J. Math. Tokushima Univ.
Vol. 20 (1986), 61–81

Resolution of Singularities Arising from Bifurcations of Periodic Solutions of Nonlinear Periodic Differential Systems with a Condition

By

Norio YAMAMOTO

(Received September 11, 1986)

§1. Introduction

We shall consider resolution of singularities arising from bifurcations of periodic solutions of a parameter-dependent nonlinear periodic system

$$\frac{dx}{dt}=f(t, x, \lambda) \quad (x, f(t, x, \lambda)\in R^n,\ \lambda\in R) \tag{1.1}$$

whose right member $f(t, x, \lambda)$ satisfies the condition

$$f\left(t+\frac{2\pi}{m},\ Px,\ \lambda\right)=Pf(t, x, \lambda) \qquad for \quad x\in R^n, \quad t, \lambda\in R \tag{1.2}$$

for a real $n\times n$ matrix P satisfying the condition

(1.3) $P\neq I_n$ *($n\times n$ unit matrix) and* $P^m=I_n$ *for a positive even integer* m (≥ 2),

where λ is a parameter, and f is a mapping from R^{n+2} to R^n and periodic in t of period 2π and $(k+2)$ times continuously differentiable with respect to (x, λ) in R^{n+2}, and $f(t, x, \lambda)$ and its first, second,..., $(k+2)$-th partial derivatives with respect to (x, λ) are all continuous on R^{n+2}.

Let $\varphi(t, \boldsymbol{x}_0)\equiv\varphi(t, x(0), \lambda)$ (where $\boldsymbol{x}_0\equiv(x(0), \lambda)\in R^{n+1}$) be a solution of (1.1) at a given λ through $x(0)\in R^n$ at $t=0$. Then, for a given λ, finding a 2π-periodic solution of (1.1) amounts to finding a vector $x(0)$ satisfying the equation

$$F(\boldsymbol{x}_0)\equiv F(x(0), \lambda)=x(0)-\varphi(2\pi, \boldsymbol{x}_0)=0. \tag{1.4}$$

In fact, when $\bar{\boldsymbol{x}}_0=(\bar{x}(0), \bar{\lambda})$ is a solution of (1.4), then $\varphi(t, \bar{\boldsymbol{x}}_0)$ becomes a 2π-periodic solution of (1.1) at $\lambda=\bar{\lambda}$. From the assumption on $f(t, x, \lambda)$, F is a C^{k+2} mapping from R^{n+1} to R^n.

It follows from (1.2) and (1.3) that when $\tilde{\boldsymbol{x}}_0=(\tilde{x}(0), \tilde{\lambda})\in R^{n+1}$ is a solution of the equation

$$E(\boldsymbol{x}_0)\equiv E(x(0), \lambda)=Px(0)-\varphi(t_1, \boldsymbol{x}_0)=0, \tag{1.5}$$

$\tilde{x}_0$ also becomes a solution of (1.4), where $t_1 = 2\pi/m$. E is also a C^{k+2} mapping from R^{n+1} to R^n.

In this paper, we consider $\hat{x}_0 = (\hat{x}(0), \hat{\lambda}) \in R^{n+1}$ such that

$$\text{(C1)} \quad E(\hat{x}_0) = 0, \quad \text{(C2)} \quad \operatorname{rank} E_x(\hat{x}_0) = n, \quad \text{(C3)} \quad \operatorname{rank} F_x(\hat{x}_0) = n-1.$$

Then by the implicit function theorem there exists only one branch of solutions of (1.5) in a neighborhood of $\hat{x}_0$ and this branch also becomes a branch of solutions of (1.4) there. We call this branch a *branch of* (1.5). If there exist two or more different branches of solutions of (1.4) which intersect at $\hat{x}_0$, then one of them coincides with the above branch and the others do not become branches of solutions of (1.5) due to the implicit function theorem. "Bifurcations of periodic solutions" mentioned in the beginning of this section actually mean this case.

In fact, if the condition (3.13) is satisfied, then two different branches of solutions of (1.4) intersect at $\hat{x}_0$ and if the condition (4.9) is satisfied, three different branches intersect at $\hat{x}_0$ and moreover, if the condition (4.18) is satisfied, four different branches intersect at $\hat{x}_0$. In each case, one of them is a branch of (1.5) and the others are not.

In this paper, we propose a method for computing $\hat{x}_0$ with high accuracy. Our method is as follows: When we consider an augmented system of nonlinear equations which contains the equation (1.5) and additional equations, then the system has an isolated solution containing $\hat{x}_0$ if a specific condition is satisfied. This is called *resolution of the singularity*. Then we can compute the isolated solution with high accuracy and therefore we can also obtain a desired approximation to $\hat{x}_0$. Our method can also be applied to computing some kinds of singular points among which $\hat{x}_0$ exists, and therefore we describe the method as that for computing such singular points. Here $\tilde{x}_0 = (\tilde{x}(0), \tilde{\lambda}) \in R^{n+1}$ is called a *singular point* of (1.4) if

$$\text{(1.6a)} \qquad F(\tilde{x}_0) = 0$$

$$\text{(1.6b)} \qquad \operatorname{rank} F_x(\tilde{x}_0) = \operatorname{rank} (F_x(\tilde{x}_0), F_\lambda(\tilde{x}_0)) = n-1$$

(cf. Brezzi et al. [1]). As shown in §3, for the above $\hat{x}_0$, we obtain

$$\text{(1.7)} \qquad \operatorname{rank} (F_x(\hat{x}_0), F_\lambda(\hat{x}_0)) = n-1.$$

Hence $\hat{x}_0$ becomes a singular point of (1.4).

From a viewpoint of bifurcation theory, we formulate this singular point $\hat{x}_0$ more precisely. In order to simplify the following arguments, we assume without loss of generality that

$$\text{(A1)} \qquad \operatorname{rank} F_x(\hat{x}_0) = \operatorname{rank} \hat{F}_0 = n-1,$$

where $\hat{F}_0$ denotes the $n \times (n-1)$ matrix obtained from $F_x(\hat{x}_0)$ by deleting the first column vector. Then the equation

$$(1.8)\qquad F_x(\hat{x}_0)h_1=0,\quad h_1^1-1=0\quad (\textit{where } h_1=(h_1^1, h_1^2,\dots, h_1^n)^T\in R^n)$$

has only one solution $\hat{h}_1\in R^n$, where $(\cdots)^T$ denotes the transposed vector of a vector $(\cdots)$. This means that $\hat{h}_1$ is an eigenvector corresponding to the eigenvalue zero of $F_x(\hat{x}_0)$. Concerning $\hat{h}_1$ we assume that

$$(\mathrm{A2})\qquad \hat{h}_1\notin V_2\equiv \mathrm{Range}\, F_x(\hat{x}_0).$$

Setting $V_1\equiv \mathrm{Ker}\, F_x(\hat{x}_0)=\{\{\hat{h}_1\}\}$, we have from (A1) and (A2)

$$(1.9)\qquad R^n=V_1\oplus V_2,$$

and so $F_x(\hat{x}_0)$ becomes an isomorphism from V_2 to V_2, where $\mathrm{Ker}\, F_x(\hat{x}_0)$ denotes the kernel of $F_x(\hat{x}_0)$, and $\{\{\hat{h}_1\}\}$ denotes the vector space spanned by $\hat{h}_1$, and $V_1\oplus V_2$ denotes the direct sum of V_1 and V_2. Then from (1.7) the equation

$$(1.10)\qquad F_x(\hat{x}_0)h_2+F_\lambda(\hat{x}_0)=0\quad (\textit{where } h_2\in V_2)$$

has only one solution $\bar{h}_2\in V_2$.

From (A1) and (A2) there exists a vector $\hat{\phi}_0\in R^n$ such that

$$(1.11)\qquad F_x(\hat{x}_0)^T\hat{\phi}_0=0,\quad \langle \hat{h}_1, \hat{\phi}_0\rangle\equiv\hat{\phi}_0^T\hat{h}_1=1,$$

where $F_x(\hat{x}_0)^T$ denotes the transposed matrix of $F_x(\hat{x}_0)$ and y^T denotes the transposed vector of $y\in R^n$. By the use of $\hat{\phi}_0$ we can write V_2 in the form

$$(1.12)\qquad V_2=\mathrm{Range}\, F_x(\hat{x}_0)=\{y\in R^n;\ \langle y, \hat{\phi}_0\rangle=0\}.$$

Let us define the projection operator Q: $R^n\to V_2$ by

$$Qw\equiv w-\langle w, \hat{\phi}_0\rangle\hat{h}_1\quad (\textit{where } w\in R^n).$$

Then the equation (1.4) is equivalent to the system

$$(1.13)\qquad QF(\boldsymbol{x}_0)=0,\quad \langle F(\boldsymbol{x}_0), \hat{\phi}_0\rangle=0.$$

According to Brezzi et al. [1], there exist two positive constants α_0, ζ_0 and a unique C^{k+2} mapping $v:[-\alpha_0, \alpha_0]\times[-\zeta_0, \zeta_0]\to V_2$ such that

$$QF(\hat{x}(0)+\alpha\hat{h}_1+v(\alpha, \zeta), \hat{\lambda}+\zeta)=0,\quad v(0, 0)=0.$$

Hence, solving the equation (1.4) in a neighborhood of the singular point $\hat{\boldsymbol{x}}_0=(\hat{x}(0), \hat{\lambda})$ amounts to solving the *bifurcation equation*

$$(1.14)\qquad g(\alpha, \zeta)\equiv\langle F(\hat{x}(0)+\alpha\hat{h}_1+v(\alpha, \zeta), \hat{\lambda}+\zeta), \hat{\phi}_0\rangle=0$$

in a neighborhood of the origin. By elementary calculations, we easily obtain

$$g(0, 0)=g_\alpha(0, 0)=g_\zeta(0, 0)=0.$$

When we set

$$A_0 \equiv g_{\alpha\alpha}(0, 0), \quad B_0 \equiv g_{\alpha\zeta}(0, 0), \quad C_0 \equiv g_{\zeta\zeta}(0, 0),$$

A_0, B_0 and C_0 can be written in the form

$$\begin{aligned} A_0 &= \langle F_{xx}(\hat{x}_0)\hat{h}_1\hat{h}_1, \hat{\phi}_0\rangle, \quad B_0 = \langle F_{xx}(\hat{x}_0)\hat{h}_1\bar{h}_2 + F_{x\lambda}(\hat{x}_0)\hat{h}_1, \hat{\phi}_0\rangle, \\ C_0 &= \langle F_{xx}(\hat{x}_0)\bar{h}_2\bar{h}_2 + 2F_{x\lambda}(\hat{x}_0)\bar{h}_2 + F_{\lambda\lambda}(\hat{x}_0), \hat{\phi}_0\rangle \end{aligned} \tag{1.15}$$

respectively. Then the singular point $\hat{x}_0 = (\hat{x}(0), \hat{\lambda})$ is called a *simple bifurcation point* of (1.4) if

$$B_0^2 - A_0 C_0 > 0 \tag{1.16}$$

(cf. Brezzi et al. [1]). In this case, two different branches of solutions of (1.4) intersect at the simple bifurcation point.

In fact, obviously $A_0 = 0$ and if the condition (3.13) is satisfied, then $B_0 \neq 0$, and therefore the above-mentioned singular point $\hat{x}_0$ becomes a simple bifurcation point. This bifurcation point is also called a *pitchfork bifurcation point* of (1.4) (cf. Kawakami et al. [3, 4, 5] and Werner et al. [7]).

In particular, in [8], when $m = 2$ and $P = -I_n$, we have already considered such a simple bifurcation point and have proposed a method for computing it with high accuracy. Therefore the results of this paper are the generalization of those of [8].

Concerning methods for computing simple bifurcation points, besides our method, Kawakami et al. [3, 4, 5] have proposed another method. The difference between their method and ours is that theirs is based on the use of the proper equation of $F_x(\hat{x}_0)$, while ours is based on the use of an eigenvector corresponding to the eigenvalue zero of $F_x(\hat{x}_0)$.

However, if the condition (3.13) is not satisfied, then $B_0 = C_0 = 0$. That is, all the second partial derivatives of g vanish at the origin. In this case, we call the singular point $\hat{x}_0$ a *singular point with a higher singularity*. Among such singular points, there exist bifurcation points at which three or more different branches of solutions of (1.4) intersect. For example, if the condition (4.9) is satisfied, three different branches intersect at $\hat{x}_0$, while if the condition (4.18) is satisfied, four different branches intersect at $\hat{x}_0$. In each case, one of them is a branch of (1.5) and the others are not.

Hence, in this paper, we consider singular points with a higher singularity and propose a method for computing them with high accuracy. This paper is based on the preceding result [12], but the latter is devoted only to the study of methods for computing singular points and we did not describe in detail about bifurcation theory. As for the problems different from what we discuss in this paper, see Cliffe et al. [2] and Yamamoto [10, 11], where methods for computing singular points with a higher singularity are given.

Generally, in the case of assuming no conditions, as shown in Weber [6] and Yamamoto [9], we consider a $(3n+2)$-dimensional equation in order to compute a simple bifurcation point. In the case of assuming the condition (1.2), however, we have only to consider a $(2n+1)$-dimensional equation. Furthermore, when we compute a singular point with a higher singularity, in the former, as shown in [8], we consider a $(6n+6)$-dimensional equation but, in the latter, we have only to consider a $(4n+2)$-dimensional equation.

In §2 basic lemmas used in the following sections are given and in §3 simple bifurcation points are considered. In §4 singular points with a higher singularity are considered and a method for computing them with high accuracy is proposed.

§ 2. Preliminaries

In this section, we give the notations and the basic lemmas used in the following sections. Throughout this section, we assume that $f(t, x, \lambda)$ is sufficiently smooth. We denote by $f_x(t, x, \lambda)=\partial f(t, x, \lambda)/\partial x$, $f_\lambda(t, x, \lambda)=\partial f(t, x, \lambda)/\partial\lambda$, $f_{xx}(t, x, \lambda)=\partial^2 f(t, x, \lambda)/\partial x^2$, $f_{x\lambda}(t, x, \lambda)=\partial^2 f(t, x, \lambda)/\partial\lambda\partial x$, $f_{\lambda\lambda}(t, x, \lambda)=\partial^2 f(t, x, \lambda)/\partial\lambda^2$,... the corresponding partial derivatives.

In the following, we consider a variational equation of (1.1), and moreover its variational equation and moreover its variational equation...successively. In order to express these variational equations simply, we introduce the following notations. For $f(t, x, \lambda)$ we define $n\times n$ matrices $Y^{(q)}$'s $(q\geq 1)$ and $V^{(r)}\in R^n (r\geq 1)$ inductively by

$$Y^{(1)}=f_x(t, x, \lambda), \quad Y^{(2j)}=\sum_{i=1}^{j} {}_{j-1}C_{i-1}Y_x^{(2j-2i+1)}h_{2i-1},$$

$$Y^{(2j+1)}=\sum_{i=1}^{j} {}_{j-1}C_{i-1}Y_x^{(2j-2i+1)}h_{2i}+Y_\lambda^{(2j-1)} \quad (j\geq 1)$$

and

$$V^{(1)}=f_\lambda(t, x, \lambda), \quad V^{(2j)}=\sum_{i=1}^{j} {}_{j-1}C_{i-1}V_x^{(2j-2i+1)}h_{2i-1},$$

$$V^{(2j+1)}=\sum_{i=1}^{j} {}_{j-1}C_{i-1}V_x^{(2j-2i+1)}h_{2i}+V_\lambda^{(2j-1)} \quad (j\geq 1)$$

respectively, where each ${}_jC_i$ denotes the binomial coefficient, and each $h_i\in R^n$ is an arbitrary vector. Then we have the following lemma.

Lemma 1 (Yamamoto [10]).

$$\sum_{j=1}^{i} {}_{i-1}C_{j-1}(Y^{(2i+2-2j)}h_{2j}+Y^{(2i+1-2j)}h_{2j+1})+V^{(2i)}$$
$$=\sum_{j=1}^{i+1} {}_iC_{j-1}Y^{(2i+3-2j)}h_{2j-1} \quad (i\geq 1).$$

Now we set

$$K_1 \equiv \{\phi(t) \in R^n;\ P^j\phi(t) = \phi(t+jt_1)\ \text{for}\ t\in R, j\in Z\},$$

$$K_{-1} \equiv \{\phi(t) \in R^n;\ P^j\phi(t) = (-1)^j\phi(t+jt_1)\ \text{for}\ t\in R, j\in Z\},$$

where Z denotes the set of all integers. Then we get the following lemma.

Lemma 2 (Yamamoto [12]). *For $x(t)\in K_1$ and $h_{2i-1}(t)\in K_{-1}$ and $h_{2i}(t)\in K_1$ $(i\geq 1)$*

(i) $V^{(2j-1)}(t)\in K_1,\quad V^{(2j)}(t)\in K_{-1}\quad (j\geq 1),$

(ii) $$\begin{matrix} Y^{(2j-1)}(t)\phi(t)\in K_1, & Y^{(2j)}(t)\phi(t)\in K_{-1} & \text{for}\ \phi(t)\in K_1 \\ Y^{(2j-1)}(t)\psi(t)\in K_{-1}, & Y^{(2j)}(t)\psi(t)\in K_1 & \text{for}\ \psi(t)\in K_{-1}\end{matrix}\quad (j\geq 1),$$

where $V^{(q)}(t)$ and $Y^{(q)}(t)$ $(q\leq 2j)$ denote the values of $V^{(q)}$ and $Y^{(q)}$ at $x=x(t)$ and $h_r=h_r(t)$ $(1\leq r\leq q-1)$, respectively.

For each positive integer i let $(\varphi(t, \boldsymbol{x}_0), \varphi_1(t, \boldsymbol{x}_1), \varphi_2(t, \boldsymbol{x}_2),\dots, \varphi_{2i-1}(t, \boldsymbol{x}_{2i-1}), \varphi_{2i}(t, \boldsymbol{x}_{2i}), \sigma_{i-1}(t, \boldsymbol{x}_{2i}))^T\in R^{(2i+2)n}$ be a solution of the system

$$\frac{dx}{dt} = f(t, x, \lambda),\quad \frac{dh_1}{dt} = Y^{(1)}h_1,\quad \frac{dh_2}{dt} = Y^{(1)}h_2 + V^{(1)},$$

$$\vdots$$

$$\frac{dh_{2i-1}}{dt} = \sum_{j=1}^{i-1} {}_{i-2}C_{j-1}(Y^{(2i-2j)}h_{2j} + Y^{(2i-1-2j)}h_{2j+1}) + V^{(2i-2)}$$

$$= \sum_{j=1}^{i} {}_{i-1}C_{j-1}Y^{(2i+1-2j)}h_{2j-1},$$

$$\frac{dh_{2i}}{dt} = \sum_{j=1}^{i} {}_{i-1}C_{j-1}Y^{(2i+1-2j)}h_{2j} + V^{(2i-1)},$$

$$\frac{d\gamma}{dt} = Y^{(1)}\gamma + \sum_{j=1}^{i} {}_{i-1}C_{j-1}Y^{(2i+2-2j)}h_{2j} + \sum_{j=1}^{i-1} {}_{i-1}C_{j-1}Y^{(2i+1-2j)}h_{2j+1}$$

$$+ V^{(2i)} = Y^{(1)}\gamma + \sum_{j=1}^{i} {}_iC_{j-1}Y^{(2i+3-2j)}h_{2j-1}$$

such that $(\varphi(0, \boldsymbol{x}_0), \varphi_1(0, \boldsymbol{x}_1), \varphi_2(0, \boldsymbol{x}_2),\dots, \varphi_{2i-1}(0, \boldsymbol{x}_{2i-1}), \varphi_{2i}(0, \boldsymbol{x}_{2i}), \sigma_{i-1}(0,\boldsymbol{x}_{2i}))^T = (x(0), h_1(0), h_2(0),\dots, h_{2i-1}(0), h_{2i}(0), 0)^T$, where $x(0)\in R^n$, $\boldsymbol{x}_0 = (x(0), \lambda)\in R^{n+1}$, $h_j(0) = (h_j^1(0), h_j^2(0),\dots, h_j^n(0))^T\in R^n$ $(1\leq j\leq 2i)$ and $\boldsymbol{x}_q = (x(0), h_1(0),\dots, h_q(0), \lambda)^T\in R^{(q+1)n+1}$ $(1\leq q\leq 2i)$. Then we set

$$\begin{pmatrix} \begin{pmatrix} Px(0) - \varphi(t_1, \boldsymbol{x}_0) \\ Ph_1(0) + \varphi_1(t_1, \boldsymbol{x}_1) \end{pmatrix} \\ \begin{pmatrix} Ph_2(0) - \varphi_2(t_1, \boldsymbol{x}_2) \\ Ph_3(0) + \varphi_3(t_1, \boldsymbol{x}_3) \end{pmatrix} \end{pmatrix}$$

$$S_i(x_{2i-1}) \equiv \begin{pmatrix} \vdots \\ \begin{pmatrix} Ph_{2i-2}(0) - \varphi_{2i-2}(t_1, x_{2i-2}) \\ Ph_{2i-1}(0) + \varphi_{2i-1}(t_1, x_{2i-1}) \end{pmatrix} \end{pmatrix} \in R^{2in},$$

$$T_i(x_{2i}) \equiv \begin{pmatrix} S_i(x_{2i-1}) \\ Ph_{2i}(0) - \varphi_{2i}(t_1, x_{2i}) \end{pmatrix} \in R^{(2i+1)n},$$

and also set vectors

$$\Xi_j^{(i)} \equiv (b_1^j, b_2^j, \ldots, b_{2i}^j, b_{2i+1}^j)^T \in R^{(2i+1)n} \quad (1 \le j \le i-1) \quad \textit{such that}$$

$$b_{2j}^j = \sigma_{i-1}(t_1, x_{2i}) \quad \textit{and} \quad b_q^j = 0 \in R^n \quad (q \ne 2j) \quad \textit{respectively.}$$

Then from Lemma 2 we easily get the following lemma.

Lemma 3. *If there exists a vector* $\tilde{x}_{2i} = (\tilde{x}(0), \tilde{h}_1(0), \tilde{h}_2(0), \ldots, \tilde{h}_{2i-1}(0), \tilde{h}_{2i}(0), \tilde{\lambda})^T \in R^{(2i+1)n+1}$ *satisfying the equation*

$$T_i(x_{2i}) = 0,$$

then $\varphi(t, \tilde{x}_0) \in K_1$ *and*

$$\varphi_{2j-1}(t, \tilde{x}_{2j-1}) \in K_{-1} \quad \textit{and} \quad \varphi_{2j}(t, \tilde{x}_{2j}) \in K_1 \quad (1 \le j \le i),$$

where $\tilde{x}_0 = (\tilde{x}(0), \tilde{\lambda})$ *and* $\tilde{x}_q = (\tilde{x}(0), \tilde{h}_1(0), \ldots, \tilde{h}_q(0), \tilde{\lambda})^T$ $(1 \le q \le 2i)$.

For each positive integer i (≥ 2) we define

$$\psi_i(x_{2i-1}) \equiv (h_1^1(0) - 1, h_3^1(0), h_5^1(0), \ldots, h_{2i-1}^1(0))^T \in R^i.$$

§3. Simple Bifurcation Points

In this section, we show that $\hat{x}_0 = (\hat{x}(0), \hat{\lambda})$ satisfying (C1)–(C3) becomes a simple bifurcation point if the condition (3.13) is satisfied, and we propose a method for computing $\hat{x}_0$ with high accuracy. This method is essentially the same as that proposed in [8].

Due to (1.2), (1.3) and (C1), we have

$$\hat{x}(t) \equiv \varphi(t, \hat{x}_0) \in K_1. \tag{3.1}$$

When $\hat{\Phi}_1(t)$ ($n \times n$ matrix) is a solution of the system

$$\frac{dz_1}{dt} = f_x(t, \hat{x}(t), \hat{\lambda}) z_1 \quad (\textit{where } z_1 \textit{ is an } n \times n \textit{ matrix})$$

such that $\hat{\Phi}_1(0) = I_n$, then we can write $E_x(\hat{x}_0)$ and $F_x(\hat{x}_0)$ in the form

$$E_x(\hat{x}_0) = P - \hat{\Phi}_1(t_1) \quad \textit{and} \quad F_x(\hat{x}_0) = I_n - \hat{\Phi}_1(2\pi) \tag{3.2}$$

respectively. Moreover, for $\hat{\Phi}_1(t)$, we have from (1.2) and (3.1)

$$\hat{\Phi}_1(t+jt_1)=P^j\hat{\Phi}_1(t)\{P^{-1}\hat{\Phi}_1(t_1)\}^j \quad \textit{for} \quad t\in R, \quad j\in Z. \tag{3.3}$$

Assuming $t=0$ and $j=m$ in (3.3), we have

$$\hat{\Phi}_1(2\pi)=\hat{U}^m \quad (\textit{where } \hat{U}\equiv P^{-1}\hat{\Phi}_1(t_1)), \tag{3.4}$$

from which it follows that

$$I_n-\hat{\Phi}_1(2\pi)=[I_n+\hat{U}]\hat{L}=\hat{L}[I_n+\hat{U}]=[I_n-\hat{U}]\hat{M}=\hat{M}[I_n-\hat{U}], \tag{3.5}$$

where

$$\hat{L}=\sum_{i=0}^{m-1}(-1)^i\hat{U}^i=I_n-\hat{U}+\hat{U}^2-\cdots+\hat{U}^{m-2}-\hat{U}^{m-1},$$

$$\hat{M}=\sum_{i=0}^{m-1}\hat{U}^i=I_n+\hat{U}+\hat{U}^2+\cdots+\hat{U}^{m-2}+\hat{U}^{m-1}.$$

By (C2) and (C3) we see that

$$\operatorname{rank}[I_n-\hat{U}]=n \quad \textit{and} \quad \operatorname{rank}\hat{M}=n-1. \tag{3.6}$$

Concerning the solution $\hat{h}_1$ of (1.8) we have the following lemma.

Lemma 4.

$$\hat{h}_1\in \operatorname{Ker}[I_n+\hat{U}]. \tag{3.7}$$

PROOF. From (3.4) we easily have

$$[I_n-\hat{\Phi}_1(2\pi)]\hat{U}\hat{h}_1=\hat{U}[I_n-\hat{\Phi}_1(2\pi)]\hat{h}_1=0,$$

which implies

$$\hat{U}\hat{h}_1=\alpha\hat{h}_1 \quad (\textit{where } \alpha\in R) \tag{3.8}$$

because $\operatorname{Ker}[I_n-\hat{\Phi}_1(2\pi)]=\{\{\hat{h}_1\}\}$. Since $\hat{h}_1=\hat{\Phi}_1(2\pi)\hat{h}_1$, we have by (3.4) and (3.8)

$$\hat{h}_1=\hat{U}^m\hat{h}_1=\alpha^m\hat{h}_1.$$

Since $\hat{h}_1$ is not a zero vector, we have

$$\alpha^m-1=0,$$

and therefore $\alpha=1$ or $\alpha=-1$. Due to (C2), $\alpha=-1$. That is, $\hat{h}_1\in\operatorname{Ker}[I_n+\hat{U}]$.

Q.E.D.

Due to (C3), (3.5) and (3.7), we easily get

$$\operatorname{rank}[I_n+\hat{U}]=\operatorname{rank}\hat{D}_0=n-1 \quad \textit{and} \quad \operatorname{rank}\hat{L}=n, \tag{3.9}$$

where $\hat{D}_0$ denotes the $n\times(n-1)$ matrix obtained from $I_n+\hat{U}$ by deleting the first column vector. In fact, $\hat{F}_0=\hat{L}\hat{D}_0$.

Now we prove (1.7). When $\xi_1(t)\in R^n$ is a solution of the system

$$\frac{d\xi_1}{dt}=f_x(t,\hat{x}(t),\hat{\lambda})\xi_1+f_\lambda(t,\hat{x}(t),\hat{\lambda})$$

such that $\xi_1(0)=0\in R^n$, then we have

$$F_\lambda(\hat{\boldsymbol{x}}_0)=-\xi_1(2\pi). \tag{3.10}$$

By (1.2) and (3.1) we readily see that

$$\xi_1(2\pi)=\hat{M}P^{-1}\xi_1(t_1), \tag{3.11}$$

which, by (3.5) and (3.6), implies (1.7). Hence $\hat{\boldsymbol{x}}_0$ becomes a singular point of (1.4).

When we set $\hat{h}_1(t)\equiv\hat{\Phi}_1(t)\hat{h}_1$, then $\hat{h}_1(t)$ becomes a 2π-periodic solution of the system

$$\frac{dh_1}{dt}=f_x(t,\hat{x}(t),\hat{\lambda})h_1$$

and also, by (3.7), satisfies

$$P\hat{h}_1(0)+\hat{h}_1(t_1)=0,$$

which, by Lemma 3, implies $\hat{h}_1(t)\in K_{-1}$. Moreover, $\hat{h}_1^1(0)=1$ because $\hat{h}_1(0)=\hat{h}_1$.

Hence, in order to obtain the singular point $\hat{\boldsymbol{x}}_0$, we consider the equation

$$G_1(\boldsymbol{x}_1)\equiv\begin{pmatrix} S_1(\boldsymbol{x}_1) \\ h_1^1(0)-1\end{pmatrix}=0. \tag{3.12}$$

As noted above, the equation (3.12) certainly has a solution $\hat{\boldsymbol{x}}_1=(\hat{x}(0),\hat{h}_1(0),\hat{\lambda})^T$. In fact, $\hat{h}_1(t)=\varphi_1(t,\hat{\boldsymbol{x}}_1)$. We denote by $G_1'(\boldsymbol{x}_1)$ the Jacobian matrix of $G_1(\boldsymbol{x}_1)$ with respect to $\boldsymbol{x}_1$.

When $\hat{Y}^{(i)}(t)$ and $\hat{V}^{(i)}(t)$ $(i=1,2)$ denote the values of $Y^{(i)}$ and $V^{(i)}$ at $x=\hat{x}(t)$, $h_1=\hat{h}_1(t)$ and $\lambda=\hat{\lambda}$, respectively, and $\hat{\Phi}_2(t)$ ($n\times n$ matrix) is a solution of the system

$$\frac{dz_2}{dt}=\hat{Y}^{(1)}(t)z_2+\hat{Y}^{(2)}(t)\hat{\Phi}_1(t)\quad (\textit{where } z_2 \textit{ is an } n\times n \textit{ matrix})$$

such that $\hat{\Phi}_2(0)=O$ ($n\times n$ zero matrix), and $\xi_2(t)\in R^n$ is a solution of the system

$$\frac{d\xi_2}{dt}=\hat{Y}^{(1)}(t)\xi_2+\hat{Y}^{(2)}(t)\xi_1(t)+\hat{V}^{(2)}(t)$$

such that $\xi_2(0)=0\in R^n$, then the matrix $G_1'(\hat{\boldsymbol{x}}_1)$ has the form

$$G_1'(\hat{x}_1)=\begin{pmatrix} P-\hat{\Phi}_1(t_1) & O & -\hat{\xi}_1(t_1) \\ \hat{\Phi}_2(t_1) & P+\hat{\Phi}_1(t_1) & \hat{\xi}_2(t_1) \\ 00\cdots 0 & 10\cdots 0 & 0 \end{pmatrix}.$$

Concerning the regularity of the matrix $G_1'(\hat{x}_1)$ we have the following theorem.

Theorem 1. *The matrix $G_1'(\hat{x}_1)$ is nonsingular if and only if*

$$\text{rank}\,(\hat{D},\ \hat{\delta}_1)=n, \tag{3.13}$$

where $\hat{D}$ $(=P\hat{D}_0)$ is the $n\times(n-1)$ matrix obtained from $P+\hat{\Phi}_1(t_1)$ by deleting the first column vector, and

$$\hat{\delta}_1=\hat{\Phi}_2(t_1)\hat{h}_2+\hat{\xi}_2(t_1) \quad (\in R^n). \tag{3.14}$$

Here $\hat{h}_2\in R^n$ is a solution of the equation

$$[P-\hat{\Phi}_1(t_1)]h_2-\hat{\xi}_1(t_1)=0 \quad (\textit{where } h_2\in R^n). \tag{3.15}$$

PROOF. For $(u_1, u_2, \rho)^T\in R^{2n+1}$ we consider the equation

$$\begin{aligned} &[P-\hat{\Phi}_1(t_1)]u_1-\rho\hat{\xi}_1(t_1)=0, \\ &\hat{\Phi}_2(t_1)u_1+[P+\hat{\Phi}_1(t_1)]u_2+\rho\hat{\xi}_2(t_1)=0, \quad u_2^1=0, \end{aligned} \tag{3.16}$$

where $u_i=(u_i^1, u_i^2,\dots, u_i^n)^T\in R^n$ $(i=1, 2)$.

When $\rho=0$, we get $u_1=0$ from the first of (3.16) because $\det[P-\hat{\Phi}_1(t_1)]=\det P\times\det[I_n-\hat{U}]\neq 0$. Letting $\rho=0$ and $u_1=0$ in the second of (3.16), we have

$$[P+\hat{\Phi}_1(t_1)]u_2=0, \quad u_2^1=0,$$

from which follows $u_2=0$. Thus we obtain a zero solution $(0, 0, 0)^T\in R^{2n+1}$ of (3.16).

When $\rho\neq 0$, we set $\rho=1$ without loss of generality. Then we get $u_1=\hat{h}_2$ from the first of (3.16). Letting $\rho=1$ and $u_1=\hat{h}_2$ in the second of (3.16), we have

$$[P+\hat{\Phi}_1(t_1)]u_2+\hat{\Phi}_2(t_1)\hat{h}_2+\hat{\xi}_2(t_1)=[P+\hat{\Phi}_1(t_1)]u_2+\hat{\delta}_1=0, \quad u_2^1=0. \tag{3.17}$$

It follows from (3.17) that if $\text{rank}\,(\hat{D}, \hat{\delta}_1)=n$, then the equation (3.16) has a zero solution only and so $\det G_1'(\hat{x}_1)\neq 0$, and conversely if $\det G_1'(\hat{x}_1)\neq 0$, then the equation (3.16) has a zero solution only and so $\text{rank}\,(\hat{D}, \hat{\delta}_1)=n$. Q. E. D.

In particular, when $m=2$ and $P=-I_n$, Theorem 1 is the same as that mentioned in Case (I) in 3.2 of [8].

Due to Theorem 1, if the condition (3.13) is satisfied, then the equation (3.12) has an isolated solution $\hat{x}_1$ containing the singular point $\hat{x}_0$ —this means that resolution of the singularity is realized— and therefore we can compute $\hat{x}_0$ with high accuracy by applying the Newton method to the equation (3.12).

Now we show that if the condition (3.13) is satisfied, then the singular point $\hat{x}_0$ becomes a simple bifurcation point. By elementary calculations, we obtain

$$\hat{v}_1 \equiv F_{xx}(\hat{x}_0)\hat{h}_1\hat{h}_1 = -\hat{\Phi}_2(2\pi)\hat{h}_1 = -\hat{M}\{P^{-1}\hat{\Phi}_2(t_1)\hat{h}_1\},$$

$$\hat{v}_2 \equiv F_{xx}(\hat{x}_0)\hat{h}_1\hat{h}_2 + F_{x\lambda}(\hat{x}_0)\hat{h}_1 = -\hat{\Phi}_2(2\pi)\hat{h}_2 - \hat{\xi}_2(2\pi)$$
$$= \hat{L}[P^{-1}\{\hat{\Phi}_2(t_1)\hat{h}_2 + \hat{\xi}_2(t_1)\}] = \hat{L}P^{-1}\hat{\delta}_1,$$

from which, by (3.5), (3.6) and (3.9), it follows that

(3.18) $$\operatorname{rank}[I_n - \hat{\Phi}_1(2\pi),\ \hat{v}_1] = n-1,$$

(3.19) $$\operatorname{rank}[I_n - \hat{\Phi}_1(2\pi),\ \hat{v}_2] = \operatorname{rank}[I_n + \hat{U},\ P^{-1}\hat{\delta}_1]$$
$$= \operatorname{rank}[P + \hat{\Phi}_1(t_1),\ \hat{\delta}_1] = \operatorname{rank}(\hat{D},\ \hat{\delta}_1).$$

Hence by (1.12), (1.15) and (3.18)

(3.20) $$A_0 = \langle \hat{v}_1, \hat{\phi}_0 \rangle = 0.$$

From (3.2), (3.5), (3.10) and (3.11) we can write the equation (1.10) in the form

$$\hat{M}\{[I_n - \hat{U}]h_2 - P^{-1}\hat{\xi}_1(t_1)\} = 0 \quad (where\ h_2 \in V_2),$$

from which it follows that there exists a constant $\hat{c} \in R$ such that

(3.21) $$[I_n - \hat{U}]\bar{h}_2 - P^{-1}\hat{\xi}_1(t_1) = \hat{c}\hat{h}_1$$

because Ker $\hat{M} = \{\{\hat{h}_1\}\}$. Since $[I_n - \hat{U}]\hat{h}_1 = 2\hat{h}_1$ due to (3.7), we obtain by (3.21)

(3.22) $$[I_n - \hat{U}]\left(\bar{h}_2 - \frac{\hat{c}}{2}\hat{h}_1\right) - P^{-1}\hat{\xi}_1(t_1) = 0.$$

The equation (3.15) is equivalent to the equation

(3.23) $$[I_n - \hat{U}]h_2 - P^{-1}\hat{\xi}_1(t_1) = 0 \quad (where\ h_2 \in R^n),$$

and $\hat{h}_2$ is only one solution of (3.23), and therefore by (3.22)

(3.24) $$\hat{h}_2 = \bar{h}_2 - \hat{d}\hat{h}_1 \quad (where\ \hat{d} \equiv \hat{c}/2).$$

Then from (1.15), (3.20) and (3.24) we have

(3.25) $$B_0 = \langle \hat{v}_2, \hat{\phi}_0 \rangle + \hat{d}\langle \hat{v}_1, \hat{\phi}_0 \rangle = \langle \hat{v}_2, \hat{\phi}_0 \rangle.$$

Therefore, if the condition (3.13) is satisfied, then we have by (1.12), (3.19) and (3.25)

(3.26) $$B_0 \neq 0,$$

which, by (3.20), implies

(3.27) $$B_0^2-A_0C_0=B_0^2>0.$$

Hence the singular point $\hat{x}_0$ becomes a simple bifurcation point.

Next we consider in what form C_0 can be written. When we set $\hat{h}_2(t)\equiv\hat{\Phi}_1(t)\hat{h}_2+\hat{\xi}_1(t)$, then $\hat{h}_2(t)$ becomes a solution of the system

$$\frac{dh_2}{dt}=\hat{Y}^{(1)}(t)h_2+\hat{V}^{(1)}(t)$$

and also satisfies

$$P\hat{h}_2(0)-\hat{h}_2(t_1)=0,$$

which, by Lemma 3, implies $\hat{h}_2(t)\in K_1$. When $\hat{Y}^{(3)}(t)$ and $\hat{V}^{(3)}(t)$ denote the values of $Y^{(3)}$ and $V^{(3)}$ at $x=\hat{x}(t)$, $h_2=\hat{h}_2(t)$ and $\lambda=\hat{\lambda}$, respectively, and $\hat{\Phi}_3(t)$ ($n\times n$ matrix) is a solution of the system

$$\frac{dz_3}{dt}=\hat{Y}^{(1)}(t)z_3+\hat{Y}^{(3)}(t)\hat{\Phi}_1(t)\quad (\textit{where } z_3 \textit{ is an } n\times n \textit{ matrix})$$

such that $\hat{\Phi}_3(0)=O$ ($n\times n$ zero matrix), and $\hat{\xi}_3(t)\in R^n$ is a solution of the system

$$\frac{d\xi_3}{dt}=\hat{Y}^{(1)}(t)\xi_3+\hat{Y}^{(3)}(t)\hat{\xi}_1(t)+\hat{V}^{(3)}(t)$$

such that $\hat{\xi}_3(0)=0\in R^n$, then by elementary calculations we obtain

(3.28) $$\hat{v}_3\equiv F_{xx}(\hat{x}_0)\hat{h}_2\hat{h}_2+2F_{x\lambda}(\hat{x}_0)\hat{h}_2+F_{\lambda\lambda}(\hat{x}_0)=-\hat{\Phi}_3(2\pi)\hat{h}_2-\hat{\xi}_3(2\pi)$$
$$=-\hat{M}[P^{-1}\{\hat{\Phi}_3(t_1)\hat{h}_2+\hat{\xi}_3(t_1)\}],$$

which, by (1.15), (3.5) and (3.6), implies

$$\text{rank}\,[I_n-\hat{\Phi}_1(2\pi),\ \hat{v}_3]=n-1,$$

and therefore by (1.12)

(3.29) $$\langle\hat{v}_3,\ \hat{\phi}_0\rangle=0.$$

By (1.15) and (3.24) we readily see that

$$C_0=\langle\hat{v}_3,\ \hat{\phi}_0\rangle+2\hat{d}\langle\hat{v}_2,\ \hat{\phi}_0\rangle+\hat{d}^2\langle\hat{v}_1,\ \hat{\phi}_0\rangle,$$

so that by (3.20), (3.25) and (3.29)

(3.30) $$C_0=2\hat{d}B_0.$$

§ 4. Singular Points with a Higher Singularity

In this section, we consider the case $\text{rank}\,(\hat{D},\ \hat{\delta}_1)=n-1$. Then $B_0=0$ due to

(1.12), (3.19) and (3.25), and so by (3.30) $C_0=0$. Hence we consider singular points with a higher singularity. Among such singular points, there exist bifurcation points at which three or more different branches of solutions of (1.4) intersect. For instance, if the condition (4.9) is satisfied, three different branches intersect, while if the condition (4.18) is satisfied, four different branches intersect.

As is well known, there exists a positive integer j_0 $(1\leq j_0\leq n)$ such that

$$\text{(4.1)} \qquad \operatorname{rank}(P+\hat{\Phi}_1(t_1),\ e_{j_0})=\operatorname{rank}(\hat{D},\ e_{j_0})=n,$$

where $e_{j_0}=(\varepsilon_1, \varepsilon_2,\dots, \varepsilon_n)^T\in R^n$ is a vector such that $\varepsilon_{j_0}=1$ and $\varepsilon_i=0$ $(i\neq j_0)$. For this vector e_{j_0} and each positive integer i (≥ 2) we set vectors

$\theta_j^{(i)}\equiv(a_1^j, a_2^j,\dots, a_{2i}^j)^T\in R^{2in}$ $(1\leq j\leq i-1)$ *such that* $a_{2j}^j=e_{j_0}$ *and* $a_q^j=0\in R^n$ $(q\neq 2j)$ *respectively.*

Due to $\operatorname{rank}(\hat{D}, \hat{\delta}_1)=n-1$, the equation

$$[P+\hat{\Phi}_1(t_1)]h_3+\hat{\delta}_1=[P+\hat{\Phi}_1(t_1)]h_3+\hat{\Phi}_2(t_1)\hat{h}_2+\hat{\xi}_2(t_1)=0,$$

$$h_3^1=0 \quad (\textit{where } h_3=(h_3^1, h_3^2,\dots, h_3^n)^T\in R^n)$$

has only one solution $\hat{h}_3\in R^n$. When we set

$$\hat{h}_3(t)\equiv\hat{\Phi}_1(t)\hat{h}_3+\hat{\Phi}_3(t)\hat{h}_1=\hat{\Phi}_1(t)\hat{h}_3+\hat{\Phi}_2(t)\hat{h}_2+\hat{\xi}_2(t),$$

then $\hat{h}_3(t)$ becomes a solution of the system

$$\frac{dh_3}{dt}=\hat{Y}^{(1)}(t)h_3+\hat{Y}^{(3)}(t)\hat{h}_1(t)=\hat{Y}^{(1)}(t)h_3+\hat{Y}^{(2)}(t)\hat{h}_2(t)+\hat{V}^{(2)}(t)$$

and also satisfies

$$P\hat{h}_3(0)+\hat{h}_3(t_1)=0,$$

which, by Lemma 3, implies $\hat{h}_3(t)\in K_{-1}$. Moreover, $\hat{h}_3^1(0)=0$ because $\hat{h}_3(0)=\hat{h}_3$.

Hence, in order to obtain the singular point $\hat{x}_0$ with a higher singularity, we introduce another parameter $\beta_1\in R$ and consider the equation

$$\text{(4.2)} \qquad G_2(y_2)\equiv\begin{pmatrix} S_2(x_3)-\beta_1\theta_1^{(2)} \\ \psi_2(x_3)\end{pmatrix}=0,$$

where $y_2=(x_3, \beta_1)^T\in R^{4n+2}$. As noted above, the equation (4.2) has a solution $\hat{y}_2=(\hat{x}(0), \hat{h}_1(0), \hat{h}_2(0), \hat{h}_3(0), \hat{\lambda}, 0)^T$, where $\hat{h}_i(0)=\hat{h}_i$ $(1\leq i\leq 3)$. In fact, $\hat{h}_i(t)=\varphi_i(t, \hat{x}_i)$ $(1\leq i\leq 3)$, where $\hat{x}_i=(\hat{x}(0), \hat{h}_1(0),\dots, \hat{h}_i(0), \hat{\lambda})^T$ $(1\leq i\leq 3)$. We denote by $G_2'(y_2)$ the Jacobian matrix of $G_2(y_2)$ with respect to y_2.

When $\hat{Y}^{(4)}(t)$ and $\hat{V}^{(4)}(t)$ denote the values of $Y^{(4)}$ and $V^{(4)}$ at $x=\hat{x}(t)$, $h_i=\hat{h}_i(t)$ $(1\leq i\leq 3)$ and $\lambda=\hat{\lambda}$, respectively, and $\hat{\Phi}_4(t)$ ($n\times n$ matrix) is a solution of the system

$$\frac{dz_4}{dt} = \hat{Y}^{(1)}(t)z_4 + \hat{Y}^{(2)}(t)\hat{\Phi}_3(t) + \hat{Y}^{(3)}(t)\hat{\Phi}_2(t) + \hat{Y}^{(4)}(t)\hat{\Phi}_1(t)$$

(*where* z_4 *is an* $n \times n$ *matrix*)

such that $\hat{\Phi}_4(0) = O$ ($n \times n$ zero matrix), and $\hat{\xi}_4(t) \in R^n$ is a solution of the system

$$\frac{d\xi_4}{dt} = \hat{Y}^{(1)}(t)\xi_4 + \hat{Y}^{(2)}(t)\hat{\xi}_3(t) + \hat{Y}^{(3)}(t)\hat{\xi}_2(t) + \hat{Y}^{(4)}(t)\hat{\xi}_1(t) + \hat{V}^{(4)}(t)$$

such that $\hat{\xi}_4(0) = 0 \in R^n$, and $\hat{h}_4 \in R^n$ is a solution of the equation

$$[P - \hat{\Phi}_1(t_1)]h_4 - \hat{\Phi}_3(t_1)\hat{h}_2 - \hat{\xi}_3(t_1) = 0 \quad (where\ h_4 \in R^n),$$

then we set

$$\hat{\delta}_2 \equiv \hat{\Phi}_4(t_1)\hat{h}_2 + \hat{\Phi}_3(t_1)\hat{h}_3 + \hat{\Phi}_2(t_1)\hat{h}_4 + \hat{\xi}_4(t_1) \quad (\in R^n).$$

Concerning the regularity of the matrix $G_2'(\hat{y}_2)$ we have the following theorem.

Theorem 2. *The matrix* $G_2'(\hat{y}_2)$ *is nonsingular if and only if*

(4.3) $$\operatorname{rank}(\hat{D}, \hat{\delta}_2) = n.$$

The proof is done in a similar way to that of Theorem 1 and will be omitted. Due to Theorem 2, if the condition (4.3) is satisfied, then the equation (4.2) has an isolated solution $\hat{y}_2$ containing the singular point $\hat{x}_0$ —this means that resolution of the singularity is realized— and therefore we can compute $\hat{x}_0$ with high accuracy by applying the Newton method to the equation (4.2).

We show that, in addition to (4.3), if the condition (4.9) is satisfied, then three different branches of solutions of (1.4) actually intersect at the singular point $\hat{x}_0$. Of course, one of the three is a branch of (1.5) and the other two are not.

When we set

(4.4) $$\hat{v}_4 \equiv -\hat{\Phi}_4(2\pi)\hat{h}_2 - \hat{\Phi}_3(2\pi)\hat{h}_3 - \hat{\Phi}_2(2\pi)\hat{h}_4 - \hat{\xi}_4(2\pi) = \hat{L}[P^{-1}\{\hat{\Phi}_4(t_1)\hat{h}_2 + \hat{\Phi}_3(t_1)\hat{h}_3 + \hat{\Phi}_2(t_1)\hat{h}_4 + \hat{\xi}_4(t_1)\}] = \hat{L}P^{-1}\hat{\delta}_2$$

and

$$\hat{\kappa} \equiv \langle \hat{v}_4, \hat{\phi}_0 \rangle,$$

then by careful calculations we obtain

(4.5) $$g_{\alpha\alpha\zeta}(0,0) = \hat{d}g_{\alpha\alpha\alpha}(0,0), \quad g_{\alpha\zeta\zeta}(0,0) = \hat{\kappa} + \hat{d}^2 g_{\alpha\alpha\alpha}(0,0) \quad and \quad g_{\zeta\zeta\zeta}(0,0) = 3\hat{d}\hat{\kappa} + \hat{d}^3 g_{\alpha\alpha\alpha}(0,0),$$

and therefore we can write the bifurcation equation (1.14) in the form

(4.6) $$g(\alpha, \zeta)=\frac{1}{3!}[(\alpha+\hat{d}\zeta)\{g_{\alpha\alpha\alpha}(0, 0)(\alpha+\hat{d}\zeta)^2+3\hat{\kappa}\zeta^2\}]+h.o.t.=0.$$

By (3.5), (3.9) and (4.4) we see that

(4.7) $$\begin{aligned}\operatorname{rank}[I_n-\hat{\Phi}_1(2\pi), \hat{v}_4]&=\operatorname{rank}[I_n+\hat{U}, P^{-1}\hat{\delta}_2]\\&=\operatorname{rank}[P+\hat{\Phi}_1(t_1), \hat{\delta}_2]=\operatorname{rank}(\hat{D}, \hat{\delta}_2),\end{aligned}$$

and therefore by (1.12) and (4.3)

(4.8) $$\hat{\kappa}\neq 0.$$

Hence, by (4.6), if the condition

(4.9) $$g_{\alpha\alpha\alpha}(0, 0)\times\hat{\kappa}<0$$

is satisfied, then three different branches of solutions of (1.4) intersect at the singular point $\hat{\boldsymbol{x}}_0$.

Next we consider singular points where the condition $\operatorname{rank}(\hat{D}, \hat{\delta}_2)=n-1$ is satisfied.

Assume that there exists a vector $(\hat{x}_0, \hat{h}_1, \hat{h}_2,\dots, \hat{h}_{2k-1}, \hat{\lambda})^T\in R^{2kn+1}$ such that the following three assumptions (I)–(III) are satisfied ($k\geq 2$):

(I) $\hat{\boldsymbol{x}}_0=(\hat{x}_0, \hat{\lambda})\in R^{n+1}$ *is a singular point of* (1.4) *satisfying* (C1)–(C3).

When $(\hat{x}(t), \hat{h}_1(t), \hat{h}_2(t),\dots, \hat{h}_{2k-1}(t))^T\in R^{2kn}$ is a solution of the system

$$\frac{dx}{dt}=f(t, x, \lambda),\quad \frac{dh_1}{dt}=Y^{(1)}h_1,\quad \frac{dh_2}{dt}=Y^{(1)}h_2+V^{(1)},$$
$$\vdots$$
$$\frac{dh_{2k-2}}{dt}=\sum_{j=1}^{k-1}{}_{k-2}C_{j-1}Y^{(2k-1-2j)}h_{2j}+V^{(2k-3)},$$
$$\begin{aligned}\frac{dh_{2k-1}}{dt}&=\sum_{j=1}^{k-1}{}_{k-2}C_{j-1}(Y^{(2k-2j)}h_{2j}+Y^{(2k-1-2j)}h_{2j+1})+V^{(2k-2)}\\&=\sum_{j=1}^{k}{}_{k-1}C_{j-1}Y^{(2k+1-2j)}h_{2j-1}\end{aligned}$$

at $\lambda=\hat{\lambda}$ such that $(\hat{x}(0), \hat{h}_1(0), \hat{h}_2(0),\dots, \hat{h}_{2k-1}(0))^T=(\hat{x}_0, \hat{h}_1, \hat{h}_2,\dots, \hat{h}_{2k-1})^T$, and $\hat{Y}^{(i)}(t)$ and $\hat{V}^{(i)}(t)$ $(1\leq i\leq 2k)$ denote the values of $Y^{(i)}$ and $V^{(i)}$ at $x=\hat{x}(t)$, $h_j=\hat{h}_j(t)$ $(1\leq j\leq i-1)$ and $\lambda=\hat{\lambda}$, respectively, and $(\hat{\Phi}_1(t), \hat{\Phi}_2(t),\dots, \hat{\Phi}_{2k-1}(t), \hat{\Phi}_{2k}(t))^T$ ($2kn\times n$ matrix) is a solution of the system

$$\frac{dz_1}{dt}=\hat{Y}^{(1)}(t)z_1,\quad \frac{dz_2}{dt}=\hat{Y}^{(1)}(t)z_2+\hat{Y}^{(2)}(t)z_1,$$
$$\vdots \qquad\qquad (\textit{where each } z_i \textit{ is an } n\times n \textit{ matrix})$$
$$\frac{dz_{2k-1}}{dt}=\sum_{j=1}^{k}{}_{k-1}C_{j-1}\hat{Y}^{(2k+1-2j)}(t)z_{2j-1},$$

$$\frac{dz_{2k}}{dt} = \sum_{j=1}^{k} {}_{k-1}C_{j-1}(\hat{Y}^{(2k+1-2j)}(t)z_{2j} + \hat{Y}^{(2k+2-2j)}(t)z_{2j-1})$$

such that $(\hat{\Phi}_1(0), \hat{\Phi}_2(0),\dots, \hat{\Phi}_{2k-1}(0), \hat{\Phi}_{2k}(0))^T = (I_n, O,\dots, O, O)^T$ (O is an $n \times n$ zero matrix), and $(\hat{\xi}_1(t), \hat{\xi}_2(t),\dots, \hat{\xi}_{2k-1}(t), \hat{\xi}_{2k}(t))^T \in R^{2kn}$ is a solution of the system

$$\frac{d\xi_1}{dt} = \hat{Y}^{(1)}(t)\xi_1 + \hat{V}^{(1)}(t), \quad \frac{d\xi_2}{dt} = \hat{Y}^{(1)}(t)\xi_2 + \hat{Y}^{(2)}(t)\xi_1 + \hat{V}^{(2)}(t),$$

$$\vdots$$

$$\frac{d\xi_{2k-1}}{dt} = \sum_{j=1}^{k} {}_{k-1}C_{j-1}\hat{Y}^{(2k+1-2j)}(t)\xi_{2j-1} + \hat{V}^{(2k-1)}(t),$$

$$\frac{d\xi_{2k}}{dt} = \sum_{j=1}^{k} {}_{k-1}C_{j-1}(\hat{Y}^{(2k+1-2j)}(t)\xi_{2j} + \hat{Y}^{(2k+2-2j)}(t)\xi_{2j-1}) + \hat{V}^{(2k)}(t)$$

such that $(\hat{\xi}_1(0), \hat{\xi}_2(0),\dots, \hat{\xi}_{2k-1}(0), \hat{\xi}_{2k}(0))^T = (0, 0,\dots, 0, 0)^T \in R^{2kn}$, and

$$\hat{\delta}_i \equiv \sum_{j=1}^{i} {}_{i-1}C_{j-1}\hat{\Phi}_{2i+2-2j}(t_1)\hat{h}_{2j} + \sum_{j=1}^{i-1} {}_{i-1}C_{j-1}\hat{\Phi}_{2i+1-2j}(t_1)\hat{h}_{2j+1} + \hat{\xi}_{2i}(t_1) \qquad (1 \le i \le k),$$

then

$$\text{(II)} \quad \begin{cases} [P+\hat{\Phi}_1(t_1)]\hat{h}_1 = 0, \quad \hat{h}_1^1 - 1 = 0, \quad [P-\hat{\Phi}_1(t_1)]\hat{h}_2 - \hat{\xi}_1(t_1) = 0, \\ [P+\hat{\Phi}_1(t_1)]\hat{h}_3 + \hat{\delta}_1 = 0, \quad \hat{h}_3^1 = 0, \quad [P-\hat{\Phi}_1(t_1)]\hat{h}_4 - \hat{\Phi}_3(t_1)\hat{h}_2 - \hat{\xi}_3(t_1) = 0, \\ \vdots \\ [P-\hat{\Phi}_1(t_1)]\hat{h}_{2k-2} - \sum_{j=1}^{k-2} {}_{k-2}C_{j-1}\hat{\Phi}_{2k-2j-1}(t_1)\hat{h}_{2j} - \hat{\xi}_{2k-3}(t_1) = 0, \\ [P+\hat{\Phi}_1(t_1)]\hat{h}_{2k-1} + \hat{\delta}_{k-1} = 0, \quad \hat{h}_{2k-1}^1 = 0 \end{cases}$$

and

(III) $\operatorname{rank}[P+\hat{\Phi}_1(t_1), \hat{\delta}_k] = \operatorname{rank}(\hat{D}, \hat{\delta}_k) = n-1$

are satisfied, where $\hat{D}$ is the $n \times (n-1)$ matrix obtained from $P+\hat{\Phi}_1(t_1)$ by deleting the first column vector, and $\hat{h}_{2k} \in R^n$ is a solution of the equation

$$[P-\hat{\Phi}_1(t_1)]h_{2k} - \sum_{j=1}^{k-1} {}_{k-1}C_{j-1}\hat{\Phi}_{2k+1-2j}(t_1)\hat{h}_{2j} - \hat{\xi}_{2k-1}(t_1) = 0 \quad (h_{2k} \in R^n).$$

Due to (I), $\hat{x}(t) = \varphi(t, \hat{x}_0) \in K_1$. Then, for $\hat{\Phi}_1(t)$ and $\hat{h}_1$, we also obtain (3.2)–(3.7) and (3.9). Since $(\hat{h}_1(t), \hat{h}_2(t),\dots, \hat{h}_{2k-1}(t))^T$ can be written in the form

$$\hat{h}_1(t) = \hat{\Phi}_1(t)\hat{h}_1, \quad \hat{h}_{2i-2}(t) = \sum_{j=1}^{i-1} {}_{i-2}C_{j-1}\hat{\Phi}_{2i-1-2j}(t)\hat{h}_{2j} + \hat{\xi}_{2i-3}(t),$$

$$\hat{h}_{2i-1}(t) = \sum_{j=1}^{i-1} {}_{i-2}C_{j-1}(\hat{\Phi}_{2i-2j}(t)\hat{h}_{2j} + \hat{\Phi}_{2i-1-2j}(t)\hat{h}_{2j+1}) + \hat{\xi}_{2i-2}(t)$$

$$= \sum_{j=1}^{i} {}_{i-1}C_{j-1}\hat{\Phi}_{2i+1-2j}(t)\hat{h}_{2j-1} \quad (2 \le i \le k),$$

we obtain by (II)

$$P\hat{h}_{2i-1}(0) + \hat{h}_{2i-1}(t_1) = 0 \quad (1 \le i \le k) \quad \textit{and} \quad P\hat{h}_{2j}(0) - \hat{h}_{2j}(t_1) = 0 \quad (1 \le j \le k-1),$$

respectively. Then from Lemma 3 we have

$$\hat{h}_{2i-1}(t) \in K_{-1} \quad (1 \le i \le k) \quad \textit{and} \quad \hat{h}_{2j}(t) \in K_1 \quad (1 \le j \le k-1).$$

Moreover, $\hat{h}_1^1(0) = 1$ and $\hat{h}_{2i-1}^1(0) = 0$ $(2 \le i \le k)$ because $\hat{h}_{2j-1}(0) = \hat{h}_{2j-1}$ $(1 \le j \le k)$.

When we set $\hat{y}_k \equiv (\hat{x}_0, \hat{h}_1, \hat{h}_2, \dots, \hat{h}_{2k-1}, \hat{\lambda}, \tilde{0})^T \in R^{2kn+k}$ ($\tilde{0} \in R^{k-1}$; a zero vector) for the above-mentioned $(\hat{x}_0, \hat{h}_1, \hat{h}_2, \dots, \hat{h}_{2k-1}, \hat{\lambda})^T \in R^{2kn+1}$, $\hat{y}_k$ becomes a solution of the equation

$$G_k(y_k) \equiv \begin{pmatrix} S_k(x_{2k-1}) - \beta_1\theta_1^{(k)} - \beta_2\theta_2^{(k)} - \cdots - \beta_{k-1}\theta_{k-1}^{(k)} \\ \psi_k(x_{2k-1}) \end{pmatrix} = 0, \tag{4.10}$$

where $\beta_i \in R$ $(1 \le i \le k-1)$ are parameters and $y_k = (x_{2k-1}, \beta_1, \beta_2, \dots, \beta_{k-1})^T \in R^{2kn+k}$. In fact, $\hat{h}_i(t) = \varphi_i(t, \hat{x}_i)$ $(1 \le i \le 2k-1)$, where $\hat{x}_i = (\hat{x}_0, \hat{h}_1, \dots, \hat{h}_i, \hat{\lambda})^T$ $(1 \le i \le 2k-1)$. We denote by $G_k'(y_k)$ the Jacobian matrix of $G_k(y_k)$ with respect to y_k. For the solution $\hat{y}_k$, the matrix $G_k'(\hat{y}_k)$ is singular, that is, $\det G_k'(\hat{y}_k) = 0$ due to (III).

We have already described the procedure for resolving the singularity and we use this procedure for the equation (4.10). In this case, the equation (4.10) plays the role of the equation (3.12) in such a procedure. Hence, if we consider an equation corresponding to the equation (4.2) in that procedure, that is, an augmented system of nonlinear equations which contains the equation (4.10) and additional equations, then the singularity is resolved. Hence we have only to consider this augmented system.

When we set

$$\hat{h}_{2k}(t) \equiv \sum_{j=1}^{k} {}_{k-1}C_{j-1}\hat{\Phi}_{2k+1-2j}(t)\hat{h}_{2j} + \hat{\xi}_{2k-1}(t),$$

then $\hat{h}_{2k}(t)$ becomes a solution of the system

$$\frac{dh_{2k}}{dt} = \hat{Y}^{(1)}(t)h_{2k} + \sum_{j=1}^{k-1} {}_{k-1}C_{j-1}\hat{Y}^{(2k+1-2j)}(t)\hat{h}_{2j}(t) + \hat{V}^{(2k-1)}(t)$$

and also satisfies

$$P\hat{h}_{2k}(0) - \hat{h}_{2k}(t_1) = 0,$$

which, by Lemma 3, implies $\hat{h}_{2k}(t) \in K_1$.

Due to (III), the equation

$$[P+\hat{\Phi}_1(t_1)]h_{2k+1}+\hat{\delta}_k=0, \quad h^1_{2k+1}=0$$

$$(where\ h_{2k+1}=(h^1_{2k+1}, h^2_{2k+1},\dots, h^n_{2k+1})^T \in R^n)$$

has only one solution $\hat{h}_{2k+1} \in R^n$. When we set

$$\hat{h}_{2k+1}(t) \equiv \sum_{j=1}^{k} {}_{k-1}C_{j-1}(\hat{\Phi}_{2k+2-2j}(t)\hat{h}_{2j}+\hat{\Phi}_{2k+1-2j}(t)\hat{h}_{2j+1})+\hat{\xi}_{2k}(t),$$

then $\hat{h}_{2k+1}(t)$ becomes a solution of the system

$$\frac{dh_{2k+1}}{dt}=\hat{Y}^{(1)}(t)h_{2k+1}+\sum_{j=1}^{k} {}_{k-1}C_{j-1}\hat{Y}^{(2k+2-2j)}(t)\hat{h}_{2j}(t)$$

$$+\sum_{j=1}^{k-1} {}_{k-1}C_{j-1}\hat{Y}^{(2k+1-2j)}(t)\hat{h}_{2j+1}(t)+\hat{V}^{(2k)}(t)$$

and also satisfies

$$P\hat{h}_{2k+1}(0)+\hat{h}_{2k+1}(t_1)=0,$$

which, by Lemma 3, implies $\hat{h}_{2k+1}(t) \in K_{-1}$. Moreover, $\hat{h}^1_{2k+1}(0)=0$ because $\hat{h}_{2k+1}(0)=\hat{h}_{2k+1}$.

From what we have discussed above, the equation

$$(4.11)\quad G_{k+1}(y_{k+1}) \equiv \begin{pmatrix} S_{k+1}(x_{2k+1})-\beta_1\theta_1^{(k+1)}-\beta_2\theta_2^{(k+1)}-\dots-\beta_k\theta_k^{(k+1)} \\ \psi_{k+1}(x_{2k+1}) \end{pmatrix}=0$$

has a solution $\hat{y}_{k+1}=(\hat{x}(0), \hat{h}_1(0), \hat{h}_2(0),\dots, \hat{h}_{2k+1}(0), \hat{\lambda}, \hat{0})^T \in R^{2(k+1)n+k+1}$, where $\beta_i \in R$ $(1 \le i \le k)$ are parameters, $y_{k+1}=(x_{2k+1}, \beta_1, \beta_2,\dots, \beta_k)^T \in R^{2(k+1)n+k+1}$, and $\hat{x}(0)=\hat{x}_0$ and $\hat{h}_i(0)=\hat{h}_i$ $(1 \le i \le 2k+1)$, and $\hat{0} \in R^k$ is a zero vector. In fact, $\hat{h}_i(t)=\varphi_i(t, \hat{x}_i)$ $(i=2k, 2k+1)$, where $\hat{x}_i=(\hat{x}(0), \hat{h}_1(0),\dots, \hat{h}_i(0), \hat{\lambda})^T$ $(i=2k, 2k+1)$. We denote by $G'_{k+1}(y_{k+1})$ the Jacobian matrix of $G_{k+1}(y_{k+1})$ with respect to y_{k+1}.

When $\hat{Y}^{(i)}(t)$ and $\hat{V}^{(i)}(t)$ $(i=2k+1, 2k+2)$ denote the values of $Y^{(i)}$ and $V^{(i)}$ at $x=\hat{x}(t)$, $h_j=\hat{h}_j(t)$ $(1 \le j \le i-1)$ and $\lambda=\hat{\lambda}$, respectively, and $(\hat{\Phi}_{2k+1}(t), \hat{\Phi}_{2k+2}(t))^T$ ($2n \times n$ matrix) is a solution of the system

$$\frac{dz_{2k+1}}{dt}=\hat{Y}^{(1)}(t)z_{2k+1}+\sum_{j=1}^{k} {}_kC_{j-1}\hat{Y}^{(2k+3-2j)}(t)\hat{\Phi}_{2j-1}(t),$$

$$\frac{dz_{2k+2}}{dt}=\hat{Y}^{(1)}(t)z_{2k+2}+\hat{Y}^{(2)}(t)z_{2k+1}$$

$$+\sum_{j=1}^{k} {}_kC_{j-1}(\hat{Y}^{(2k+3-2j)}(t)\hat{\Phi}_{2j}(t)+\hat{Y}^{(2k+4-2j)}(t)\hat{\Phi}_{2j-1}(t))$$

$$(where\ z_i\text{'s}\ (i=2k+1, 2k+2)\ are\ n \times n\ matrices)$$

such that $(\hat{\Phi}_{2k+1}(0), \hat{\Phi}_{2k+2}(0))^T=(O, O)^T$ (O is an $n \times n$ zero matrix), and $(\hat{\xi}_{2k+1}(t),$

$\hat{\xi}_{2k+2}(t))^T \in R^{2n}$ is a solution of the system

$$\frac{d\xi_{2k+1}}{dt} = \hat{Y}^{(1)}(t)\xi_{2k+1} + \sum_{j=1}^{k} {}_kC_{j-1}\hat{Y}^{(2k+3-2j)}(t)\hat{\xi}_{2j-1}(t) + \hat{V}^{(2k+1)}(t),$$

$$\frac{d\xi_{2k+2}}{dt} = \hat{Y}^{(1)}(t)\xi_{2k+2} + \hat{Y}^{(2)}(t)\xi_{2k+1} + \sum_{j=1}^{k} {}_kC_{j-1}(\hat{Y}^{(2k+3-2j)}(t)\hat{\xi}_{2j}(t)$$
$$+ \hat{Y}^{(2k+4-2j)}(t)\hat{\xi}_{2j-1}(t)) + \hat{V}^{(2k+2)}(t)$$

such that $(\hat{\xi}_{2k+1}(0), \hat{\xi}_{2k+2}(0))^T = (0, 0)^T \in R^{2n}$, and $\hat{h}_{2k+2} \in R^n$ is a solution of the equation

$$[P - \hat{\Phi}_1(t_1)]h_{2k+2} - \sum_{j=1}^{k} {}_kC_{j-1}\hat{\Phi}_{2k+3-2j}(t_1)\hat{h}_{2j} - \hat{\xi}_{2k+1}(t_1) = 0 \quad (h_{2k+2} \in R^n),$$

then we set

$$\hat{\delta}_{k+1} \equiv \sum_{j=1}^{k+1} {}_kC_{j-1}\hat{\Phi}_{2k+4-2j}(t_1)\hat{h}_{2j} + \sum_{j=1}^{k} {}_kC_{j-1}\hat{\Phi}_{2k+3-2j}(t_1)\hat{h}_{2j+1} + \hat{\xi}_{2k+2}(t_1).$$

Then, concerning the regularity of the matrix $G'_{k+1}(\hat{y}_{k+1})$, we have the following theorem.

Theorem 3. *The matrix $G'_{k+1}(\hat{y}_{k+1})$ is nonsingular if and only if*

$$\text{rank}(\hat{D}, \hat{\delta}_{k+1}) = n. \tag{4.12}$$

The proof is done in a similar way to that of Theorem 1 and will be omitted. Due to Theorem 3, if the condition (4.12) is satisfied, then the equation (4.11) has an isolated solution $\hat{y}_{k+1}$ containing the singular point $\hat{x}_0$—this means that resolution of the singularity is realized— and therefore we can compute $\hat{x}_0$ with high accuracy by applying the Newton method to the equation (4.11).

Now we consider another method which need not look for the vector e_{j_0} satisfying the condition (4.1).

When we set

$$\hat{h}_{2k+2}(t) \equiv \sum_{j=1}^{k+1} {}_kC_{j-1}\hat{\Phi}_{2k+3-2j}(t)\hat{h}_{2j} + \hat{\xi}_{2k+1}(t),$$

then $\hat{h}_{2k+2}(t)$ becomes a solution of the system

$$\frac{dh_{2k+2}}{dt} = \hat{Y}^{(1)}(t)h_{2k+2} + \sum_{j=1}^{k} {}_kC_{j-1}\hat{Y}^{(2k+3-2j)}(t)\hat{h}_{2j}(t) + \hat{V}^{(2k+1)}(t)$$

and also satisfies

$$P\hat{h}_{2k+2}(0) - \hat{h}_{2k+2}(t_1) = 0,$$

which, by Lemma 3, implies $\hat{h}_{2k+2}(t) \in K_1$.

Then the equation

$$(4.13)\qquad H_{k+1}(z_{k+1})\equiv\begin{pmatrix} T_{k+1}(\boldsymbol{x}_{2k+2})-\beta_1\Xi_1^{(k+1)}-\beta_2\Xi_2^{(k+1)}-\cdots-\beta_k\Xi_k^{(k+1)} \\ \psi_{k+1}(\boldsymbol{x}_{2k+1}) \end{pmatrix}=0$$

has a solution $\hat{z}_{k+1}=(\hat{\boldsymbol{x}}_{2k+2},\hat{0})^T=(\hat{x}(0),\hat{h}_1(0),\hat{h}_2(0),\dots,\hat{h}_{2k+2}(0),\hat{\lambda},\hat{0})^T\in R^{(2k+3)n+k+1}$, where $z_{k+1}=(\boldsymbol{x}_{2k+2},\beta_1,\beta_2,\dots,\beta_k)^T\in R^{(2k+3)n+k+1}$. In fact, $\hat{h}_{2k+2}(t)=\varphi_{2k+2}(t,\hat{\boldsymbol{x}}_{2k+2})$. Hence, when we consider the equation (4.13) instead of (4.11), we need not look for the vector e_{j_0} satisfying the condition (4.1). We denote by $H'_{k+1}(z_{k+1})$ the Jacobian matrix of $H_{k+1}(z_{k+1})$ with respect to z_{k+1}. Concerning the regularity of the matrix $H'_{k+1}(\hat{z}_{k+1})$ we have the following corollary.

Corollary. *The matrix $H'_{k+1}(\hat{z}_{k+1})$ is nonsingular if and only if the condition (4.12) is satisfied.*

Remark. If $g_{\alpha\alpha\alpha}(0,0)=0$ and $\mathrm{rank}(\hat{D},\hat{\delta}_2)=n-1$, then $g_{\alpha\alpha\alpha}(0,0)=g_{\alpha\alpha\zeta}(0,0)=g_{\alpha\zeta\zeta}(0,0)=g_{\zeta\zeta\zeta}(0,0)=0$ due to (4.5) and (4.7). In this case, when we set

$$\begin{aligned}(4.14)\qquad \hat{v}_5&\equiv-\hat{\Phi}_6(2\pi)\hat{h}_2-\hat{\Phi}_5(2\pi)\hat{h}_3-2(\hat{\Phi}_4(2\pi)\hat{h}_4+\hat{\Phi}_3(2\pi)\hat{h}_5)-\hat{\Phi}_2(2\pi)\hat{h}_6-\hat{\xi}_6(2\pi)\\ &=\hat{L}[P^{-1}\{\hat{\Phi}_6(t_1)\hat{h}_2+\hat{\Phi}_5(t_1)\hat{h}_3+2(\hat{\Phi}_4(t_1)\hat{h}_4+\hat{\Phi}_3(t_1)\hat{h}_5)+\hat{\Phi}_2(t_1)\hat{h}_6\\ &\quad+\hat{\xi}_6(t_1)\}]=\hat{L}P^{-1}\hat{\delta}_3\end{aligned}$$

and $\hat{\mu}\equiv\langle\hat{v}_5,\hat{\phi}_0\rangle$, then by careful calculations we obtain

$$\begin{aligned}(4.15)\qquad & g_{\alpha\alpha\alpha\alpha}(0,0)=0,\quad g_{\alpha\alpha\zeta\zeta}(0,0)=2\hat{a}g_{\alpha\alpha\alpha\zeta}(0,0),\\ & g_{\alpha\zeta\zeta\zeta}(0,0)=\hat{\mu}+3\hat{a}^2g_{\alpha\alpha\alpha\zeta}(0,0)\quad \textit{and}\quad g_{\zeta\zeta\zeta\zeta}(0,0)=4\hat{a}\hat{\mu}+4\hat{a}^3g_{\alpha\alpha\alpha\zeta}(0,0),\end{aligned}$$

and therefore we can write the bifurcation equation (1.14) in the form

$$(4.16)\qquad g(\alpha,\zeta)=\frac{1}{4!}[4\zeta(\alpha+\hat{a}\zeta)\{g_{\alpha\alpha\alpha\zeta}(0,0)(\alpha+\hat{a}\zeta)^2+\hat{\mu}\zeta^2\}]+h.o.t.=0.$$

By (3.5), (3.9) and (4.14) we see that

$$\begin{aligned}(4.17)\qquad \mathrm{rank}\,[I_n-\hat{\Phi}_1(2\pi),\hat{v}_5]&=\mathrm{rank}\,[I_n+\hat{U},P^{-1}\hat{\delta}_3]\\ &=\mathrm{rank}\,[P+\hat{\Phi}_1(t_1),\hat{\delta}_3]=\mathrm{rank}\,(\hat{D},\hat{\delta}_3).\end{aligned}$$

Therefore, if $\mathrm{rank}(\hat{D},\hat{\delta}_3)=n$, then $\hat{\mu}\neq0$. Hence, by (4.16), if the condition

$$(4.18)\qquad g_{\alpha\alpha\alpha\zeta}(0,0)\times\hat{\mu}<0$$

is satisfied, then four different branches of solutions of (1.4) intersect at the singular point $\hat{\boldsymbol{x}}_0$.

Acknowledgement

The author would like to express his deep gratitude to Professor Seiiti Huzino of Kyushu University for his polite guidance and very helpful suggestions throughout this work.

Department of Applied Mathematics
Faculty of Engineering
Tokushima University

References

[1] F. Brezzi, J. Rappaz and P. A. Raviart, Finite dimensional approximation of nonlinear problems. Part III: Simple bifurcation points, *Numer. Math.*, **38** (1981), 1–30.
[2] K. A. Cliffe and A. Spence, The calculation of high order singularities in the finite Taylor problem, *Internat. Ser. Numer. Math.*, **70** (1984), 129–144.
[3] Y. Katsuta, Tajigen hisenkeikei no teiseiteki kenkyu (Qualitative study on finite dimensional nonlinear systems), *Master's Thesis in Electronics, Tokushima Univ.*, (1982) (in Japanese).
[4] Y. Katsuta, H. Kawakami and N. Kawaguchi, Taishyosei o motsu jiritsukei ni okeru shyukikai no bunki (Bifurcations of periodic solutions of autonomous systems with a symmetric condition), *National Convention Record, IECE Japan*, (1982), Part I-1, p. 9 (in Japanese).
[5] H. Kawakami, Bifurcation of periodic responses in forced dynamic nonlinear circuits: Computation of bifurcation values of the system parameters, *IEEE Trans. Circuits and Systems*, **31** (1984), 248–260.
[6] H. Weber, On the numerical approximation of secondary bifurcation problems, *Lecture Notes in Math.*, **878** (1981), 407–425.
[7] B. Werner and A. Spence, The computation of symmetry-breaking bifurcation points, *SIAM J. Numer. Anal.*, **21** (1984), 388–399.
[8] N. Yamamoto, Newton's method for singular problems and its application to boundary value problems, *J. Math. Tokushima Univ.*, **17** (1983), 27–88.
[9] N. Yamamoto, A method for computing singular points of nonlinear equations involving parameters, *J. Inform. Process.*, **7** (1984), 75–80.
[10] N. Yamamoto, An analysis of bifurcation points of nonlinear equations satisfying a condition, *J. Math. Tokushima Univ.*, **19** (1985), 63–99.
[11] N. Yamamoto, An analysis of bifurcations of periodic solutions of nonlinear periodic differential systems satisfying some condition. I, **RMC 59–11** (1984), Kyushu University.
[12] N. Yamamoto, An analysis of bifurcations of periodic solutions of nonlinear periodic differential systems satisfying some condition. II, **RMC 59–12** (1984), Kyushu University.