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On the Stability of Maximal Submanifolds in Pseudo-Riemannian Manifolds

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§ 1. Introduction

There are many studies on the stability of minimal submanifolds in Riemannian manifolds (see, for example, [1], [2]). In the present paper, we investigate the stability of spacelike maximal submanifolds in a pseudo-Riemannian manifold. Let f: $M \to N$ be an immersion of a Riemannian manifold into a pseudo-Riemannian manifold. If f is a maximal immersion, then it represents a critical points for the area function on the space of all immersions of M into N. To study the stability of maximal immersions, we need a formula related with the second derivatives of the area function, the second variational formula.

In §2, we descrive local formulas for immersions into pseudo-Riemannian manifolds. The second variational formula is given in §3. In §4, we will prove the stability of maximal immersions into pseudo-Riemannian manifolds of non-positive sectional curvature. We investigate the instability of a compact Riemannian manifold imbedded into the unit sphere S_p^{n+p} with index p in the lase section.

§ 2. Local formulas

Let N be an $(n+p)$-dimensional pseudo-Riemannian manifold with index p. Let M be an m-dimensional Riemannian manifold isometrically immersed in N. As the pseudo-Riemannian metric of N induces the Riemannian meteric of M, we must assume that $m \leq n$ and we may call it the spacelike immersion. We choose a local field of pseudo-Riemannian orthonormal frames $e_1, e_2, \dots, e_{n+p}$ in N such that at each point of M, $e_1, e_2, \dots, e_m$ span the tangent space of M and forms an orthonormal frame there. We make use of the following convension on the ranges of incices if otherwise stated:

$$1 \leq A, B, C \leq n+p, \ 1 \leq i, j, k \leq m, \ m+1 \leq \alpha, \beta, \gamma \leq n+p$$

and we shall agree that repeated indices are summed over the respective ranges. Let $\{\omega_A\}$ be the coframe field dual to $\{e_A\}$. Then the pseudo-Riemannian metric of N is given locally by

$$ds_N^2 = \sum_{a=1}^{n} \omega_a^2 - \sum_{s=n+1}^{n+p} \omega_s^2 = \sum \varepsilon_A \omega_A^2,$$

where $\varepsilon_a = 1$ for $1 \leq a \leq n$ and $\varepsilon_s = -1$ for $n+1 \leq s \leq n+p$. The structure equations of N are given by

$$\text{(2.1)} \qquad \begin{aligned} d\omega_A &= \sum \varepsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \\ d\omega_{AB} &= \sum \varepsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum \varepsilon_C \varepsilon_D K_{ABCD} \omega_C \wedge \omega_D. \end{aligned}$$

We denote by θ_A, θ_{AB} the restrictions of ω_A, ω_{AB} to M. Then $\theta_\alpha = 0$ for $m+1 \leq \alpha \leq n+p$ and the Riemannian memetric is given by $ds_M^2 = \sum \theta_i^2$. We may put

$$\text{(2.2)} \qquad \omega_{\alpha i} = \sum h_{\alpha ij} \omega_j.$$

Then $h_{\alpha ij}$ are the components of the second fundamental form of the immersion. From (2.1), we obtain the structure equations of M;

$$\text{(2.3)} \qquad \begin{aligned} d\theta_i &= \sum \theta_{ij} \wedge \theta_j, \\ d\theta_{ij} &= \sum \theta_{ik} \wedge \theta_{kj} - \frac{1}{2} \sum R_{ijkl} \theta_k \wedge \theta_l \end{aligned}$$

and the Gauss formula

$$\text{(2.4)} \qquad R_{ijkl} = K_{ijkl} + \sum \varepsilon_\alpha (h_{\alpha ik} h_{\alpha jl} - h_{\alpha il} h_{\alpha jk}).$$

We call $H = \frac{1}{m} \sum_\alpha (\sum_i h_{\alpha ii})\, e_\alpha$ the *mean curvature normal*. In the present paper, we study an immersion with *vanishing mean curvature*, that is, $H=0$. When $n=m$, an immersion with vanishing mean curvature is said to be *maximal*.

§3. Variational formulas

We will follow the method in [3]. Let $f\colon M \to N$ be an immersion as in §2. If M is compact, possibly with boundary, its total volume is given by the integral

$$\text{(3.1)} \qquad V = \int_M \theta_1 \wedge \cdots \wedge \theta_m.$$

Let I be the interval $-\frac{1}{2} < t < \frac{1}{2}$. Let $F\colon M \times I \to N$ be a differentiable mapping such that its restriction to $M \times t$, $t \in I$, is an immersion and that $F(m, 0) = f(m)$, $m \in M$. Put $f_t(m) = F(m, t)$. We call f_t the variation of f. Let $\{e_A(m, t)\}$ be a local frame field over $M \times I$ such that for every $t \in I$, $e_i(m, t)$ are tangent to $F(M \times t)$ and hence $e_\alpha(M, t)$ are normal vectors. The forms ω_A, ω_{AB} can be written

$$(3.1) \qquad \omega_i=\theta_i+a_i dt, \quad \omega_\alpha=a_\alpha dt, \quad \omega_{i\alpha}=\theta_{i\alpha}+a_{i\alpha}dt,$$

where θ_i, $\theta_{i\alpha}$ are linear differential forms in M with coefficients which may depend on t. For $t=0$ they are reduced to the forms with the same notation on M. The vector $\sum \varepsilon_A a_A e_A$ at $t=0$ is called the deformation vector. The operater d on $M\times I$ is written as

$$(3.2) \qquad d=d_M+dt\frac{\partial}{\partial t}.$$

Using (2.1), we get

$$(3.3) \qquad d(\omega_1\wedge\cdots\wedge\omega_m)=\sum \varepsilon_\alpha\omega_\alpha\Omega_\alpha,$$

where

$$(3.4) \qquad \Omega_\alpha=-\sum_i \omega_1\wedge\cdots\wedge\omega_{i-1}\wedge\omega_{i\alpha}\wedge\omega_{i+1}\wedge\cdots\wedge\omega_m.$$

Substituting (3.1) into (3.3) and considering the coefficient of dt, we get

$$(3.5) \qquad \frac{\partial}{\partial t}(\theta_1\wedge\cdots\wedge\theta_m)=d_M\sum(-1)^{i-1}a_i\theta_1\wedge\cdots\wedge\theta_{i-1}\wedge\theta_{i+1}\wedge\cdots\wedge\theta_m + \sum \varepsilon_\alpha a_\alpha \Theta_\alpha$$

where

$$(3.6) \qquad \Theta_\alpha=-\sum_i \theta_1\wedge\cdots\wedge\theta_{i-1}\wedge\theta_{i\alpha}\wedge\theta_{i+1}\wedge\cdots\wedge\theta_m.$$

Thus the first variation of volume is written as

$$(3.7) \qquad V'(0)=\int_M \sum \varepsilon_\alpha a_\alpha\Theta_\alpha+\int_{\partial M}\sum(-1)^{i-1}a_i\theta_1\wedge\cdots\wedge\theta_{i-1}\wedge\theta_{i+1}\wedge\cdots\wedge\theta_m.$$

If the deformation vector is normal to M along the booudary ∂M, the second term at the right hand side of (3.5) vanishers. This condition is satisfied if the boundary ∂M remains fixed. The first integral is zero for arbitrary a_α if and only if the mean curvature normal H vanishes.

To obtain the second variational formula, we also follow Chern (see §8 in [3]). By exterior differentiation of Ω_α, we have

$$(3.8) \qquad -d\Omega_\alpha=-\sum\sum \varepsilon_\beta\omega_\beta\wedge\omega_1\wedge\cdots\wedge\omega_{j-1}\wedge\omega_{j\beta}\wedge\omega_{j+1}\wedge\cdots\wedge\omega_{i-1}\wedge\omega_{i\alpha}\wedge\omega_{i+1} \wedge\cdots\wedge\omega_m+\sum\varepsilon_\beta\omega_{\beta\alpha}\wedge\omega_\beta+\sum\varepsilon_\beta\tilde{K}_{\alpha\beta}\omega_\beta\wedge\omega_1\wedge\cdots\wedge\omega_m + \text{terms quadratic in } \omega_\alpha, \omega_\beta,$$

where

$$(3.9) \qquad \tilde{K}_{\alpha\beta}=\sum_i K_{i\alpha i\beta}.$$

Substituting (3.1) into (3.4), we get

$$\Omega_\alpha = \Theta_\alpha + dt \wedge \Phi_\alpha , \tag{3.10}$$

where

$$\Phi_\alpha = \sum (-1)^i a_{i\alpha} \theta_1 \wedge \cdots \wedge \theta_{i-1} \wedge \theta_{i+1} \wedge \cdots \wedge \theta_m + \sum \sum (-1)^j a_j \theta_1 \wedge \cdots \wedge \theta_{j-1} \wedge \theta_{j+1} \wedge \cdots \wedge \theta_{i-1} \wedge \theta_{i+1} \wedge \cdots \wedge \theta_m . \tag{3.11}$$

From (3.10), it follows

$$d\Omega_\alpha = dt \wedge \frac{\partial \Theta_\alpha}{\partial t} - dt \wedge d_M \Phi_\alpha + d_M \Theta_\alpha . \tag{3.12}$$

Substituting (3.1) into (3.8), we have

$$-d\Omega_\alpha = -dt \wedge (\sum \varepsilon_\beta \omega_{\beta\alpha} \wedge \Phi_\alpha + \Delta_\alpha) + \textit{other terms}, \tag{3.13}$$

where

$$\Delta_\alpha = -\sum \varepsilon_\beta \tilde{k}_{\alpha\beta} a_\beta \theta_1 \wedge \cdots \wedge \theta_m + \sum \sum \varepsilon_\beta a_\beta \theta_1 \wedge \cdots \wedge \theta_{j-1} \wedge \theta_{j\beta} \wedge \theta_{j+1} \wedge \cdots \wedge \theta_{i-1} \wedge \theta_{i\alpha} \wedge \theta_{i+1} \wedge \cdots \wedge \theta_m . \tag{3.14}$$

From (3.12) and (3.13), we obtain

$$\frac{\partial \Theta_\alpha}{\partial t} = d_M \Phi_\alpha + \sum \varepsilon_\beta \omega_{\beta\alpha} \wedge \Phi_\beta + \Delta_\alpha . \tag{3.15}$$

Taking the exterior derivative of the second equation of (3.1) and using (2.1), we get

$$d_M a_\alpha = -\sum (a_i \theta_{i\alpha} - a_{i\alpha} \theta_i) - \sum \varepsilon_\beta a_\beta \omega_{\beta\alpha} . \tag{3.16}$$

Combining (3.15) and (3.16), we have

$$\frac{\partial}{\partial t} (\sum \varepsilon_\alpha a_\alpha \Theta_\alpha) = \sum \varepsilon_\alpha \frac{\partial a_\alpha}{\partial t} \Theta_\alpha + d_M (\sum \varepsilon_\alpha a_\alpha \Phi_\alpha) + \sum \varepsilon_\alpha a_\alpha \Delta_\alpha + \sum \varepsilon_\alpha (a_i \theta_{i\alpha} - a_{i\alpha} \theta_i) \wedge \Phi_\alpha . \tag{3.17}$$

In the sequel, we assume that $f\colon M \to N$ is an immersion with vanising mean curvature normal, that is, $\Theta_\alpha |_{t=0} = 0$. Differentiating (3.5) and setting $t=0$, we get

$$\begin{aligned} V''(0) &= \int \frac{\partial^2}{\partial t^2} (\theta_1 \wedge \cdots \wedge \theta_m) |_{t=0} \\ &= \int_{\partial M} \left(\frac{\partial}{\partial t} \sum (-1)^{i-1} a_i \theta_1 \wedge \cdots \wedge \theta_{i-1} \wedge \theta_{i+1} \wedge \cdots \wedge \theta_m + \sum \varepsilon_\alpha a_\alpha \Phi_\alpha \right) \\ &\quad + \int_M (\sum \varepsilon_\alpha (a_i \theta_{i\alpha} - a_{i\alpha} \theta_i) \wedge \Phi_\alpha + \sum \varepsilon_\alpha a_\alpha \Delta_\alpha). \end{aligned} \tag{3.18}$$

Moreover assume that the variation vector is normal and the boundary ∂M is fixed, that is, $a_i=0$ and $a_\alpha(m, 0)=0$, $m\in\partial M$. Then on M, we have

$$\Phi_\alpha=\sum(-1)^i a_{i\alpha}\theta_1\wedge\cdots\wedge\theta_{i-1}\wedge\theta_{i+1}\wedge\cdots\wedge\theta_m.$$

and

$$\sum\varepsilon_\alpha a_\alpha\Delta_\alpha=-(\sum_{\alpha,\beta}\varepsilon_\alpha\varepsilon_\beta a_\alpha a_\beta\tilde{K}_{\alpha\beta}+\sum\varepsilon_\alpha\varepsilon_\beta a_\alpha a_\beta\sigma_{\alpha\beta})dM$$

where $dM=\theta_1\wedge\cdots\wedge\theta_m$ and

(3.19) $$\sigma_{\alpha\beta}=\sum_{i,j}h_{\alpha ij}h_{\beta ij}.$$

Thus we obtain

(3.20) $$V''(0)=\int(\sum\varepsilon_\alpha a_{i\alpha}^2-\sum\varepsilon_\alpha\varepsilon_\beta a_\alpha a_\beta(\tilde{k}_{\alpha\beta}+\sigma_{\alpha\beta}))dM.$$

From (3.16), we have on M

(3.21) $$\sum a_{i\alpha}\theta_i=da_\alpha+\sum\varepsilon_\beta a_\beta\omega_{\beta\alpha}.$$

Hence $a_{i\alpha}$ are the coefficients of the covariant derivativs of $a=\sum\varepsilon_\alpha a_\alpha e_\alpha$. Its second covariant derivative is given by

(3.22) $$\sum a_{i\alpha i}\theta_i=da_{i\alpha}+\sum a_{j\alpha}\theta_{ji}+\sum\varepsilon_\beta a_{i\beta}\omega_{\beta\alpha}$$

and the Laplacian of a

(3.23) $$\Delta a_\alpha=\sum a_{i\alpha i}.$$

Hence we have

(3.24) $$d(\sum\varepsilon_\alpha a_\alpha a_{i\alpha}*\theta_i)=(\sum\varepsilon_\alpha a_{i\alpha}^2+\langle\Delta a, a\rangle)dM.$$

Thus, by putting

(3.25) $$La_\alpha=-\Delta a_\alpha-\sum\varepsilon_\beta a_\beta(\tilde{K}_{\alpha\beta}+\sigma_{\alpha\beta}),$$

we obtain

(3.26) $$V''(0)=\int_M\langle La, a\rangle dM.$$

The operator introduced in (3.25) is a strongly elliptic operator and has distinct real eigenvalues $\lambda_1<\lambda_2<\cdots<\lambda_p<\cdots\cdots\to\infty$. Moreover, the dimension of each eigenspace is finite. The *index* of M is the sum of the dimensions of the eigenspaces which correspond to negative eigenvalues. The *nullity* to M is the dimension of the null eigenspace.

§4. The stability of spacelike maximal immersions

In this section we assume that $n=m$. Then we have $\varepsilon_\alpha=-1$ for all $n+1\leq\alpha\leq n+p$. The condition $H=0$ implies that the immersion is maximal. In this case, the second variational formula (3.20) is reduced to

$$V''(0)=-\int(\sum a_{i\alpha}^2+\sum(\tilde{k}_{\alpha\beta}+\sigma_{\alpha\beta})a_\alpha a_\beta)dM. \tag{4.1}$$

We can put

$$\|h(a)\|^2=\sum\sigma_{\alpha\beta}a_\alpha a_\beta \tag{4.2}$$

and

$$\sum\tilde{K}_{\alpha\beta}a_\alpha a_\beta=-\sum_i K(e_i, a)\|a\|^2, \tag{4.3}$$

where $\|a\|^2=\sum a_\alpha^2$ and each $K(e_i, a)$ is the sectional curvature of N for the plane spanned by e_i and a. Now we Let $f: M\to N$ be a maximal isometric immersion. Let D be a domain on M with $\bar{D}$ compact. Then the domain is called *stable* if $V_D''(0)\leq 0$, where $V_D''(0)$ is the second variation for the immersion $f|_D: D\to N$, the restriction of f to$D$. The immersion is stable if every such domain is stable. Thus from (4.1), we have

$$V_D''(0)=-\int_D(\sum_{i,\alpha}a_{i\alpha}^2+\|h(a)\|^2-\sum_i K(e_i, a)\|a\|^2)dM. \tag{4.4}$$

Thus we obtain

Proposition 4.1. *Let $f: M\to N$ be an maximal isometric immersion of a Riemannian manifold M into a pseudo-Riemannian manifold with non-positive sectional curvature. Then the maximal immersion is stable.*

Let $H_p^{n+p}(r)$ be the pseudo-hyperbolic space of radius $r(>0)$ (see, for example, [4] or [5]). We constructed the maximal isometric immersion of $H^2(\sqrt{3})$ into $H_2^4(1)$ and the maximal isometric immersion of $H^{n_1}(\sqrt{n_1/n})\times\cdots\times H^{n_{p+1}}(\sqrt{n_{p+1}/n})$ into $H_p^{n+p}(1)$, where $n_1+\cdots+n_{p+1}=n$. From the above proposition, it is evident that these immersions are stable. Let M be a compact hyperserface of N, that is, $p=1$. In this case, a deformation vector a is written as $a=ue_{n+1}$, where $e_1,\dots, e_{n+1}$ is a local orthonormal frame field of N such that on M, e_{n+1} is a timelike normal unit vector field. Let D be a relative compact domain on M. From (4.4), it follows

$$\begin{aligned} V_D''(0)&=-\int_D(|\nabla u|^2+(\|h\|^2-\sum_i K(e_i, e_{n+1}))u^2)dM \\ &=\int_D(u\Delta u-(\|h\|^2-\sum_i K(e_i, e_{n+1}))u^2)dM, \end{aligned} \tag{4.5}$$

where Δ is the Laplacian on D. As it is well known, the first eigenvalue $\lambda_1(D)$ satisfies

$$\int_D |\nabla u|^2 dM \geq \lambda_1(D) \int_D u^2 dM.$$

Hence, from (4.5), we obtain

Proposition 4.2. *If M is a n-dimensional spacelike maximal hypersurface of a Lorentzian manifold N. Assume that the sectional curvature is bounded above by the positive constant c. Let D be a relaive compact domain on M. If the first eigenvalue of the Laplacian of D satisfies $\lambda_1(D) \geq nc$, then D is stable.*

Now assume that M is a maximal sapcelike surface immersed in a 3-dimensional Lorentzian manifold. Then from the Gauss formula (2.4), we get

$$2(R - K(e_1, e_2)) = \|h\|^2, \tag{4.7}$$

where R is the curvature of M. If the sectional curvature of N is bounded above by a positive constant c, we have from (4.6), for a relative compact domain D on M,

$$V_D''(0) \leq -\int_D (|\nabla u|^2 + 2(R - 2c)) dM. \tag{4.8}$$

Thus we obtain

Proposition 4.3. *Let M be a maximal spacelike surface immersed in a 3-dimensional Lorentzian manifold. Assume that the sectional curvature is bounded above by a positive constnat c. If the curvature of M is bounded below by the constant $2c$, the immersion is stable.*

§5. Compact maximal submanifolds in S_p^{n+p}

Let $N = S_p^{n+p}$ be the pseudosphere with index p, that is, $S_p^{n+p} = \{x \in R_p^{n+p+1} : \langle x, x\rangle = x_1^2 + \cdots + x_{n+1}^2 - x_{n+2}^2 - \cdots - x_{n+p+1}^2\}$. In this case, $\tilde{K}_{\alpha\beta}$ defined by (3.9) are given by $\tilde{K}_{\alpha\beta} = n\varepsilon_\alpha \delta_{\alpha\beta}$. In this section, if a Riemannian manifold M is immersed in S_p^{n+p} with vanishing mean curvature normal, we say that M is immersed maximally in S_p^{n+p}, though it is an abuse of langage. Now, we consider the standard totally geodesic imbedding of S^m into S_n^{n+p} $(m \leq n)$. Then the corresponding operater defined by (3.25) is reduced to

$$La_\alpha = -\Delta a_\alpha - n a_\alpha. \tag{5.1}$$

Thus, by using the same argument in the proof of Proposition 5.1.1 in [6], we have

Proposition 5.1. *When S^m is regarded as a maximal submanifold of S_p^{n+p}, its index is $n+p-m$ and its nullity is $(m+1)(n+p-m)$.*

Let $\{f_{A'}\}$ be the frame field on R_p^{n+p+1} given by the parallel translation of the standard base of R_p^{n+p+1}. It satisfies $\langle f_{n+p+1}, f_{n+p+1}\rangle=1$ and

$$(5.2)\qquad \langle f_A, f_B\rangle=\varepsilon_A\delta_{AB},\quad \varepsilon_i=1 \quad \text{for}\quad 0\leq i\leq n+p,$$
$$\varepsilon_\alpha=-1 \quad \text{for}\quad n+1\leq\alpha\leq n+p.$$

Let $\{\theta_{A'}\}$ be the dual coframe field. Then we have

$$(5.3)\qquad \theta_{A'}(f_{B'})=\varepsilon_{A'}\delta_{A'B'} \quad \text{for}\quad 1\leq A',\quad B'\leq n+p+1.$$

Take a local frame field $\{e_{A'}\}$ on S_p^{n+p} such that $\langle e_{A'}, e_{B'}\rangle=\varepsilon_{A'}\delta_{A'B'}$ and for each $x\in S_p^{n+p}$, $e_{n+p+1}(x)=x$. Its dual coframe field $\{\omega_{A'}\}$ satisfies

$$(5.4)\qquad \omega_{A'}(e_{B'})=\varepsilon_{A'}\delta_{A'B'}.$$

Put

$$(5.5)\qquad f_{A'}=\sum_{B'=1}^{n+p+1}\varepsilon_{B'}\lambda_{A'B'}e_{B'},\quad \theta_{A'}=\sum_{B'=1}^{n+p+1}\varepsilon_{B'}\mu_{A'B'}\omega_{B'}.$$

Then we have

$$(5.6)\qquad \sum\varepsilon_{B'}\lambda_{A'B'}\mu_{B'C'}=\varepsilon_{A'}\delta_{A'C'}.$$

We can put

$$(5.7)\qquad dx=\sum\varepsilon_{A'}\omega_{A'}e_{A'},\quad de_{A'}=\sum\varepsilon_{B'}\omega_{A'B'}e_{B'}.$$

By the exterior differentiation of the first equation at (5.5), we get

$$d\lambda_{A'B'}+\sum\varepsilon_{C'}\lambda_{A'C'}\omega_{C'B'}=0.$$

Using (5.6), we have

$$(5.8)\qquad \omega_{A'B'}=-\sum\varepsilon_{C'}\mu_{C'A'}d\lambda_{C'B'}.$$

As we have $\langle dx, x\rangle=0$, we have $\omega_{n+p+1}=0$ on S_p^{n+p}. From (5.7), it follows

$$de_{n+p+1}=\sum\varepsilon_A\omega_{n+1A}e_A=\sum\varepsilon_A\omega_Ae_A.$$

Thus we have $\omega_{n+1A}=\omega_A$ on S_p^{n+p}. Hence the second fundamental form H_{AB} of S_p^{n+p} in R_p^{n+p+1} is given by

$$(5.9)\qquad H_{AB}=\varepsilon_A\delta_{AB}.$$

Let $\tilde{Z}=\sum\varepsilon_{A'}c_{A'}f_{A'}$ be a parallel vector field on R_p^{n+p+1} where $c_{A'}$ are constants. Denote by $Z=\sum\varepsilon_Az_Ae_A$ the tangential projection onto S_p^{n+p} of $\tilde{Z}$. Then we have

$$(5.10)\qquad z_B=\sum\varepsilon_{A'}c_{A'}\lambda_{A'B}.$$

The coefficients of the covariant derivative of Z are given by

(5.11) $$\sum \varepsilon_A z_{BA}\omega_A = dz_B + \sum \varepsilon_A z_A \omega_{AB}.$$

Then, using (5.8), we obtain

(5.12) $$z_{AB} = \lambda \varepsilon_A \delta_{AB},$$

where we put $\lambda = -\sum \varepsilon_{A'A'}\lambda_{A'n+p+1}$.

Let M be a comact m-dimensional Riemannian manifold imbedded maximally in S_p^{n+p}. We may assume that the frame field satisfies that $e_1, \ldots, e_m$ are tangent to M. Now put on M

(5.13) $$Z^T = \sum z_i e_i, \quad Z^N = \sum \varepsilon_\alpha z_\alpha e_\alpha.$$

The covariant derivative of Z^T is given by

(5.13) $$\sum z^N_{\alpha i}\omega_i = dz_\alpha + \sum \varepsilon_\beta z_\beta \omega_{\beta\alpha}.$$

From (5.12), it follows

(5.14) $$z^N_{\alpha i} = -\sum z_j h_{\alpha ji}.$$

Similarlly, the covariant derivative of Z^T is given by

(5.15) $$\sum z^T_{ji}\omega_i = dz_j + \sum z_i \omega_{ij}.$$

Hence, we have

(5.16) $$z^T_{ij} = \lambda \delta_{ij} + \sum \varepsilon_\alpha z_\alpha h_{\alpha ij}.$$

The second covariant derivative of Z^N is also given by

(5.17) $$\sum z^N_{\alpha ij}\omega_j = dz_{\alpha i} + \sum z^N_{\alpha j}\omega_{ji} + \sum \varepsilon_\beta z^N_{\beta i}\omega_{\beta\alpha}.$$

By a calculation, we get

(5.18) $$Z^N_{\alpha ij} = -\sum z_k h_{\alpha kij} - \lambda h_{\alpha ij} + \sum \varepsilon_\beta z_\beta h_{\alpha ki} h_{\beta kj},$$

where $h_{\alpha kij}$ are the components of the covariant derivative of the second fundamental form h of M and satisfy

(5.19) $$h_{\alpha kij} = h_{\alpha kij}, \quad h_{\alpha kij} = h_{\alpha ikj}.$$

As M is a maximal submanifold, that is, $\sum h_{\alpha ii} = 0$, it follows from (5.19) that $\sum h_{\alpha kii} = 0$. Thus we obtain

(5.20) $$\Delta z_\alpha = \sum z^N_{\alpha ii} = -\sum \varepsilon_\beta \sigma_{\alpha\beta} z_\beta.$$

The operater L defined (3.25) is reduced to

(5.21) $$L z_\alpha = -\Delta z_\alpha - n z_\alpha - \sum \varepsilon_\beta \sigma_{\alpha\beta} z_\beta = -n z_\alpha.$$

In other words, we obtain $LZ^N = -nZ^N$. Thus we have

Lemma 5.2. *Let M be a m-diemnsional compact Riemannian manifold imbedded maximally in S_p^{n+p}. Then it holds the index of $M \geq n+p-m$.*
By the same argument in the proof of Proposition 5.1.6 in [6], we have

Lemma 5.3. *Under the same assumption of Lemma 5.2, the index of M is $n+p-m$ if and only if M is isometric to S^n, and imbedded in the standard way as a totally geodesic submanifold.*

A Killig vector field X on a pseudo-Riemannian manifold is a vector field for which the Lie derivative of the metric tensor vanishes. Let $X = \sum X_A e_A$ be the vector field on S_p^{n+p}. Then it is a Killing vector field if and only if it is skew-adjoint relative to the metric, that is,

$$X_{AB} + X_{BA} = 0, \tag{5.22}$$

where X_{AB} are the componets of the covariant derivative of X (see p. 250 of [5]). Let X_{ABC} be the components of the covariant derivative of the killing vector field X. Then they satisfy

$$X_{ABC} - X_{ACB} = \sum \varepsilon_D X_D K_{DABC}, \tag{5.23}$$

where $K_{DABC} = \varepsilon_D \varepsilon_A (\delta_{DB}\delta_{AC} - \delta_{AD}\delta_{BC})$. Let M be a compact Riemannian manifold immersed maximally in S_p^{n+p}. Let X^N be the normal vector field on M by normal projection of a Killing vector field X. Then the componets of the covariant derivative of X^N are given by

$$X_{\alpha i}^N = X_{\alpha i} - \sum X_j h_{\alpha ji} \tag{5.24}$$

and the components of its covariant derivative are given by

$$\begin{aligned} X_{\alpha ij}^N = & X_{\alpha ij} + \sum \varepsilon_\beta X_{\alpha\beta} h_{\beta ij} + \sum X_{ki} h_{\alpha kj} - \sum X_{kj} h_{\alpha ki} \\ & - \sum X_k h_{\alpha kij} - \sum \varepsilon_\beta X_\beta h_{\beta kj} h_{\alpha ki}. \end{aligned} \tag{5.25}$$

Hence the Laplacian of X^N is given by

$$\Delta X_\alpha^N = \sum X_{\alpha ii}^N = \sum X_{\alpha ii} - \sum \varepsilon_\beta \sigma_{\alpha\beta} X_\beta. \tag{5.26}$$

On the other hand, we get from (5.22) and (5.23)

$$\sum X_{\alpha ii} = -nX_\alpha. \tag{5.27}$$

Thus as similarly as (5.21), we obtain

$$LZ^N = 0, \tag{5.28}$$

that is, Z^N is a *Jacobi field* on M.

Lemma 5.4. *Let Ω denote the vector space of Killing vector fields on S_p^{n+p}. For any $X \in \Omega$, X^N is a Jacobi field on M.*

By using the same arguments in the proofs of Lemmas 5.1.8 and 5.1.9 of [6], we have

Lemma 5.5. *Put $\Omega^N = \{X^N : X \in \Omega\}$. Then dim $\Omega^N \geq (n+p-m)(n+1)$. Dim $\Omega^N = (n+p-m)(n+1)$ if and only if M is diffeomorphic to S^m and imbedded in the standard way as a totally geodesic submanifold.*

Consecuently, we have an analogue of Theorem 5.1.1 of [6].

Theorem 5.6. *Let M be a compact m-dimensional Riemannian manifold imbedded in $S_p^{n+p}(m \leq n)$ such that the mean curvature normal vanishes. Then the index of M is greater than or equal to $n+p-m$, and equality holds only when M is S^m. The nullity of M is greater than or equal to $(n+p-m)(m+1)$ and equality holds only when M is S^m.*

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