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Theory of (Vector Valued) Sato Hyperfunctions. Their Realization as Boundary Values of (Vector Valued) Holomorphic Functions

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Introduction

In Schapira [20] and Ito [8], [9], we defined a (vector valued) Sato hyperfunctions as a residue class of analytic functionals (or analytic linear mappings). In this paper, we will realize these (vector valued) Sato hyperfunctions as "boundary values" of (vector valued) holomorphic functions. Thus we will prove the main theorems (Theorems 1.6.1 and 2.6.1).

The proof of Theorem 1.6.1 goes as follows. At first we will prove the Dolbeault-Grothendieck resolution of the sheaf $\mathcal{O}$ using the sheaves $\mathscr{L}^{p,q}$ of germs of differential forms of type (p, q) with locally square integrable functions as coefficients over C^n. Then, using the above resolution of $\mathcal{O}$, we will prove the Oka-Cartan Theorem B, Malgrange's Theorem and Serre's duality theorem. Then, using these results, we will prove Martineau-Harvey's Theorem and, as a result, Sato's Theorem. Here, in order to prove Martineau-Harvey's Theorem, we need the Dolbeault-Grothendieck resolution of the above type, the Oka-Cartan Theorem B, Malgrange's Theorem and Serre's duality theorem. But, in order to prove Sato's Theorem, we have only to make a direct use of the general theory of cohomology groups and Martineau-Harvey's Theorem. This method of proof is essentially new and simplified in the point that we use the Dolbeault-Grothendieck resolution of the above type. The merit of this method is that the duality argument of complexes are simplified and confined within the classical analysis.

Here we found the following fact. In the proof of Martineau-Harvey's Theorem, we used the long exact sequence of (relative) cohomology groups. But, because of using canonical flabby resolutions to prove this long exact sequence, we can only say that the isomorphisms $H^n_K(\Omega, \mathcal{O}) \cong H^{n-1}(\Omega \backslash K, \mathcal{O}) \cong \mathcal{O}(K)'$ in this theorem are algebraic. But, because Sato's Theorem is independent of the topology, it is sufficient, for this theory, to have these algebraic isomorphisms.

By virtue of Sato's Theorem, we can see that Sato hyperfunctions can be realized as "boundary values" of holomorphic functions.

As for vector valued Sato hyperfunctions, similar assertions can be made.

Therefrom we can see that the mathematical substance named (vector valued) Sato hyperfunctions has the duplexity of having two realizations as dual objects of functions and as "boundary values" of (vector valued) holomorphic functions.

In chapter 1, we will treat the scalar valued case, and in chapter 2, we will treat the vector valued case.

Chapter 1. Case of Sato Hyperfunctions

1.1. The Dolbeault-Grothendieck Resolutions of $\mathcal{O}$

In this section we will recall the soft resolutions of the sheaf $\mathcal{O}$ of germs of holomorphic functions over the n-dimensional complex Euclidean space $\boldsymbol{C}^n$.

If $\mathscr{F}$ is a sheaf over $\boldsymbol{C}^n$, we will define the sheaf $\mathscr{F}^{p,q}$ to be the sheaf of germs of differential forms of type (p, q) with coefficients in $\mathscr{F}$ and denote the Cauchy-Riemann operator by $\bar{\partial}$, where p and q are nonnegative integers. We will denote $\mathscr{F}^p=\mathscr{F}^{p,0}$. Then we have the following

Theorem 1.1.1 (The Dolbeault-Grothendieck resolution). *The sequence of sheaves over* $\boldsymbol{C}^n$

$$0 \longrightarrow \mathcal{O}^p \longrightarrow \mathscr{E}^{p,0} \xrightarrow{\bar{\partial}} \mathscr{E}^{p,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathscr{E}^{p,n} \longrightarrow 0$$

is exact, where $\mathscr{E}$ *denote the soft nuclear Fréchet sheaf of germs of* C^∞*-functions over* $\boldsymbol{C}^n$ *and* p *is a nonnegative integer.*

Proof. Since the assertion is locally, this easily follows from Hörmander [4], Theorem 2.3.3. Q. E. D.

Corollary. *For an open set* Ω *in* $\boldsymbol{C}^n$, *we have the following isomorphism:*

$$H^q(\Omega, \mathcal{O}^p)\cong\{f\in\mathscr{E}^{p,q}(\Omega);\ \bar{\partial}f=0\}/\{\bar{\partial}g;\ g\in\mathscr{E}^{p,q-1}(\Omega)\}\quad (p\geqq 0,\ q\geqq 1).$$

Proof. It follows from Theorem 1.1.1 and Komatsu [14], Theorems II.2.9 and II.2.19. Q. E. D.

Next we will prove one another soft resolution of $\mathcal{O}$.

Let $L=L_{2,\mathrm{loc}}$ be a soft FS* sheaf of germs of locally L_2-functions over $\boldsymbol{C}^n$. Then we define the sheaf $\mathscr{L}^{p,q}=\mathscr{L}^{p,q}_{2,\mathrm{loc}}$ to be the sheafification of the presheaf $\{\mathscr{L}^{p,q}(\Omega);\ \Omega\subset \boldsymbol{C}^n \text{ open}\}$, where, for an open set Ω in $\boldsymbol{C}^n$, the section module $\mathscr{L}^{p,q}(\Omega)$ is the space of all $f\in L^{p,q}(\Omega)=L^{p,q}_{2,\mathrm{loc}}(\Omega)$ such that $\bar{\partial}f\in L^{p,q+1}(\Omega)=L^{p,q+1}_{2,\mathrm{loc}}(\Omega)$. We put $\mathscr{L}=\mathscr{L}^{0,0}$. Then $\mathscr{L}^{p,q}$ is a soft FS* sheaf. Then we have the following

Theorem 1.1.2 (The Dolbeault-Grothendieck resolution). *The sequence of sheaves over* $\boldsymbol{C}^n$

$$0 \longrightarrow \mathcal{O}^p \longrightarrow \mathscr{L}^{p,0} \xrightarrow{\bar\partial} \mathscr{L}^{p,1} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{L}^{p,n} \longrightarrow 0$$

is exact.

Proof. The exactness of the sequence

$$0 \longrightarrow \mathcal{O}^p \longrightarrow \mathscr{L}^{p,0} \xrightarrow{\bar\partial} \mathscr{L}^{p,1}$$

is evident because of the ellipticity of the operator $\bar\partial$.

Next we have to prove the exactness of the sequence

$$\mathscr{L}^{p,0} \xrightarrow{\bar\partial} \mathscr{L}^{p,1} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{L}^{p,n} \longrightarrow 0.$$

We will reason as in Hörmander [4], p. 32. Thus it follows from the following

Lemma. *Let D be an open polydisc in $\boldsymbol{C}^n$ and let $f \in \mathscr{L}^{p,q+1}(D)$ $(p, q \geqq 0)$ satisfy the condition $\bar\partial f = 0$. If D' is a relatively compact open polydisc in D, we can find $u \in \mathscr{L}^{p,q}(D')$ such that $\bar\partial u = f$ in D'.*

This completes the proof. Q. E. D.

Corollary. *For an open set Ω in $\boldsymbol{C}^n$, we have the following isomorphism:*

$$H^q(\Omega, \mathcal{O}^p) \cong \{f \in \mathscr{L}^{p,q}_{2,\mathrm{loc}}(\Omega);\ \bar\partial f = 0\}/\{\bar\partial g;\ g \in \mathscr{L}^{p,q-1}_{2,\mathrm{loc}}(\Omega)\}, \quad (p \geqq 0,\ q \geqq 1).$$

1.2. The Oka-Cartan Theorem B

In this section we will prove the Oka-Cartan Theorem B for the sheaf $\mathcal{O}$ by using the soft resolution of Theorem 1.1.2. Thus we have the following

Theorem 1.2.1 (The Oka-Cartan Theorem B). *For any pseudoconvex open set V in $\boldsymbol{C}^n$, we have $H^s(V, \mathcal{O}^p) = 0$ $(p \geqq 0,\ s \geqq 1)$.*

Proof. This is an immediate consequence of Theorem 1.1.2 and Hörmander [4], Theorem 4.2.2. Q. E. D.

Corollary. *For any pseudoconvex open set Ω in $\boldsymbol{C}^n$, the equation $\bar\partial u = f$ has a solution $u \in \mathscr{L}^{p,q}(\Omega)$ for every $f \in \mathscr{L}^{p,q+1}(\Omega)$ such that $\bar\partial f = 0$. Here p and q are nonnegative integers.*

Proof. This is an immediate consequences of Corollary to Theorems 1.1.1 and 1.2.1. Q. E. D.

Let $\mathscr{A}$ be the sheaf of germs of real analytic functions over $\boldsymbol{R}^n$ defined by $\mathscr{A} = \mathcal{O}|\boldsymbol{R}^n$. Then we have the following

Theorem 1.2.2 (Malgrange). *For any subset Ω of $\boldsymbol{R}^n$, we have $H^s(\Omega, \mathscr{A}^p) = 0$ $(p \geqq 0,\ s \geqq 1)$.*

Proof. See Komatsu [14], Corollary V.2.6 and Schapira [20], Lemma 411. Q. E. D.

1.3. Malgrange's Theorem

In this section we will prove the following

Theorem 1.3.1. *Let Ω be an open set in C^n. Then we have $H^n(\Omega, \mathcal{O})=0$.*

Proof. By virtue of Corollary to Theorem 1.1.2, we have only to prove the exactness of the sequence

$$\mathscr{L}^{0,n-1}(\Omega) \xrightarrow{\bar{\partial}} \mathscr{L}^{0,n}(\Omega) \longrightarrow 0$$

or equivalently that of the sequence

$$L^{0,n-1}(\Omega) \longrightarrow L^{0,n}(\Omega) \longrightarrow 0.$$

This can be proved by a similar method to Komatsu [14]. Q. E. D.

1.4. Serre's duality theorem

In this section we will prove Serre's duality theorem.

Theorem 1.4.1. *Let Ω be an open set in C^n such that* $\dim H^p(\Omega, \mathcal{O})<\infty$ *holds* $(p\geqq 1)$. *Then we have the isomorphism* $[H^p(\Omega, \mathcal{O})]'=H_c^{n-p}(\Omega, \mathcal{O})$ $(0\leqq p\leqq n)$.

Proof. By virtue of Corollary to Theorem 1.1.2, cohomology groups $H^p(\Omega, \mathcal{O})$ and $H_c^{n-p}(\Omega, \mathcal{O})$ are cohomology groups respectively of the complexes

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathscr{L}^{0,0}(\Omega) & \xrightarrow{\bar{\partial}} & \mathscr{L}^{0,1}(\Omega) & \xrightarrow{\bar{\partial}} & \cdots & \xrightarrow{\bar{\partial}} & \mathscr{L}^{0,n}(\Omega) & \longrightarrow & 0 \\ & & \updownarrow & & \updownarrow & & & & \updownarrow & & \\ 0 & \longleftarrow & \mathscr{L}_c^{0,n}(\Omega) & \xleftarrow{-\bar{\partial}} & \mathscr{L}_c^{0,n-1}(\Omega) & \xleftarrow{-\bar{\partial}} & \cdots & \xleftarrow{-\bar{\partial}} & \mathscr{L}_c^{0,0}(\Omega) & \longleftarrow & 0. \end{array}$$

Here the upper complex is composed of FS* spaces and the lower complex is composed of DFS* spaces. Since the ranges of operators $\bar{\partial}$ in the upper complex are all closed by virtue of Schwartz' Lemma (cf. Komatsu [13]), the ranges of operators $-\bar{\partial}=(\bar{\partial})'$ in the lower complex are also all closed. Hence we have the isomorphism

$$[H^p(\Omega, \mathcal{O})]'\cong H_c^{n-p}(\Omega, \mathcal{O})$$

by virtue of Serre's Lemma (cf. Komatsu [13]). Q. E. D.

1.5. Martineau-Harvey's Theorem

In this section we will prove Martineau-Harvey's Theorem.

Theorem 1.5.1. *Let K be a compact set in C^n such that $H^p(K, \mathcal{O})=0$ $(p\geqq 1)$ holds. Then, for an open neighborhood V of K, we have $H_K^p(V, \mathcal{O})=0$ $(p\neq n)$ and algebraic isomorphisms $H_K^n(V, \mathcal{O})\cong H^{n-1}(V\backslash K, \mathcal{O})\cong \mathcal{O}(K)'$.*

Remark. If a compact set K in $\boldsymbol{C}^n$ has a fundamental system of pseudoconvex open neighborhoods, it satisfies the assumptions in Theorem 1.5.1.

Proof. By the excision theorem, $H_K^p(V, \mathcal{O})$ is independent of an open neighborhood V of K. So, we may assume that V is a pseudoconvex open neighborhood. Then in the long exact sequence of cohomology groups (cf. Komatsu [14], Theorem II. 3.2):

$$\begin{aligned} 0 &\longrightarrow H_K^0(V, \mathcal{O}) \longrightarrow H^0(V, \mathcal{O}) \longrightarrow H^0(V\backslash K, \mathcal{O}) \\ &\longrightarrow H_K^1(V, \mathcal{O}) \longrightarrow H^1(V, \mathcal{O}) \longrightarrow H^1(V\backslash K, \mathcal{O}) \\ &\longrightarrow \cdots \\ &\longrightarrow H_K^n(V, \mathcal{O}) \longrightarrow H^n(V, \mathcal{O}) \longrightarrow H^n(V\backslash K, \mathcal{O}) \longrightarrow \cdots, \end{aligned}$$

we have $H^p(V, \mathcal{O})=0$ for $p\geqq 1$ and $H_K^0(V, \mathcal{O})=0$ by the unique continuation theorem. Hence we have isomorphisms

$$\begin{aligned} &H_K^1(V, \mathcal{O}) \cong \mathcal{O}(V\backslash K)/\mathcal{O}(V), \\ &H_K^p(V, \mathcal{O}) \cong H^{p-1}(V\backslash K, \mathcal{O}), \quad p\geqq 2. \end{aligned}$$

We also have the long exact sequence of cohomology groups with compact support (cf. Komatsu [14], Theorem II.3.15):

$$\begin{aligned} 0 &\longrightarrow H_c^0(V\backslash K, \mathcal{O}) \longrightarrow H_c^0(V, \mathcal{O}) \longrightarrow H^0(K, \mathcal{O}) \\ &\longrightarrow H_c^1(V\backslash K, \mathcal{O}) \longrightarrow H_c^1(V, \mathcal{O}) \longrightarrow H^1(K, \mathcal{O}) \\ &\longrightarrow \cdots \\ &\longrightarrow H_c^p(V\backslash K, \mathcal{O}) \longrightarrow H_c^p(V, \mathcal{O}) \longrightarrow H^p(K, \mathcal{O}) \longrightarrow \cdots. \end{aligned}$$

Here $H^p(K, \mathcal{O})=0$ $(p\geqq 1)$ by the assumption on K. We also have $H_c^p(V, \mathcal{O})=0$ by Theorem 1.4.1 $(p\neq n)$. Therefore we obtain the isomorphisms:

$$\begin{aligned} &\mathcal{O}(K) \cong H_c^1(V\backslash K, \mathcal{O}), \\ &H_c^p(V\backslash K, \mathcal{O}) \cong H_c^p(V, \mathcal{O})=0, \quad p\neq 1,\ n, \\ &H_c^n(V\backslash K, \mathcal{O}) \cong H_c^n(V, \mathcal{O})=\mathcal{O}(V)'. \end{aligned}$$

Now we consider the following dual complexes:

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathcal{L}^{0,0}(V\backslash K) & \xrightarrow{\bar\partial_0} & \mathcal{L}^{0,1}(V\backslash K) & \xrightarrow{\bar\partial_1} \cdots \xrightarrow{\bar\partial_{n-2}} & \mathcal{L}^{0,n-1}(V\backslash K) & \xrightarrow{\bar\partial_{n-1}} * \\ & \updownarrow & & \updownarrow & & \updownarrow & \\ 0 \longleftarrow & \mathcal{L}_c^{0,n}(V\backslash K) & \xleftarrow{-\bar\partial_{n-1}} & \mathcal{L}_c^{0,n-1}(V\backslash K) & \xleftarrow{-\bar\partial_{n-2}} \cdots \xleftarrow{-\bar\partial_1} & \mathcal{L}_c^{0,1}(V\backslash K) & \xleftarrow{-\bar\partial_0} ** \end{array}$$

$$\begin{array}{ccc} * \longrightarrow & \mathcal{L}^{0,n}(V\backslash K) & \longrightarrow 0 \\ & \updownarrow & \\ ** \longleftarrow & \mathcal{L}_c^{0,0}(V\backslash K) & \longleftarrow 0. \end{array}$$

Then, since $H_c^p(V\backslash K, \mathcal{O})=0$ $(p\neq 1, n)$, the range of $-\bar{\partial}_j=(\bar{\partial}_{n-j-1})'$ is closed except for $j=0, n-1$. However $\bar{\partial}_{n-1}$ is of closed range by Malgrange's Theorem. Hence, by the closed range theorem, $-\bar{\partial}_0$ is of closed range (cf. Komatsu [13], Theorem 19, p. 381).

In order to prove the closedness of the range of $-\bar{\partial}_{n-1}$, we consider the following diagram:

$$\begin{tikzcd} 0 & \mathscr{L}_c^{0,n}(V\backslash K) \arrow[l] \arrow[d, hook, "i"] & \mathscr{L}_c^{0,n-1}(V\backslash K) \arrow[l, "-\bar{\partial}_{n-1}^{V\backslash K}"'] \arrow[d, hook] \\ 0 & \mathscr{L}_c^{0,n}(V) \arrow[l] & \mathscr{L}_c^{0,n-1}(V), \arrow[l, "-\bar{\partial}_{n-1}^{V}"'] \end{tikzcd}$$

where the map i is the natural injection. However, in the dual complexes for V, $\bar{\partial}_0^V$ is of closed range since $H^1(V, \mathcal{O})=0$. Thus, by the closed range theorem, $\mathrm{Im}\,(-\bar{\partial}_{n-1}^{V\backslash K})=i^{-1}(\mathrm{Im}\,(-\bar{\partial}_{n-1}^V))$ is closed. Hence all $-\bar{\partial}_j^{V\backslash K}$ are of closed range. Hence by the Serre-Komatsu duality theorem, we have the isomorphisms $[H^p(V\backslash K, \mathcal{O})]'\cong H_c^{n-p}(V\backslash K, \mathcal{O})$ for $0\leqq p\leqq n$. Thus we have $\mathcal{O}(V\backslash K)'\cong H_c^n(V\backslash K, \mathcal{O})\cong H_c^n(V, \mathcal{O})\cong \mathcal{O}(V)'$. Here $\mathcal{O}(V\backslash K)$ and $\mathcal{O}(V)$ are both FS spaces, a posteriori, reflexive. Hence we have the isomorphism $\mathcal{O}(V)\cong\mathcal{O}(V\backslash K)$. Thus $H_K^1(V, \mathcal{O})\cong \mathcal{O}(V\backslash K)/\mathcal{O}(V)=0$. Hence, for $p\geqq 2$, $p\neq n$, we have $0=[H_c^{n-p+1}(V, \mathcal{O})]'\cong[H_c^{n-p+1}\cdot(V\backslash K, \mathcal{O})]'\cong H^{p-1}(V\backslash K, \mathcal{O})\cong H_K^p(V, \mathcal{O})$. Thus $H_K^p(V, \mathcal{O})=0$. In the case $p=n$, we have the isomorphisms $H_K^n(V, \mathcal{O})\cong H^{n-1}(V\backslash K, \mathcal{O})\cong[H_c^1(V\backslash K, \mathcal{O}])'\cong[H^0(K, \mathcal{O})]'\cong\mathcal{O}(K)'$. Here all the isomorphisms except for the first are topological and the first one is only algebraic. Hence we have the algebraic isomorphism $H_K^n(V, \mathcal{O})\cong \mathcal{O}(K)'$. Here we use the topological isomorphisms $H_c^1(V\backslash K, \mathcal{O})\cong H^0(K, \mathcal{O})$ and $H_c^p(V, \mathcal{O})\cong H_c^p(V\backslash K, \mathcal{O})$ $(p\geqq 2)$. These isomorphisms can be proved by a similar method to Nagamachi [16], Proposition 5.4. Q. E. D.

1.6. Sato's Theorem

In this section we will prove the pure-codimensionality of $\boldsymbol{R}^n$ with respect to $\mathcal{O}$. Then we will realize Sato hyperfunctions as "boundary values" of holomorphic functions or as (relative) cohomology classes of holomorphic functions.

Theorem 1.6.1 (Sato's Theorem). *Let Ω be an open set in $\boldsymbol{R}^n$ and V an open set in $\boldsymbol{C}^n$ which contains Ω as its closed subset. Then we have the following*

(1) *$\boldsymbol{R}^n$ is purely n-codimensional with respect to $\mathcal{O}$.*

(2) *The presheaf over $\boldsymbol{R}^n$, $\Omega\to H_\Omega^n(V, \mathcal{O})$, is a sheaf.*

(3) *This sheaf (2) is isomorphic to the sheaf $\mathscr{B}$ of Sato hyperfunctions.*

Proof. (1) We have to prove the vanishing of the derived sheaf $\mathscr{H}_{\boldsymbol{R}^n}^p(\mathcal{O})$ for $p\neq n$. This is local in nature. Thus, it is sufficient to prove $H_\Omega^p(V, \mathcal{O})=0$ $(p\neq n)$ for a relatively compact open set Ω in $\boldsymbol{R}^n$. But this can be shown by using Martineau-Harvey's Theorem.

(2) By (1) and Komatsu [14], Theorem II.3.18, we have the conclusion.

(3) We have only to prove this isomorphism stalkswise. This is local in nature. Consider the following exact sequence of relative cohomology groups for a relatively compact open set Ω in $\boldsymbol{R}^n$

$$\begin{aligned} 0 &\longrightarrow H^0_{\partial\Omega}(V, \mathcal{O}) \longrightarrow H^0_{\Omega^a}(V, \mathcal{O}) \longrightarrow H^0_{\Omega}(V, \mathcal{O}) \\ &\longrightarrow H^1_{\partial\Omega}(V, \mathcal{O}) \longrightarrow \cdots \longrightarrow H^{n-1}_{\Omega}(V, \mathcal{O}) \\ &\longrightarrow H^n_{\partial\Omega}(V, \mathcal{O}) \longrightarrow H^n_{\Omega^a}(V, \mathcal{O}) \longrightarrow H^n_{\Omega}(V, \mathcal{O}) \\ &\longrightarrow H^{n+1}_{\partial\Omega}(V, \mathcal{O}) \longrightarrow \cdots. \end{aligned}$$

(Here Ω^a denotes the closure of Ω). Then, by (1) and by Martineau-Harvey's Theorem, we have $H^{n-1}_{\Omega}(V, \mathcal{O})=0$ and $H^{n+1}_{\partial\Omega}(V, \mathcal{O})=0$. Thus we have the exact sequence

$$0 \longrightarrow H^n_{\partial\Omega}(V, \mathcal{O}) \longrightarrow H^n_{\Omega^a}(V, \mathcal{O}) \longrightarrow H^n_{\Omega}(V, \mathcal{O}) \longrightarrow 0.$$

Since, by Martineau-Harvey's Theorem, we have isomorphisms

$$H^n_{\partial\Omega}(V, \mathcal{O}) \cong \mathscr{A}(\partial\Omega)', \quad H^n_{\Omega^a}(V, \mathcal{O}) \cong \mathscr{A}(\Omega^a)',$$

we obtain the isomorphism

$$H^n_{\Omega}(V, \mathcal{O}) \cong \mathscr{A}(\Omega^a)'/\mathscr{A}(\partial\Omega)' = \mathscr{B}(\Omega).$$

Thus the sheaf $\Omega \to H^n_{\Omega}(V, \mathcal{O})$ is isomorphic to the sheaf $\mathscr{B}$ of Sato hyperfunctions over $\boldsymbol{R}^n$ (cf. Ito [9]). Q.E.D.

Let Ω be an open set in $\boldsymbol{R}^n$. Then there exists a pseudoconvex open neighborhood V of Ω such that $V \cap \boldsymbol{R}^n = \Omega$ (cf. Grauert [2]). We put $V_0 = V$ and $V_j = \{z \in V;\ \operatorname{Im} z_j \neq 0\}$, $j=1, 2,\dots, n$. Then $\mathfrak{U}=\{V_0, V_1,\dots, V_n\}$ and $\mathfrak{U}'=\{V_1,\dots, V_n\}$ cover V and $V\backslash\Omega$ respectively. Since V_j and their intersections are also pseudoconvex open sets, the covering $(\mathfrak{U}, \mathfrak{U}')$ satisfies the conditions of Leray's Theorem (cf. Komatsu [14]). Thus, by Leray's Theorem, we obtain the isomorphism $H^n_{\Omega}(V, \mathcal{O}) = H^n(\mathfrak{U}, \mathfrak{U}', \mathcal{O})$. Since the covering $\mathfrak{U}$ is composed of only $n+1$ open sets V_j $(j=0, 1,\dots, n)$, we easily obtain the isomorphisms

$$Z^n(\mathfrak{U}, \mathfrak{U}', \mathcal{O}) \cong \mathcal{O}(\bigcap_j V_j),$$

$$C^{n-1}(\mathfrak{U}, \mathfrak{U}', \mathcal{O}) \cong \bigoplus_{j=1}^{n} \mathcal{O}(\bigcap_{i \neq j} V_i).$$

Hence we have

$$\delta C^{n-1}(\mathfrak{U}, \mathfrak{U}', \mathcal{O}) \cong \sum_{j=1}^{n} \mathcal{O}(\bigcap_{i \neq j} V_i) \mid V_1 \cap \cdots \cap V_n.$$

Thus we have the isomorphisms

$$\begin{aligned} H^n_\Omega(V, \mathcal{O}) &\cong H^n(\mathfrak{U}, \mathfrak{U}', \mathcal{O}) \\ &\cong Z^n(\mathfrak{U}, \mathfrak{U}', \mathcal{O})/\delta C^{n-1}(\mathfrak{U}, \mathfrak{U}', \mathcal{O}) \\ &\cong \mathcal{O}(\bigcap_j V_j)/\sum_{j=1}^n \mathcal{O}(\bigcap_{i\neq j} V_i). \end{aligned}$$

Thus we have the following

Theorem 1.6.2. *We use notations as above. Then we have the isomorphisms*

$$H^n_\Omega(V, \mathcal{O}) \cong H^n(\mathfrak{U}, \mathfrak{U}', \mathcal{O}) \cong \mathcal{O}(\bigcap_j V_j) \setminus \sum_{j=1}^n \mathcal{O}(\bigcap_{i\neq j} V_i).$$

Chapter 2. Case of Vector Valued Sato Hyperfunctions

2.1. The Dolbeault-Grothendieck Resolution of ${}^E\mathcal{O}$

In this chapter we will recall Ion-Kawai's Theory of vector valued Sato hyperfunctions for unification. But, at some points of view, our theory is different from Ion-Kawai's one. Namely, our vector valued hyperfunctions are defined first as residue classes of analytic linear mappings and then realized as "boundary values" of vector valued holomorphic functions, while vector valued hyperfunctions in the sense of Ion-Kawai are nothing else but "boundary values" of vector valued holomorphic functions.

In this section, E denotes a quasi-complete locally convex topological vector space (LCTVS) (always assumed to be Hausdorff) and $\mathcal{T} = \mathcal{T}_E$ denotes the family of continuous seminorms of E defining a locally convex topology on E.

We define the sheaf ${}^E\mathcal{O}$ of germs of E-valued holomorphic functions over $\boldsymbol{C}^n$ by the sheafification of the presheaf $\{\mathcal{O}(\Omega; E)\}$, where, for an open set Ω in $\boldsymbol{C}^n$, the section module $\mathcal{O}(\Omega; E)$ is the space of all E-valued holomorphic functions on Ω.

We also define the sheaf ${}^E\mathcal{E}$ of germs of E-valued C^∞-functions over $\boldsymbol{C}^n$ by the sheafification of the presheaf $\{\mathcal{E}(\Omega; E)\}$, where, for an open set Ω in $\boldsymbol{C}^n$, the section module $\mathcal{E}(\Omega; E)$ is the space of all E-valued C^∞-functions on Ω.

Then we have the following

Proposition 2.1.1. *The sheaf ${}^E\mathcal{E}$ is soft.*

Proof. Since ${}^E\mathcal{E}$ is obviously an $\mathcal{E}$-module and $\mathcal{E} = {}^C\mathcal{E}$ is a soft sheaf, we have the conclusion by virtue of Bredon [1], Chapter II, Theorem 9.12, p. 50. Q. E. D.

Then we have the following

Theorem 2.1.2 (The Dolbeault-Grothendieck resolution of ${}^E\mathcal{O}^p$). *Let E be a quasi-complete* LCTVS. *Then the sequence of sheaves $\boldsymbol{C}^n$*

$$0 \longrightarrow {}^E\mathcal{O}^p \longrightarrow {}^E\mathcal{E}^{p,0} \xrightarrow{\bar\partial} {}^E\mathcal{E}^{p,1} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} {}^E\mathcal{E}^{p,n} \longrightarrow 0$$

is exact.

Proof. The exactness of the sequence

$$0 \longrightarrow {}^E\mathcal{O}^p \longrightarrow {}^E\mathcal{E}^{p,0} \xrightarrow{\bar\partial} {}^E\mathcal{E}^{p,1}$$

is evident.

Next we have to prove the exactness of the sequence

$${}^E\mathcal{E}^{p,0} \xrightarrow{\bar\partial} {}^E\mathcal{E}^{p,1} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} {}^E\mathcal{E}^{p,n} \longrightarrow 0.$$

We will reason as in Hörmander [4], p. 32. Thus it follows from the following

Lemma. *Let D be an open polydisc in $\boldsymbol{C}^n$ and let $f \in \mathcal{E}^{p,q+1}(D; E)$ $(p, q \geqq 0)$ satisfy the condition $\bar\partial f = 0$. If D' is a relatively compact open polydisc in D, we can find $u \in \mathcal{E}^{p,q}(D'; E)$ such that $\bar\partial u = f$ in D'.*

This completes the proof. Q. E. D.

Corollary. *For an open set Ω in $\boldsymbol{C}^n$, we have the following isomorphism*:

$$H^q(\Omega, {}^E\mathcal{O}^p) \cong \{f \in \mathcal{E}^{p,q}(\Omega; E);\ \bar\partial f = 0\}/\{\bar\partial g;\ g \in \mathcal{E}^{p,q-1}(\Omega; E)\}, \quad (p \geqq 0,\ q \geqq 1).$$

Proof. It follows from Theorem 2.1.2 and Komatsu [14], Theorems II.2.9 and II.2.19. Q. E. D.

2.2. The Oka-Cartan Theorem B

We will recall the Oka-Cartan Theorem B for the sheaf ${}^E\mathcal{O}$. In the sequel of this chapter, we will always assume that E is a Fréchet space.

Theorem 2.2.1 (The Oka-Cartan Theorem B). *Let E be a Fréchet space. For any pseudoconvex open set Ω in $\boldsymbol{C}^n$, we have $H^q(\Omega, {}^E\mathcal{O}^p) = 0$ for $p \geqq 0$ and $q \geqq 1$.*

Proof. By virtue of the Oka-Cartan Theorem B for the sheaf $\mathcal{O}$, we have

$$H^q(\Omega, \mathcal{O}^p) = 0, \quad p \geqq 0 \quad \text{and} \quad q \geqq 1.$$

Thus the complex obtained from Theorem 1.1.1

$$\mathcal{E}^{p,0}(\Omega) \xrightarrow{\bar\partial} \mathcal{E}^{p,1}(\Omega) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathcal{E}^{p,n}(\Omega) \longrightarrow 0$$

is exact. Since $\mathcal{E}^{p,q}(\Omega)$'s are nuclear Fréchet spaces and E is a Fréchet space, the complex

$$\mathcal{E}^{p,0}(\Omega; E) \xrightarrow{\bar\partial} \mathcal{E}^{p,1}(\Omega; E) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathcal{E}^{p,n}(\Omega; E) \longrightarrow 0$$

is also exact by virtue of the isomorphism

$$\mathscr{E}^{p,q}(\Omega;\, E) \cong \mathscr{E}^{p,q}(\Omega)\hat{\otimes} E$$

and Ion and Kawai [5], Theorem 1.10, p. 9. Hence we obtain

$$H^q(\Omega,\, {}^E\mathcal{O}^p)=0 \quad (p \geqq 0 \text{ and } q \geqq 1).$$

This completes the proof. Q. E. D.

Corollary. *We use notations in Theorem 2.2.1. Then the equation $\bar{\partial}u=f$ has a solution $u \in \mathscr{E}^{p,q}(\Omega;\, E)$ for every $f \in \mathscr{E}^{p,q+1}(\Omega;\, E)$ such that $\bar{\partial}f=0$. Here p and q are nonnegative integers.*

Proof. It follows from Theorem 2.2.1 and Corollary to Theorem 2.1.2. Q. E. D.

2.3. Malgrange Theorem

In this section we will prove the Malgrange Theorem.

Theorem 2.3.1. *Let Ω be an open set in $\boldsymbol{C}^n$. Then we have $H^n(\Omega,\, {}^E\mathcal{O})=0$.*

Proof. By virtue of Theorems 1.1.1 and 1.3.1, we have an exact sequence

$$\mathscr{E}^{0,n-1}(\Omega) \xrightarrow{\bar{\partial}} \mathscr{E}^{0,n}(\Omega) \longrightarrow 0.$$

Thus, by Trèves [21], Proposition 43.9, we have the exact sequence

$$\mathscr{E}^{0,n-1}(\Omega)\hat{\otimes} E \xrightarrow{\bar{\partial}} \mathscr{E}^{0,n}(\Omega)\hat{\otimes} E \longrightarrow 0$$

or

$$\mathscr{E}^{0,n-1}(\Omega;\, E) \xrightarrow{\bar{\partial}} \mathscr{E}^{0,n}(\Omega;\, E) \longrightarrow 0.$$

Hence we obtain the conclusion. Q. E. D.

Corollary. Flabby dim ${}^E\mathcal{O} \leqq n$.

2.4. Serre Duality Theorem

In this section we will prove the Serre Duality Theorem.

Theorem 2.4.1. *Let Ω be an open set in $\boldsymbol{C}^n$ such that* dim $H^p(\Omega,\, \mathcal{O})<\infty$ *holds* $(p \geqq 1)$. *Then we have the isomorphism $H^p(\Omega,\, {}^E\mathcal{O}) \cong L(H_c^{n-p}(\Omega,\, \mathcal{O});\, E)$, $0 \leqq p \leqq n$.*

Proof. Since we can easily obtain the isomorphism $H^p(\Omega,\, {}^E\mathcal{O}) \cong H^p(\Omega,\, \mathcal{O})\hat{\otimes}_\pi E$, we have the following isomorphisms by Theorem 1.4.1,

$$\begin{aligned} H^p(\Omega,\, {}^E\mathcal{O}) &\cong H^p(\Omega,\, \mathcal{O})\hat{\otimes}_\pi E \cong [H_c^{n-p}(\Omega,\, \mathcal{O})]'\hat{\otimes}_\pi E \\ &\cong L(H_c^{n-p}(\Omega,\, \mathcal{O});\, E). \end{aligned}$$

Q. E. D.

2.5. Martineau-Harvey Theorem

In this section we will prove the Martineau-Harvey Theorem.

Theorem 2.5.1. *Let K be a compact set in $\boldsymbol{C}^n$ such that $H^p(K, \mathcal{O})=0$ $(p \geqq 1)$ holds. Then we have $H^p_K(\Omega, {}^E\mathcal{O})=0$ for $p \neq n$ and algebraic isomorphisms $H^n_K(\Omega, {}^E\mathcal{O}) \cong H^{n-1}(\Omega \backslash K, {}^E\mathcal{O}) \cong L(\mathcal{O}(K); E)$.*

Proof. We can assume that Ω is a pseudoconvex open neighborhood of K. Then, in the long exact sequence of cohomology groups (cf. Komatsu [14], Theorem II.3.2):

$$
\begin{aligned}
0 &\longrightarrow H^0_K(\Omega, {}^E\mathcal{O}) \longrightarrow H^0(\Omega, {}^E\mathcal{O}) \longrightarrow H^0(\Omega \backslash K, {}^E\mathcal{O}) \\
&\longrightarrow H^1_K(\Omega, {}^E\mathcal{O}) \longrightarrow H^1(\Omega, {}^E\mathcal{O}) \longrightarrow H^1(\Omega \backslash K, {}^E\mathcal{O}) \\
&\longrightarrow \cdots \\
&\longrightarrow H^n_K(\Omega, {}^E\mathcal{O}) \longrightarrow H^n(\Omega, {}^E\mathcal{O}) \longrightarrow H^n(\Omega \backslash K, {}^E\mathcal{O}) \longrightarrow \cdots,
\end{aligned}
$$

we have $H^p(\Omega, {}^E\mathcal{O})=0$ for $p \geqq 1$ and $H^0_K(\Omega, {}^E\mathcal{O})=0$ by the unique continuation theorem. Hence we have isomorphisms

$$
\begin{aligned}
&H^1_K(\Omega, {}^E\mathcal{O}) \cong \mathcal{O}(\Omega \backslash K; E)/\mathcal{O}(\Omega; E), \\
&H^p_K(\Omega, {}^E\mathcal{O}) \cong H^{p-1}(\Omega \backslash K, {}^E\mathcal{O}), \quad p \geqq 2.
\end{aligned}
$$

Since we have isomorphisms $H^p(V, {}^E\mathcal{O}) = H^p(V, \mathcal{O}) \hat{\otimes}_\pi E$, $0 \leqq p \leqq n$, for an open set V in $\boldsymbol{C}^n$ and isomorphisms $\mathcal{O}(\Omega \backslash K; E) \cong \mathcal{O}(\Omega \backslash K) \hat{\otimes} E \cong \mathcal{O}(\Omega) \hat{\otimes} E \cong \mathcal{O}(\Omega; E)$, we have $H^1_K(\Omega, {}^E\mathcal{O})=0$ and we have, for $p \geqq 2$, $p \neq n$, by Theorem 1.5.1,

$$
H^p_K(\Omega, {}^E\mathcal{O}) \cong H^{p-1}(\Omega \backslash K, {}^E\mathcal{O}) \cong H^{p-1}(\Omega \backslash K, \mathcal{O}) \hat{\otimes}_\pi E = 0.
$$

Then, by a similar method to Nagamachi [16], Proposition 5.4, we have the topological isomorphism $H^1_c(\Omega \backslash K, \mathcal{O}) \cong \mathcal{O}(K)$. Thus, by virtue of Theorem 2.4.1, we have algebraic isomorphisms

$$
H^n_K(\Omega, {}^E\mathcal{O}) \cong H^{n-1}(\Omega \backslash K, {}^E\mathcal{O}) \cong L(H^1_c(\Omega \backslash K); E) \cong L(\mathcal{O}(K); E). \qquad \text{Q.E.D.}
$$

2.6. Sato Theorem

In this section we will prove the pure-codimensionality of $\boldsymbol{R}^n$ with respect to ${}^E\mathcal{O}$. Then we will realize E-valued Sato hyperfunctions as "boundary values" of E-valued holomorphic functions or as (relative) cohomology classes of E-valued holomorphic functions.

Theorem 2.6.1 (Sato Theorem). *Let Ω be an open set in $\boldsymbol{R}^n$ and V an open set in $\boldsymbol{C}^n$ which contains Ω as its closed subset. Then we have the following*

(1) *$\boldsymbol{R}^n$ is purely n-codimensional with respect to ${}^E\mathcal{O}$.*

(2) *The presheaf over $\boldsymbol{R}^n$, $\Omega \to H^n_\Omega(V, {}^E\mathcal{O})$, is a sheaf.*

(3) *This sheaf (2) is isomorphic to the sheaf ${}^E\mathcal{B}$ of E-valued Sato hyperfunctions.*

Proof. (1) We have to prove the vanishing of the derived sheaf $\mathcal{H}^p_{\boldsymbol{R}^n}({}^E\mathcal{O})$ for $p \neq n$. This is local in nature. Thus, it is sufficient to prove $H^p_\Omega(V, {}^E\mathcal{O})=0$ $(p \neq n)$ for a relatively compact open set Ω in $\boldsymbol{R}^n$. Thus, by the excision theorem, we may assume that V is a pseudoconvex open set in $\boldsymbol{C}^n$. Consider the following exact sequence of relative cohomology groups

$$\begin{aligned} 0 &\longrightarrow H^0_{\partial\Omega}(V, {}^E\mathcal{O}) \longrightarrow H^0_{\Omega^a}(V, {}^E\mathcal{O}) \longrightarrow H^0_\Omega(V, {}^E\mathcal{O}) \\ &\longrightarrow H^1_{\partial\Omega}(V, {}^E\mathcal{O}) \longrightarrow \cdots \longrightarrow H^{n-1}_\Omega(V, {}^E\mathcal{O}) \\ &\longrightarrow H^n_{\partial\Omega}(V, {}^E\mathcal{O}) \longrightarrow H^n_{\Omega^a}(V, {}^E\mathcal{O}) \longrightarrow H^n_\Omega(V, {}^E\mathcal{O}) \\ &\longrightarrow H^{n+1}_{\partial\Omega}(V, {}^E\mathcal{O}) \longrightarrow \cdots. \end{aligned}$$

By Theorems 1.2.2 and 2.5.1, we may conclude that $H^p_{\partial\Omega}(V, {}^E\mathcal{O})=H^p_{\Omega^a}(V, {}^E\mathcal{O})=0$ for $p \neq n$. So that, we have $H^p_\Omega(V, {}^E\mathcal{O})=0$ for $p \neq n-1$, n. On the other hand, by Theorems 1.2.2 and 2.5.1, we also have the exact sequence

$$0 \longrightarrow H^{n-1}_\Omega(V, {}^E\mathcal{O}) \longrightarrow L(\mathcal{A}(\partial\Omega); E) \xrightarrow{j} L(\mathcal{A}(\Omega^a); E).$$

Since j is injective, we have $H^{n-1}_\Omega(V, {}^E\mathcal{O})=0$.

(2) By (1) and Komatsu [14], Theorem II.3.18, we have the conclusion.

(3) We have only to prove this isomorphism stalkswise. This is local in nature. By the proof of (1), we have the exact sequence for a relatively compact open set Ω in $\boldsymbol{R}^n$

$$0 \longrightarrow H^n_{\partial\Omega}(V, {}^E\mathcal{O}) \longrightarrow H^n_{\Omega^a}(V, {}^E\mathcal{O}) \longrightarrow H^n_\Omega(V, {}^E\mathcal{O}) \longrightarrow 0.$$

Since, by the Martineau-Harvey Theorem, we have isomorphisms

$$H^n_{\partial\Omega}(V, {}^E\mathcal{O}) \cong L(\mathcal{A}(\partial\Omega); E),$$

$$H^n_{\Omega^a}(V, {}^E\mathcal{O}) \cong L(\mathcal{A}(\Omega^a); E),$$

we obtain the isomorphism

$$H^n_\Omega(V, {}^E\mathcal{O}) \cong L(\mathcal{A}(\Omega^a); E)/L(\mathcal{A}(\partial\Omega); E) = \mathcal{B}(\Omega; E).$$

Thus the sheaf $\Omega \to H^n_\Omega(V, {}^E\mathcal{O})$ is isomorphic to the sheaf ${}^E\mathcal{B}$ of E-valued Sato hyperfunctions over $\boldsymbol{R}^n$. Q.E.D.

In the same notations as in Theorem 1.6.2, we have the following

Theorem 2.6.2. *$H^n_\Omega(V, {}^E\mathcal{O}) \cong H^n(\mathfrak{U}, \mathfrak{U}', {}^E\mathcal{O}) \cong \mathcal{O}(\bigcap_j V_j; E)/\sum_{j=1}^n \mathcal{O}(\bigcap_{i \neq j} V_i; E)$ hold.*

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