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A Note on Some Positive Definite 1-*dimensional Schrödinger Operators with Rapidly Decreasing Potentials*

By

Mayumi OHMIYA

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§ 1. Introduction

In this short note, we study the 1-dimensional Schrödinger operator

$$H_u = -(d/dx)^2 + u(x), \quad -\infty < x < \infty$$

with the real valued potential $u = u(x)$ which decreases sufficiently rapidly. Throughout the paper we assume that the potential $u(x)$ is in L_2^1, where

$$L_\lambda^1 = \left\{ w(x) \mid \text{real valued and} \int_{-\infty}^{\infty} (1 + |x|^\lambda)|w(x)|dx < \infty \right\}, \quad \lambda \geqslant 0.$$

We consider the unique selfadjoint extension of H_u considered in the space of twice continuously differentiable functions on $(-\infty, \infty)$ with compact support and denote it again by H_u. Then, it is well known that H_u has purely absolutely continuous spectrum $[0, \infty)$ and a finite number of negative discrete eigenvalues. See e.g., [2], [3] and [4] for detail.

On the other hand, we say that H_u is *decomposable* if and only if there exists an absolutely continuous real valued function $v(x) \in L_1^1$ such that

$$(1) \qquad H_u = AA^*,$$

where

$$A = d/dx + v(x)$$

and $A^* = -d/dx + v(x)$ is the formal adjoint operator of A. This implies that $v(x)$ solves the Riccati equation

$$v'(x) + v(x)^2 = u(x), \quad ' = d/dx.$$

The main purpose of the present paper is to characterize the decomposable

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1-dimensional Schrödinger operator H_u in connection with scattering theory. More precisely, our aim is to prove the following.

Theorem. *H_u, $u \in L_2^1$ is decomposable if and only if H_u has no discrete eigenvalues and the reflection coefficients $r_\pm(\xi)$ of H_u satisfy the condition*

$$|r_\pm(\xi)| \leqslant \text{Const.} < 1, \quad \xi \neq 0.$$

This problem was first studied by M. J. Ablowitz, M. Kruskal and H. Segur [1] to examine the range of Miura's transformation

$$v(x) \longrightarrow v'(x) + v(x)^2.$$

They studied it in connection with the method of inverse scattering transform for soliton equations. In this paper, we will study the problem from somewhat different standpoint of view and by different method.

In §2 we summarize the results from scattering theory for H_u. In §3 we prove Theorem.

Throughout the paper c^* denotes the complex conjugate of c, if c is a complex number.

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§ 2. Scattering data.

Here we summarize results from scattering theory for H_u, $u \in L_2^1$. We refer to [2], [3] and [4] for detail.

Consider the differential equation

$$H_u f = -f'' + u(x) f = \zeta^2 f, \quad \zeta = \xi + i\eta, \tag{2}$$

where ξ and η are real numbers and $i = \sqrt{-1}$. If $\eta = \operatorname{Im} \zeta > 0$, then there exist unique solutions $f_\pm(x; \zeta)$ of (2) which behave like $\exp(\pm i\zeta x)$ as $x \to \pm\infty$ respectively. The solutions $f_+(x; \zeta)$ and $f_-(x; \zeta)$ are called the right and left Jost solutions respectively. Jost solutions $f_\pm(x; \zeta)$ are analytic in ζ, $\operatorname{Im} \zeta > 0$. Suppose that $\zeta = \xi$ is real. Then one easily verifies

$$[f_+(x; \xi), f_+(x; \xi)^*] = -2i\xi,$$

where $[y, z] = yz' - y'z$ is the Wronskian. Thus, $f_+(x; \xi)$ and $f_+(x; \xi)^*$ are linearly independent solutions of (2) for $\xi \neq 0$. Hence we have

$$f_-(x; \xi) = a(\xi) f_+(x; \xi)^* + b(\xi) f_+(x; \xi), \quad \xi \neq 0.$$

This implies

$$a(\xi) = (2i\xi)^{-1} [f_-(x; \xi), f_+(x; \xi)], \tag{3}$$

(4) $$b(\xi)=(2i\xi)^{-1}[f_+(x;\ \xi)^*, f_-(x;\ \xi)]$$

and

$$|a(\xi)|^2=1+|b(\xi)|^2.$$

By (3), $a(\xi)$ is the boundary value of the analytic function $a(\zeta)$, $\operatorname{Im}\zeta>0$. Put

$$r_\pm(\xi)=\pm b(\pm\xi)/a(\xi),\quad \xi\neq 0.$$

The functions $r_+(\xi)$ and $r_-(\xi)$ are called the right and left reflection coefficients respectively. They are defined for $\xi\neq 0$ and

$$|r_\pm(\xi)|<1$$

are valid for $\xi\neq 0$. Jost solutions $f_\pm(x;\ \xi)$ are expressed as

(5) $$f_\pm(x;\ \xi)=\exp(\pm i\xi x)\left\{1\pm\int_0^{\pm\infty}B_\pm(x,\ y)\exp(\pm 2i\xi y)dy\right\},$$

where $B_\pm(x, y)$ are real valued. By (5), $f_\pm(x;\ \xi)$ are meaningful even for $\xi=0$. Moreover $f'_\pm(x;\ 0)$ obey the estimates

(6) $$|f'_\pm(x;\ 0)|\leqslant\pm K\int_x^{\pm\infty}|u(y)|dy,\quad 0\leqslant\pm x<\infty$$

respectively, where K is a positive constant (see [2; Lemma 1, p. 130]).

On the other hand, the following integral representations for the scattering data hold:

(7) $$a(\xi)=1-(2i\xi)^{-1}\int_{-\infty}^{\infty}u(x)f_+(x;\ \xi)\exp(-i\xi x)dx$$

and

(8) $$b(\xi)=(2i\xi)^{-1}\int_{-\infty}^{\infty}u(x)f_+(x;\ \xi)\exp(i\xi x)dx.$$

Put

$$\nu=\int_{-\infty}^{\infty}u(x)f(x;\ 0)dx.$$

Then, by (3), (4), (7) and (8), we obtain

$$\nu=-2i\xi a(\xi)|_{\xi=0}=[f_-(x;\ 0), f_+(x;\ 0)]$$

and

$$\nu=2i\xi b(\xi)|_{\xi=0}=[f_+(x;\ 0)^*, f_-(x;\ 0)].$$

The following is shown by P. Deift and E. Trubowitz [2; Theorem 1, p. 147]: *If* $u(x)\in L^1_2$, *then* $r_\pm(\xi)$ *are continuous at* $\xi=0$. *Moreover, if* $\nu=0$, *then the condition*

I. $|r_\pm(\xi)| \leqslant \text{Const.} < 1, \ \xi \neq 0$

holds and if $v \neq 0$, *then the condition*

II. $r_\pm(\xi) = -1 + \alpha_\pm \xi + o(\xi)$, as $\xi \to 0$

holds, where $\alpha_\pm$ *are constants.* In particular, the condition I is valid if and only if $f_+(x; 0)$ and $f_-(x; 0)$ are linearly dependent.

§3. Proof of Theorem.

In this section, we prove Theorem stated in §1. Our proof is devided into several steps.

Suppose that H_u, $u \in L_2^1$ is decomposable, i.e., there exists an absolutely continuous real valued function $v(x) \in L_1^1$ such as

$$u(x) = v'(x) + v(x)^2.$$

Put

$$L = \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix},$$

where $A = d/dx + v(x)$. Then we have

$$L^2 = \begin{bmatrix} H_{u_+} & 0 \\ 0 & H_{u_-} \end{bmatrix},$$

where $u_+(x) = u(x)$ and

$$u_-(x) = -v'(x) + v(x)^2 \in L_2^1,$$

i.e., $H_{u_+} = AA^*$ and $H_{u_-} = A^*A$. Let $f_+^{(\pm)}(x; \xi)$ and $f_-^{(\pm)}(x; \xi)$ be the right and left Jost solutions of $H_{u_\pm}$ respectively. Moreover, let $r_+^{(\pm)}(\xi)$ and $r_-^{(\pm)}(\xi)$ be the right and left reflection coefficients of $H_{u_\pm}$ respectively. Unless explicitly stated otherwise, we adopt the following convention: $r^{(\pm)}(\xi)$ stand for $r_\pm^{(\pm)}(\xi)$ and $r(\xi)$ for $r_\pm(\xi)$.

Then the following is obtained by S. Tanaka [5].

Lemma 1. $r^{(+)}(\xi) = -r^{(-)}(\xi)$.

Proof. We first assume $\xi \neq 0$. Put

$$\phi^{(+)}(x; \xi) = \xi^{-1} A f_+^{(-)}(x; \xi)$$

and

$$\phi^{(-)}(x; \xi) = -\xi^{-1} A^* f_+^{(+)}(x; \xi).$$

Then we have

(9) $$A^*\phi^{(+)}(x;\xi)=\xi f_+^{(-)}(x;\xi)$$

and

(10) $$A\phi^{(-)}(x;\xi)=-\xi f_+^{(+)}(x;\xi).$$

Hence we obtain

$$H_{u_+}\phi^{(+)}(x;\xi)=\xi A f_+^{(-)}(x;\xi)=\xi^2\phi^{(+)}(x;\xi)$$

and

$$H_{u_-}\phi^{(-)}(x;\xi)=-\xi A^* f_+^{(+)}(x;\xi)=\xi^2\phi^{(-)}(x;\xi).$$

Consider the asymptotic behaviour of $\phi^{(\pm)}(x;\xi)$ as $x\to\infty$, then, by the uniqueness of Jost solutions, we have

$$\phi^{(\pm)}(x;\xi)=if_+^{(\pm)}(x;\xi).$$

Hence, from (9) and (10),

$$Af_+^{(-)}(x;\xi)=i\xi f_+^{(+)}(x;\xi)$$

and

$$A^*f_+^{(+)}(x;\xi)=-i\xi f_+^{(-)}(x;\xi)$$

follow. This implies

$$LY_+(x;\xi)=i\xi JY_+(x;\xi),$$

where $Y_+(x;\xi)={}^t(f_+^{(+)}(x;\xi),f_+^{(-)}(x;\xi))$ and $J=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$. Hence we have

$$LY_+(x;\xi)^*=-i\xi JY_+(x;\xi)^*$$

and

$$LY_+(x;\xi)^\#=-i\xi JY_+(x;\xi)^\#,$$

where $Y_+(x;\xi)^\#=JY_+(x;\xi)={}^t(f_+^{(+)}(x;\xi),\ -f_+^{(-)}(x;\xi))$. Similarly to the above, we have

$$LY_-(x;\xi)=-i\xi JY_-(x;\xi),$$

where $Y_-(x;\xi)={}^t(f_-^{(+)}(x;\xi),f_-^{(-)}(x;\xi))$. Thus Y_+^*, $Y_+^\#$ and Y_- solve the differential equation

$$LY=-i\xi JY.$$

We have

$$\det(Y_+(x;\xi)^*,\ Y_+(x;\xi)^\#)=-2,$$

since the left hand side of the above is the Wronskian and $f_{+}^{(\pm)}(x;\xi)$ behave like $\exp(i\xi x)$ as $x\rightarrow\infty$. Hence Y_{+}^{*} and $Y_{+}^{\#}$ are linearly independent and consequently there exist $c_j(\xi)$ $(j=1, 2)$ such that

$$Y_{-}(x;\xi)=c_1(\xi)Y_{+}(x;\xi)^{*}+c_2(\xi)Y_{+}(x;\xi)^{\#}.$$

This implies

$$f_{-}^{(\pm)}(x;\xi)=c_1(\xi)f_{+}^{(\pm)}(x;\xi)^{*}\pm c_2(\xi)f_{+}^{(\pm)}(x;\xi).$$

Thus, we obtain

$$r^{(\pm)}(\xi)=\pm c_2(\xi)/c_1(\xi).$$

This completes the proof. Q. E. D.

Next we have

Lemma 2. *If H_u, $u\in L_2^1$ is decomposable then the reflection coefficient $r(\xi)$ of H_u satisfies the condition I.*

Proof. By the definition, there exists an absolutely continuous real valued function $v(x)\in L_1^1$ such that

$$u(x)=v'(x)+v(x)^2.$$

Put

$$u_{-}(x)=-v'(x)+v(x)^2\in L_2^1$$

and let $r^{(-)}(\xi)$ be the reflection coefficient of $H_{u_{-}}$. Then, by Lemma 1, we have

$$r^{(-)}(\xi)=-r(\xi),$$

where $r(\xi)$ is the reflection coefficient of H_u. Suppose that $r(\xi)$ satisfies the condition II. Then we have

$$r^{(-)}(\xi)=1-\alpha\xi+o(\xi), \quad \text{as} \quad \xi\longrightarrow 0.$$

This cannot occur. Hence $r(\xi)$ satisfies the condition I. Q. E. D.

Moreover we have

Lemma 3. *If H_u, $u\in L_2^1$ has no discrete eigenvalues and the reflection coefficient $r(\xi)$ of H_u satisfies the condition* I *then H_u is decomposable.*

Proof. As mentioned at the end of the preceding section,

$$v=\int_{-\infty}^{\infty}u(x)f_{+}(x;0)dx=[f_{+}(x;0), f_{-}(x;0)]=0$$

holds. Hence we have

$$f'_+(x; 0)/f_+(x; 0) = f'_-(x; 0)/f_-(x; 0).$$

Put

$$v(x) = f'_\pm(x; 0)/f_\pm(x; 0)$$

then, by (5), $v(x)$ is real valued and

$$v'(x) + v(x)^2 = u(x)$$

holds. On the other hand, P. Deift and E. Trubowitz [2; Theorem 3, p. 163] obtained the following: H_u, $u \in L^1_2$ *has no discrete eigenvalues if and only if* $f_\pm(x; 0)$ *do not vanish for any* x. By virtue of this result, $v(x)$ is absolutely continuous. Moreover, by (6), we have

$$|v(x)| \leqslant C|f'_\pm(x; 0)| \leqslant \pm K \int_x^{\pm\infty} |u(x)|dy,$$

since $1/f_\pm(x; 0)$ are bounded. Then we have the following estimates:

$$\int_0^\infty |v(x)|dx \leqslant K \int_0^\infty dx \int_x^\infty |u(y)|dy = K \int_0^\infty y|u(y)|dy$$

and

$$\int_0^\infty x|v(x)|dx \leqslant 2^{-1}K \int_0^\infty y^2|u(y)|dy.$$

Hence we have

$$\int_0^\infty (1+x)\,|v(x)|dx < \infty,$$

since $u(x) \in L^1_2$. Similarly to the above, we have

$$\int_{-\infty}^0 (1+|x|)\,|v(x)|dx < \infty,$$

i.e., $v(x) \in L^1_1$. Thus H_u is decomposable. Q. E. D.

Now we can prove Theorem.

Proof of Theorem. First suppose that H_u is decomposable. Then, by (1), H_u is positive definite. On the other hand, the discrete eigenvalues of H_u, $u \in L^1_2$ are all negative (cf. [2]). Hence H_u has no discrete eigenvalues. Moreover, by Lemma 2, the reflection coefficients $r_\pm(\xi)$ of H_u satisfy the condition I. Furthermore the converse statement is nothing but Lemma 3. This completes the proof. Q. E. D.

Department of Mathematics
College of General Education
Tokushima University

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