+(u)=1$.*

(2) *H_u is of type* II *if and only if $u(x)\in L_2^1$, $\sigma_p(H_u)=\emptyset$ and there exists a bijective mapping*

$$\mu\colon \tilde{S}_+(u) \longrightarrow \bar{\boldsymbol{R}}_+=\boldsymbol{R}_+\cup\{\infty\},$$

where $\boldsymbol{R}_+=[0,\infty)$.

PROOF. First suppose that H_u is of type I. Then, $f_+(x,0)$ and $f_-(x,0)$ are linearly dependent. Note that $f_\pm(x,0)$ are positive solutions of the equation (1), because $\sigma_p(H_u)=\emptyset$. Hence $K=f_+(x,0)/f_-(x,0)$ is the positive real number. Therefore

$$f_+(x,0)\longrightarrow 1,\qquad x\longrightarrow\infty,$$

$$f_+(x,0)\longrightarrow K,\qquad x\longrightarrow -\infty$$

follow. Put

$$\phi(x)=f_+(x,0)\int_0^x f_+(y,0)^{-2}dy$$

then we have $W(f_+(x,0),\phi(x))=1$, i.e., $f_+(x,0)$ and $\phi(x)$ form a fundamental system of solutions of (1). From (2.2),

$$\phi(x)=\begin{cases} x+O(1), & x\longrightarrow\infty \\ K^{-1}x+O(1), & x\longrightarrow -\infty \end{cases}$$

follow, especially, $\phi(x)$ tends to $-\infty$ as $x\to-\infty$. Hence

$$g(x)=k_1f_+(x,0)+k_2\phi(x)$$

belongs to $S_+(u)$ if and only if $k_1>0$ and $k_2=0$. Thus

$$S_+(u)=\{kf_+(x,0)\mid k\in\boldsymbol{R}_+\backslash\{0\}\},$$

i.e., $\sharp\tilde{S}_+(u)=1$ follows. Next suppose that H_u is of type II. Then, $f_+(x, 0)$ and $f_-(x, 0)$ are positive and linearly independent. Consider the linear combination

$$g(x)=k_1 f_+(x, 0)+k_2 f_-(x, 0).$$

Suppose $k_1<0$. Then, by lemma 2.1 and corollary 2.2, $f_+(x, 0)$ tends to $-\infty$ as $x\to-\infty$. Hence $g(x)\notin S_+(u)$ follows. Similarly, if $k_2<0$ then $g(x)\notin S_+(u)$ follows. Thus we have

$$S_+(u)=\{k_1 f_+(x, 0)+k_2 f_-(x, 0)|(k_1, k_2)\in \boldsymbol{R}_+^2\backslash\{(0, 0)\}\}.$$

Put

$$(2.3)\qquad g_\alpha(x)=\begin{cases} f_+(x, 0)+\alpha f_-(x, 0), & \alpha\in\boldsymbol{R}_+, \\ f_-(x, 0), & \alpha=\infty. \end{cases}$$

If $\alpha\neq\beta$ then g_α and g_β are linearly independent. Hence the mapping μ from $\tilde{S}_+(u)$ to $\bar{\boldsymbol{R}}_+$ corresponding the equivalent class $\tilde{g}_\alpha=\{kg_\alpha|k>0\}\in\tilde{S}_+(u)$ to $\alpha\in\bar{\boldsymbol{R}}_+$ is bijective. This completes the proof. Q.E.D.

In this paper, as stated in the introduction, we consider only the Darboux transformation by positive solutions. Hence, provided that $u(x)\in L_2^1$ and $\sigma_p(H_u)=\emptyset$, the Darboux transformation of H_u turns out to be as follows: If H_u is of type I then the Darboux transformation of H_u is nothing but $\hat{H}_{u,f_{+0}}$; otherwise, if H_u is of type II then the Darboux transformation of H_u is the 1-parameter family $\hat{H}_{u,g_\alpha}$ parametrized by $\alpha\in\bar{\boldsymbol{R}}_+$, where g_α is defined by (2.3). More precisely, we have

Lemma 2.4. *H_u is of type* II *if and only if $u(x)\in L_2^1$, $\sigma_p(H_u)=\emptyset$ and*

$$\hat{u}_{g_\alpha}(x)\neq\hat{u}_{g_\beta}(x)$$

for any α, $\beta\in\bar{\boldsymbol{R}}_+$ $(\alpha\neq\beta)$.

PROOF. Suppose α, $\beta\neq\infty$ and $\alpha\neq\beta$. By direct calculation, we have

$$\begin{aligned}\hat{u}_{g_\alpha}(x)-\hat{u}_{g_\beta}(x)&=2\{(g'_\alpha(x)/g_\alpha(x))^2-(g'_\beta(x)/g_\beta(x))^2\}\\ &=2(\beta-\alpha)W(f_{+0}, f_{-0})(1/g_\alpha(x)g_\beta(x))'.\end{aligned}$$

Hence, provided that H_u is of type II,

$$\hat{u}_{g_\alpha}(x)-\hat{u}_{g_\beta}(x)=0$$

if and only if

$$1/g_\alpha(x)g_\beta(x)=c,$$

where c is a constant. This implies

$$W(g_\alpha, 1/g_\alpha)=cW(g_\alpha, g_\beta)=c(\beta-\alpha)W(f_{+0}, f_{-0}).$$

Put

$$k=-c(\beta-\alpha)W(f_{+0}, f_{-0})/2$$

then we have

$$g'_\alpha(x)/g_\alpha(x)=k.$$

Hence, by (4), we have $u(x)=k^2$. On the other hand, since $u(x)\in L^1_2$, $k=0$, i.e., $u(x)\equiv 0$ follows. However, $H_0=-\partial_x^2$ is obviously of type I. This is contradiction. A proof in case of $\alpha\neq\infty$, $\beta=\infty$ is similar to the above. Q. E. D.

3. Integrability of $\hat{u}_{g_\alpha}(x)$.

First we investigate $\hat{u}_{g_\alpha}(x)$ $(0<\alpha<\infty)$ in case of type II. By lemma 2.1, we have

$$g_\alpha(x)=f_+(x, 0)\left(1+\alpha c_+ +\alpha v\int_0^x f_+(y, 0)^{-2}dy\right)$$

and

$$g'_\alpha(x)=f'_+(x, 0)\left(1+\alpha c_+ +\alpha v\int_0^x f_+(y, 0)^{-2}dy\right)+\alpha v f_+(x, 0)^{-1}.$$

Suppose $x\geqq 0$ then, by (1.2), we have

$$\begin{aligned}|f'_+(x, 0)\int_0^x f_+(y, 0)^{-2}dy| &\leqq K_1\int_x^\infty |u(z)|dz\int_0^x f_+(y, 0)^{-2}dy\\ &\leqq K_1\int_x^\infty |u(z)|\int_0^z f_+(y, 0)^{-2}dydz\\ &\leqq K_2\int_x^\infty z|u(z)|dz\\ &\leqq K_2\int_x^\infty (1+z^2)\,|u(z)|dz.\end{aligned}$$

Hence,

(3.1) $$g'_\alpha(x)\longrightarrow \alpha v, \quad x\longrightarrow\infty$$

and

$$g_\alpha(x)=\alpha v x+O(1), \quad x\longrightarrow\infty$$

follow. Thus $(g'_\alpha(x)/g_\alpha(x))^2$ behaves like x^{-2} as $x\to\infty$. Similarly to the above, we can show that $(g'_\alpha(x)/g_\alpha(x))^2$ behaves like x^{-2} as $x\to-\infty$. Therefore $(g'_\alpha(x)/g_\alpha(x))^2$ turns out to belong to $L^1_0\backslash L^1_1$. Since

$$\hat{u}_{g_\alpha}(x) = -u(x) + 2(g'_\alpha(x)/g_\alpha(x))^2$$

and $u(x) \in L^1_2 \subset L^1_0$, $\hat{u}_{g_\alpha}(x)$ also belongs to L^1_0. On the other hand, if we assume $\hat{u}_{g_\alpha}(x) \in L^1_1$, then

$$2(g'_\alpha(x)/g_\alpha(x))^2 = u(x) + \hat{u}_{g_\alpha}(x) \in L^1_1$$

follows. This is contradiction, *i.e.*, $\hat{u}_{g_\alpha}(x) \in L^1_0 \backslash L^1_1$.

Next we consider $\hat{u}_{g_0}(x)$. Since $g_0(x) = f_+(x, 0)$, by (1.2), one verifies

$$\int_0^\infty (g'_0(x)/g_0(x))^2 dx \leqq K \int_0^\infty |u(x)| dx.$$

Hence

$$\int_0^\infty |\hat{u}_{g_0}(x)| dx < \infty$$

follows. On the other hand, by lemma 2.1, $(f'_+(x, 0)/f_+(x, 0))^2$ turns out to behave like x^{-2} as $x \to -\infty$. This implies that $\hat{u}_{g_0}(x)$ belongs to $L^1_0 \backslash L^1_1$. Similarly to the above, we can show that $\hat{u}_{g_\infty}(x)$ also belongs to $L^1_0 \backslash L^1_1$. Thus we have proved

Theorem 3.1. *If H_u is of type* II *then $\hat{u}_{g_\alpha}(x)$ belongs to $L^1_0 \backslash L^1_1$ for any $\alpha \in \bar{\boldsymbol{R}}_+$.*

Next we consider H_u of type I. We have

Theorem 3.2. *If H_u is of type* I *then $\hat{u}_{f_{+0}}(x)$ belongs to L^1_2. Conversely, if $u(x) \in L^1_2$, $\sigma_p(H_u) = \emptyset$ and $\hat{u}_g(x) \in L^1_2$ for some $g(x) \in S_+(u)$ then H_u is of type* I *and $g(x) = Kf_+(x, 0)$ holds for some positive constant.*

Proof. Since $1/f_+(x, 0)$ is bounded and

$$\hat{u}_{f_{+0}}(x) = -u(x) + 2(f'_+(x, 0)/f_+(x, 0))^2,$$

we have

$$\begin{aligned} &\int_{-\infty}^\infty (1+x^2) |\hat{u}_{f_{+0}}(x)| dx \\ &\quad \leqq \int_{-\infty}^\infty (1+x^2) |u(x)| dx + K_1 \int_{-\infty}^\infty (1+x^2) f'_+(x, 0)^2 dx. \end{aligned}$$

Since H_u is of type I, $f_+(x, 0)$ and $f_-(x, 0)$ are linearly dependent. Hence, by (1.2), we have

$$\begin{aligned} \int_{-\infty}^\infty (1+x^2) f'_+(x, 0)^2 dx &\leqq K_2 \left\{ \int_0^\infty (1+x^2) \left(\int_x^\infty |u(y)| dy \right)^2 dx \right. \\ &\quad \left. + \int_{-\infty}^0 (1+x^2) \left(\int_{-\infty}^x |u(y)| dy \right)^2 dx \right\} \\ &\leqq K_3 \left(\int_{-\infty}^\infty (1+x^2) |u(x)| dx \right)^2 < \infty. \end{aligned}$$

Thus, $\hat{u}_{f_{+0}}(x) \in L_2^1$ follows.
Conversely, suppose that $u(x) \in L_2^1$, $\sigma_p(H_u) = \emptyset$ and $\hat{u}_g(x) \in L_2^1$ for some $g(x) \in S_+(u)$. Then, by Theorem 3.1, H_u turns out to be of type I. Moreover, since H_u is of type I, by Theorem 2.3, there exists $K > 0$ such that $g(x) = Kf_+(x, 0)$. Q. E. D.

By the above, we can refine on the result obtained in [5]. We have

Theorem 3.3. *If H_u is of type* I, *then there exists uniquely $v(x) \in L_1^1 \cap C_a$, where C_a is the totality of absolutely continuous functions, such that*

$$H_u = (\partial_x + v(x))(-\partial_x + v(x)) \tag{3.2}$$

holds. Conversely, if there exists $v(x) \in L_1^1 \cap C_a$ such that (3.2) *is valid, then H_u is of type I. Moreover, existence of such $v(x)$ is unique.*

PROOF. First assume that H_u is of type I. Then, by [5; Theorem], there exists $v(x) \in L_1^1 \cap C_a$ such that (3.2) is valid. If we assume that there exists another $w(x) \in L_1^1 \cap C_a$ such that (3.2) is valid. Put

$$g(x) = \exp\left(\int_{-\infty}^{x} v(y)\,dy\right)$$

and

$$h(x) = \exp\left(\int_{-\infty}^{x} w(y)\,dy\right).$$

Then, both $g(x)$ and $h(x)$ belong to $S_+(u)$. Since H_u is of type I, by Theorem 2.3, there exists $K > 0$ such that $h(x) = Kg(x)$. This implies $v(x) = w(x)$. Thus we have proved that such a $v(x)$ as (3.2) exists uniquely.

Next assume that there exists $v(x) \in L_1^1 \cap C_a$ such that (3.2) is valid. Then, by [5; Theorem], H_u is of type I. Hence, unique existence of $v(x) \in L_1^1 \cap C_a$ follows from the first half of this theorem. Q. E. D.

Moreover, we have

Theorem 3.4. *If H_u is of type* I *then $\hat{H}_{u,f_{+0}}$ is also of type* I.

PROOF. By Theorem 3.2, $\hat{H}_{u,f_{+0}}$ is non-negative definite. On the other hand, it is well known that the non-negative 1-dimensional Schrödinger operator with the L_2^1-potential has a purely absolutely continuous spectrum and no bound states. Thus, $\sigma_p(\hat{H}_{u,f_{+'}}) = \emptyset$ follows. On the other hand, by [5; lemma 1],

$$r_\pm(\xi; u) = -r_\pm(\xi; \hat{u}_{f_{+0}})$$

are valid. Therefore

$$|r_\pm(\xi; \hat{u}_{f_{+0}})| \leqq \delta < 1$$

hold. Thus, $\hat{H}_{u,f_{+0}}$ is of type I. Q. E. D.

4. Spectrum of $\hat{H}_{u,g_\alpha}$.

As is mentioned in the proof of Theorem 3.4, if H_u is of type I, then the spectrum of $\hat{H}_{u,f+0}$ consists of only purely absolutely continuous spectrum $[0, \infty)$. In this section, we investigate the spectrum of $\hat{H}_{u,g_\alpha}$ precisely when H_u is of type II.

First of all, by the commutation relation (6), we have immediately

$$(0, \infty)\subset\sigma_c(\hat{H}_{u,g_\alpha})$$

and

$$\sigma_p(\hat{H}_{u,g_\alpha})\cap(-\infty, 0)=\emptyset,$$

where $\sigma_c(*)$ denotes the continuous spectrum. Thus, it suffices to consider which of the following three cases is valid;

(i) $0\in\rho(\hat{H}_{u,g_\alpha})$,

(ii) $0\in\sigma_c(\hat{H}_{u,g_\alpha})$,

(iii) $0\in\sigma_p(\hat{H}_{u,g_\alpha})$,

where $\rho(*)$ denotes the resolvent set. First, we have

Theorem 4.1. *If H_u is of type* II, *then* (iii) *is valid for any* $\alpha\in(0, \infty)$.

PROOF. Put

$$\phi(x; \alpha)=1/g_\alpha(x),$$

Then $\phi(x; \alpha)$ solves the equation

$$\hat{H}_{u,g_\alpha}y=0.$$

If H_u is of type II, then, by lemma 2.1, we have

$$f_-(x, 0)=x+O(1), \quad x\longrightarrow\infty$$

and

$$f_+(x, 0)=1+o(1), \quad x\longrightarrow\infty.$$

Hence, if $0<\alpha<\infty$ then $\phi(x; \alpha)$ behaves like $1/\alpha x$ as $x\to\infty$. Similarly, it turns out that $\phi(x; \alpha)$ behaves like $1/x$ as $x\to-\infty$. Thus, $\phi(x; \alpha)$ is square integrable for any $\alpha\in(0, \infty)$. This implies that 0 is the eigenvalue of $\hat{H}_{u,g_\alpha}$. Q.E.D.

On the other hand, one verifies that $\phi_1(x)=1/f_+(x, 0)$ and

$$\phi_2(x)=\phi_1(x)\int_0^x\phi_1(y)^{-2}dy$$

are the fundamental system of solutions of

(4.1) $$\hat{H}_{u,g_0}y=0.$$

We have

(4.2) $$\phi_1(x)=1+o(1), \quad x \longrightarrow \infty,$$
$$\phi_2(x)=x+O(1), \quad x \longrightarrow \infty.$$

Hence, the equation (4.1) has no square integrable solution, i.e., $0 \notin \sigma_p(\hat{H}_{u,g_0})$. Similarly to the above, we can show $0 \notin \sigma_p(\hat{H}_{u,g_\infty})$. More precisely, we have

Theorem 4.2. *If H_u is of type* II *then* 0 *belongs to the continuous spectrum* $\sigma_c(\hat{H}_{u,g_\mu})$ *for* $\mu=0, \infty$.

PROOF. We prove the statement in case of $\mu=0$. Put

$$(R_0f)(x)=-\int_{-\infty}^{x}\phi_1(y)\phi_2(x)f(y)dy-\int_{x}^{\infty}\phi_1(x)\phi_2(y)f(y)dy$$

for a infinitely differentiable function f with the compact support. Then, by straight forward calculation, we have

$$\hat{H}_{u,g_0}R_0f=R_0\hat{H}_{u,g_0}f=f.$$

Let $\psi(x)$ be the infinitely differentiable function such that $\psi(x)>0$ on the interval $I=(a, b)$ and $\psi(x)\equiv 0$ on the complement I^c. Then if $x \geqq b$,

$$(R_0\psi)(x)=-K\phi_2(x)$$

holds, where

$$K=\int_{a}^{b}\phi_1(y)\psi(y)dy$$

is the positive constant. Hence, from (4.2),

$$(R_0\psi)(x)=-Kx+O(1), \quad x \longrightarrow \infty$$

follows. Therefore, the operator R_0 is not bounded, i.e., 0 does not belong to the resolvent set $\rho(\hat{H}_{u,g_0})$. Since 0 is not an eigenvalue of $\hat{H}_{u,g_0}$, 0 turns out to belong to the continuous spectrum $\sigma_c(\hat{H}_{u,g_0})$. The proof in case of $\mu=\infty$ is complete parallel to the above. Q.E.D.

On the other hand, as for the reflection coefficient of $\hat{H}_{u,g_\alpha}$, we have the following.

Theorem 4.3. (Tanaka's lemma). *The identities*

(4.3) $$r_\pm(\xi;\, \hat{u}_{g_\alpha})=-r_\pm(\xi;\, u)$$

are valid for any $\alpha \in \bar{\boldsymbol{R}}_+$.

PROOF. If H_u is of type I, the proof has been already given in [5; pp. 44–46]. On the other hand, if we take the argument in §3 into consideration, the proof in case of type II turns out to be completely parallel to [5; pp. 44–46]. Q. E. D.

We have immediately

Corollary. *There exist the potentials* $u_\mu(x)$ $(\mu=0, \infty)$ *in* $L_0^1\backslash L_1^1$ *such that* $u_0(x)\neq u_\infty(x)$, $\sigma_p(H_{u_0})=\sigma_p(H_{u_\infty})=\emptyset$ *and*

$$r_\pm(\xi; u_0)=r_\pm(\xi; u_\infty). \tag{4.4}$$

PROOF. Suppose that H_u is of type II. Put

$$u_\mu(x)=\hat{u}_{g_\mu}(x), \quad \mu=0, \infty.$$

Then, from lemma 2.4, $u_0(x)\neq u_\infty(x)$ follows. By Theorem 3.2, $u_\mu(x)$ $(\mu=0, \infty)$ belong to $L_0^1\backslash L_1^1$. Moreover, by Theorem 4.2, we have

$$\sigma_p(H_{u_\mu})=\emptyset, \quad \mu=0, \infty.$$

On the other hand, by Theorem 4.3,

$$r_\pm(\xi; u_\mu)=-r_\pm(\xi; u), \quad \mu=0, \infty$$

hold. Therefore, (4.4) follows. Q. E. D.

Thus, it turns out that if the potential is in $L_0^1\backslash L_1^1$ then the reflection coefficient does not determine the potential uniquely in general even though without bound states. This implies that Levinson's theorem [3; Corollary, p. 208] does not hold in the class of potentials in $L_0^1\backslash L_1^1$ any more.

Department of Mathematics
College of General Education
Tokushima University

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