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On the Radius of Convergence of the Simplest Power Series Solution of Painlevé-I equation II(*A Singular Solution*)

By

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In our previous paper [1] we considered the simplest power series solution of the Painlevé-I equation which is regular at the origin. This note is a sequel to it. Here we consider another simplest Laurent series solution which is singular at the origin. Important feature of this solution is the location of the singularities. The location of the nearest singularity from the origin is given by the radius S of convergence of this Laurent series. The value of S is calculated numerically by the same method as in [1]. We obtained $S=2.56\dots$. Various theoretical bounds for S are also obtained.

The mathematical part of this work was done by Kametaka and the numerical part by Noda.

1. Introduction

In this note we treat the Painlevé-I (P-I) equation

$$v''=6v^2+z \quad ('=d/dz,\ v=v(z)). \tag{1.1}$$

Painlevé observed a two parameter (z_0 and h) family of Laurent series solutions:

$$\begin{aligned} v=(z-z_0)^{-2}-\frac{z_0}{10}(z-z_0)^2-\frac{1}{6}(z-z_0)^3 \\ +h(z-z_0)^4+\frac{z_0^2}{300}(z-z_0)^6+\cdots \end{aligned} \tag{1.2}$$

which has a moving double pole $z=z_0$ [2]. Here we consider the simplest case $z_0=0$ and $h=0$. That is we treat a special Laurent series solution

$$v=z^{-2}-\frac{1}{6}z^3+\frac{1}{264}z^8-\frac{1}{19008}z^{13}+\frac{5}{7683984}z^{18}+\cdots \tag{1.3}$$

which satisfies the initial conditions

$$(z^2v(z)|_{z=0}=1),\quad \left(\frac{d}{dz}\right)^6(z^2v(z))|_{z=0}=0. \tag{1.4}$$

The solution (1.3) has a positive finite radius S of convergence. The location of the nearest singularity from the origin is given by S. The numerical value of S is obtained by the same method as in [1].

(1.5) $$S=2.56\ldots$$

2. Comparison with a Jacobi elliptic function $tn^2(z\,;\,k)$

Inserting the expansion

(2.1) $$v=z^{-2}-\frac{1}{6}z^3+\frac{1}{264}\sum_{j=0}^{\infty}(-1)^j B_j z^{5j+8} \quad (|z|<S)$$

into the equation (1.1) we obtain the recurrence relation

(2.2) $B_0=1, \quad B_1=1/72,$

$$B_{j+2}=\frac{1}{(j+2.8)(j+4.2)}\left\{\frac{1}{1100}\sum_{i=0}^{j}B_iB_{j-i}+\frac{2}{25}B_{j+1}\right\},$$

$j=0, 1, 2,\ldots$

On the other hand the function

(2.3) $$t(z)=tn^2(z;\,k)=(sn(z;\,k)/cn(z;\,k))^2 \quad (0\le k<1).$$

has a power series expansion:

(2.4) $$t(z)=\sum_{j=0}^{\infty}t_j z^{2j+2},$$

whose radius of convergence is $K(k)$ where

(2.5) $$K(k)=\int_0^1\{(1-x^2)(1-k^2x^2)\}^{-1/2}dx$$

is the complete elliptic integral of the first kind. Since $t(z)$ satisfies a differential equation:

(2.6) $$t''=6(1-k^2)t^2+4(2-k^2)t+2,$$

the coefficients t_j satisfy the recurrence relation:

(2.7) $t_0=1, \quad t_1=(2-k^2)/3,$

$$t_{j+2}=\frac{1}{(j+2.5)(j+3)}\left\{\frac{3(1-k^2)}{2}\sum_{i=0}^{j}t_it_{j-i}+(2-k^2)t_{j+1}\right\}$$

$j=0, 1, 2,\ldots$

Notice that B_j and t_j are strictly positive. Comparing (2.2) with (2.7) we can show the inequalities:

(2.8) $$c_2(k)^{-j}t_j \leq B_j \leq c_1(k)^{-j}t_j, \quad j=0, 1, 2,\dots$$

where positive valued functions $c_1(k)$ and $c_2(k)$ are given by

(2.9) $$c_1(k)=\mathrm{Min}\,\{25(2-k^2)/2,\ 5\{66(1-k^2)\}^{1/2}\}$$
$$=\begin{cases} 25(2-k^2)/2 & (0\leq k^2\leq k_1^2), \\ 5\{66(1-k^2)\}^{1/2} & (k_1^2\leq k^2<1), \end{cases}$$

where $k_1^2=2(\sqrt{2706}-41)/25=0.8815\dots$,

(2.10) $$c_2(k)=\mathrm{Max}\,\{\{12936(1-k^2)/5\}^{1/2},\ 24(2-k^2)\}$$
$$=\begin{cases} \{12936(1-k^2)/5\}^{1/2} & (0\leq k^2\leq k_2^2), \\ 24(2-k^2) & (k_2^2\leq k^2<1), \end{cases}$$

where $k_2^2=(\sqrt{9145}-59)/48=0.76311\dots$
$c=c_1(k)$ satisfies $(2-k^2)/3c\geq 1/72$, $3(1-k^2)/2c^2\geq 1/1100$ and $(2-k^2)/c\geq 2/25$. On the other hand $c=c_2(k)$ satisfies $(2-k^2)/3c\leq 1/72$, $(196/125)3(1-k^2)/2c^2\leq 1/1100$ and $(196/125)(2-k^2)/c\leq 2/25$. These inequalities and comparison (2.2) with (2.7) assures (2.8). We omit further discussion because the similar process discussed in detail in [1].

Since

(2.11) $$\lim_{j\to\infty} B_j^{-1/5j}=S, \quad \lim_{j\to\infty} t_j^{-1/j}=K^2(k),$$

it follows from (2.8)

Theorem 2.1

(2.12) $$(c_1(k)K^2(k))^{1/5}\leq S\leq (c_2(k)K^2(k))^{1/5} \quad (0\leq k<1).$$

The maximal value of the left side and the minimal value of the right side of (2.12) are obtained numerically.

Theorem 2.2

(2.13) $$2.44449\dots\leq S\leq 2.59522\dots$$

3. Differential inequalities and elliptic integrals

The function

(3.1) $$b(z)=\sum_{j=0}^{\infty} B_j z^{2j+2} \quad (|z|<S^{5/2})$$

satisfies the differential equation:

(3.2) $$b''+3z^{-1}b'-(24/25)z^{-2}b=(b^2+88b+1936)/275.$$

Since B_j are positive, in the interval $0\leq z<S^{5/2}$ we have

(3.3) $$b''\leq(b^2+88b+550)/275,$$

(3.4) $$b''\geq(b^2+88b+1056)/528.$$

Multiply both sides of these inequalities by b' (≥ 0) and integrate on the interval $(0, z)$ we have

(3.5) $$b'^2\leq(2/825)(b^3+132b^2+1650b),$$

(3.6) $$b'^2\geq(b^3+132b^2+3168b)/792.$$

It follows

(3.7) $$(825/2)^{1/2}(b^3+132b^2+1650b)^{-1/2}b'\leq 1,$$

(3.8) $$792^{1/2}(b^3+132b^2+3168b)^{-1/2}b'\geq 1.$$

Integrating these inequalities on the interval $(0, S^{5/2})$ we have

(3.9) $$J_1\leq S^{5/2}\leq J_2$$

where

(3.10) $$J_1=(5\sqrt{66}/2)\int_0^{\infty}(b^3+132b^2+1650b)^{-1/2}db,$$

(3.11) $$J_2=\sqrt{792}\int_0^{\infty}(b^3+132b^2+3168b)^{-1/2}db.$$

By the change of variable $b=\sqrt{1650}\,x^2$ ($b=\sqrt{3168}\,x^2$)

(3.12) $$J_1=\sqrt{5\sqrt{66}}\int_0^{\infty}(x^4+(2\sqrt{66}/5)x^2+1)^{-1/2}dx,$$

(3.13) $$J_2=2\sqrt{3\sqrt{22}}\int_0^{\infty}(x^4+(\sqrt{22}/2)x^2+1)^{-1/2}dx.$$

By further change of variable $x=\{(\sqrt{66}-\sqrt{41})/5\}^{1/2}\tan(\theta)$ $(x=\{(\sqrt{22}-\sqrt{6})/4\}^{1/2}\tan(\theta))$ under the sign of integration the elliptic integral J_1 (J_2) takes its canonical form.

(3.14) $$J_1=c_1K(k_1),$$

$$c_1=\{66-\sqrt{2706}\}^{1/2}=3.7390872\dots,$$

$$k_1^2=2(\sqrt{2706}-41)/25=0.8815382\dots.$$

(3.15) $$J_2=c_2K(k_2),$$

$$c_2=\{6(11-\sqrt{33})\}^{1/2}=5.6153917\dots,$$

$$k_2^2=(\sqrt{33}-3)/4=0.6861407\dots.$$

By the numerical table of K. Hayashi and S. Moriguchi [3] we have

$$K(\sqrt{0.882})=2.5004844,\quad K(\sqrt{0.881})=2.4965426,$$

$$K(\sqrt{0.687})=2.0566350,\quad K(\sqrt{0.686})=2.0552292.$$

By the linear interpolation we have

$$K(k_1)=2.4986640,\quad K(k_2)=2.0554269.$$

Using these values we have

Theorem 3.1

(3.16) $$J_1^{2/5}=2.444\dots\le S\le J_2^{2/5}=2.66\dots.$$

4. Method of E. Hille

The method used in this section is due to E. Hille [4]. The function

(4.1) $$w(z)=\sum_{j=0}^{\infty}B_jz^{5j+8}\quad(|z|<S),$$

satisfies the differential equation:

(4.2) $$w''=w^2/44+(12z^{-2}+2z^3)w+44z^6.$$

Since B_j are positive, in the interval $0\le z<S$ we have

(4.3) $$w''\ge w^2/44;$$

after integration,

(4.4) $$w'^2\ge w^3/66.$$

Integrating the inequality

(4.5) $$(-w^{-1/2})'\ge 1/2\sqrt{66},$$

which follows from (4.4), on the interval (z, S) and using the fact $\lim_{z\to S-0} w^{-1/2}(z)=0$, we have

(4.6) $$w^{-1/2}(z) \geq (S-z)/2\sqrt{66}.$$

So we have the inequality:

(4.7) $$w(z) \leq 264(S-z)^{-2}, \quad (0 \leq z < S),$$

especially

(4.8) $$B_j z^{5j+8} \leq 264(S-z)^{-2}, \quad (0 \leq z < S), \quad j=0, 1, 2,\dots$$

Putting $z=Sx$, we have

(4.9) $$S \leq (264 f_j(x)/B_j)^{1/(5j+10)},$$

$$(0<x<1), \quad j=0, 1, 2,\dots$$

where $f_j(x)=(1-x)^{-2}x^{-(5j+8)}$. The minimal value of $f_j(x)$ is attained at $x=(5j+8)/(5j+10)$. Inserting this value of x into x we finally have

Theorem 4.1

(4.10) $$S \leq S_j, \quad j=0, 1, 2,\dots$$

where

(4.11) $$S_j=\{(5j+10)/(5j+8)\}\{66(5j+8)^2/B_j\}^{1/(5j+10)}.$$

By (2.11) and (4.10) we have

Theorem 4.2

(4.12) $$S=\inf_{j \geq k} S_j$$

for any fixed $k=0, 1, 2,\dots$.

5. Cauchy-Hadamard and d'Alembert's method

Define C_j and D_j by

(5.1) $$C_j=B_j^{-1/(5j+10)},$$

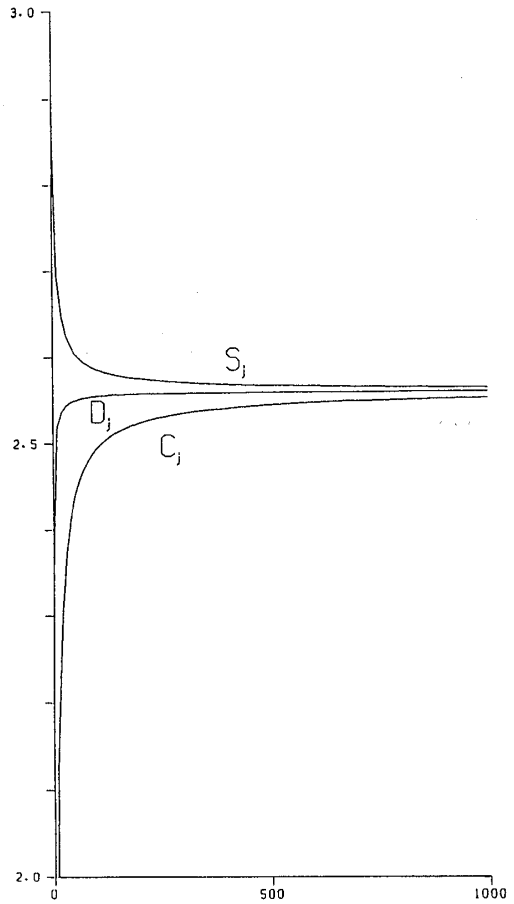
(5.2) $$D_j=(B_j/B_{j+1})^{1/5}, \quad j=0, 1, 2,\dots$$

we have

(5.3) $$S=\varliminf_{j\to\infty} C_j \quad \text{(Cauchy-Hadamard)},$$

(5.4) $$S=\lim_{j\to\infty} D_j \quad \text{(d'Alembert)}.$$

d'Alembert's formula is valid only when the right side has meaning. We can not



Numerical behaviour of C_j (Cauchy-Hadamard), D_j (d'Alembert) and S_j (modified Cauchy-Hadamard).

prove the existence of the limit value. But numerically, as shown in the next figure, C_j and D_j converges slowly.

In the figure 5.1 the behaviour of S_j is also indicated. As a conclusion of this numerical calculation and the theorem 4.1 we obtain

Theorem 5.1

$$S \leq \min_{0 \leq j \leq 1000} S_j = S_{1000} = 2.56374\ldots \tag{5.5}$$

6. Numerical calculation of S by Briot-Bouquet equation

The functions:

$$\alpha(z) = \frac{1}{6600} \sum_{j=0}^{\infty} B_j z^{j+2}, \tag{6.1}$$

$$\beta(z)=\frac{1}{13200}\sum_{j=0}^{\infty}(j+2)B_j z^{j+2}, \quad (|z|<S^5).$$

satisfies the system of differential equations:

$$(6.2) \qquad \begin{cases} z\alpha'=2\beta, \\ z\beta'=3\alpha^2+\beta+(z+3)\alpha/25+z^2/7500. \end{cases}$$

As stated in the section 2, $z=S$ is a double pole of $v(z)$. So $z=S^5=z_0$ is a double pole of $\alpha(z)$. Since $\alpha(z)$ is strictly increasing on the interval $(0, S^5)$, its inverse function $\alpha^{-1}(x)$ is also strictly increasing on the interval $(0, \infty)$. Using this inverse function $\alpha^{-1}(x)$, we introduce new functions:

$$(6.3) \qquad \begin{cases} t(\xi)=\log\alpha^{-1}(\xi^{-2}), \\ \eta(\xi)=\xi^3\beta(\alpha^{-1}(\xi^{-2}))-1. \end{cases}$$

By (6.2) these functions satisfy

$$(6.4) \qquad \begin{cases} t'=-(1+\eta)^{-1}, \\ \xi(1+\eta)'=(1+\eta)^{-1}\{3(1+\eta)^2-3-(1+\eta)\xi \\ \qquad -(e^t+3)\xi^2/25-e^{2t}\xi^4/7500\}, \end{cases}$$

where $'=d/d\xi$. Since

$$(6.5) \qquad \lim_{z\to z_0-0}\alpha(z)=+\infty,$$

we have

$$(6.6) \qquad \lim_{\xi\to+0}t(\xi)=\log z_0=t_0.$$

Moreover we can show (see [1])

Lemma 6.1

$$(6.7) \qquad \lim_{z\to z_0-0}\alpha^3(z)/\beta^2(z)=1.$$

This means that

$$(6.8) \qquad \lim_{\xi\to+0}\eta(\xi)=0.$$

The function $t(\xi)$ can be expressed as a Taylor series

$$(6.9) \qquad t_0-t(\xi)=\xi-\xi^2/10-(2+\exp(t_0))\xi^3/300+\cdots$$

which converges for sufficiently small ξ. For sufficiently small ξ we have an estimate

$$(6.10) \qquad |t_0-(t(\xi)+\xi-\xi^2/10)|\leq\text{const.}\,|\xi|^3$$

where const. denotes a suitable positive constant. The value of $t(\xi)+\xi-\xi^2/10$ for sufficiently small ξ is a good approximation to t_0.

To calculate the values of $t(\xi)$ we need to solve numerically the system of differential equations (6.4). We explain how to choose suitable initial "time" and how to settle initial conditions.

(i) Choose relatively small z_1 which satisfies $0<z_1<z_0$. (ii) Calculate approximate values of $\alpha(z_1)$ and $\beta(z_1)$ by

$$\alpha(z_1)=\frac{1}{6600}\sum_{j=0}^{N} B_j z_1^{j+2}, \tag{6.11}$$

$$\beta(z_1)=\frac{1}{13200}\sum_{j=0}^{N}(j+2)B_j z_1^{j+2}$$

taking a sufficiently large N. (iii) Settle the initial time by

$$\xi_1=\alpha(z_1)^{-1/2}. \tag{6.12}$$

(iv) The initial conditions are

$$t(\xi_1)=\log z_1, \quad 1+\eta(\xi_1)=\xi_1^3\beta(z_1). \tag{6.13}$$

Finally the algorithm calculating S is as follows.

(i) Solve the system of differential equations (6.4) under the initial conditions (6.13) by the Runge-Kutta method.

(ii) Choose sufficiently small ξ_2 and determine $t(\xi_2)$.

(iii) Put $t_0=t(\xi_2)+\xi_2-\xi_2^2/10$.

(iv) $S=\exp(t_0/5)$ is what we wanted.

By means of the above algorithm we calculated for several values of z_1 and ξ_2. The conclusion is as follows:

Numerical result

$$S=2.5599\ldots \tag{6.14}$$

7. Relation between R and S

In [1] we obtained a numerical value of the radius R of convergence of a special power series solution:

$$u=\frac{1}{6}z^3+\frac{1}{336}\sum_{j=0}^{\infty}A_j z^{5j+8} \quad (|z|<R) \tag{7.1}$$

of the Painlevé-I equation. The coefficients A_j satisfy

(7.2) $A_0=1, \quad A_1=1/78,$

$$A_{j+2}=\frac{1}{(j+3.4)(j+3.6)}\left\{\frac{1}{1400}\sum_{i=0}^{j}A_iA_{j-i}+\frac{2}{25}A_{j+1}\right\} \qquad j=0, 1, 2,\dots$$

Comparing (2.2) with (7.2) it follows

(7.3) $$A_j\leqq B_j, \quad j=0, 1, 2,\dots;$$

(7.4) $$S\leqq R.$$

For a positive constant c

(7.5) $$\tilde{A}_j=c^jA_j$$

satisfy

(7.6) $\tilde{A}_0=1, \quad \tilde{A}_1=c/78,$

$$\tilde{A}_{j+2}=\frac{1}{(j+3.4)(j+3.6)}\left\{\frac{c^2}{1400}\sum_{i=0}^{j}\tilde{A}_i\tilde{A}_{j-i}+\frac{2c}{25}\tilde{A}_{j+1}\right\}, \qquad j=0, 1, 2,\dots.$$

Putting $c=\sqrt{102/77}$, we have

(7.7) $\tilde{A}_0=B_0, \quad \tilde{A}_1\geqq B_1,$

$$\tilde{A}_{j+2}\geqq\frac{1}{(j+2.8)(j+4.2)}\left\{\frac{1}{1100}\sum_{i=0}^{j}\tilde{A}_i\tilde{A}_{j-i}+\frac{2}{25}\tilde{A}_{j+1}\right\},$$

using an inequality:

$$(j+3.4)(j+3.6)\leqq(51/49)(j+2.8)(j+4.2).$$

Comparering (2.2) with (7.7) we obtain

(7.8) $$B_j\leqq\tilde{A}_j=c^jA_j, \quad j=0, 1, 2,\dots;$$

(7.9) $$R\leqq c^{1/5}S.$$

We obtained

Theorem 7.1

(7.10) $$c_1R\leqq S\leqq R\leqq c_2S,$$

$$c_1=(77/102)^{1/10}=0.972\dots,$$

$$c_2=(102/77)^{1/10}=1.0285\dots.$$

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