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An Analysis of Bifurcation Points of Nonlinear Equations Satisfying a Condition

By

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We consider bifurcation points of a parameter-dependent nonlinear equation $F(x, B)=0$ whose left member $F(x, B)$ satisfies the condition $F(Sx, B)=SF(x, B)$ for a matrix S which has eigenvalues ± 1. If the x-component $\hat{x}$ of a bifurcation point $(\hat{x}, \hat{B})$ is an eigenvector corresponding to the eigenvalue 1 (or -1) of the matrix S, then we can compute $(\hat{x}, \hat{B})$ with high accuracy in a way using an augmented system of nonlinear equations which contains the equation $F(x, B)=0$. Moreover we also give a necessary and sufficient condition for guaranteeing the isolatedness of such a bifurcation point.

§1. Introduction

We consider a bifurcation point $(\hat{x}, \hat{B})$ of a parameter-dependent nonlinear equation

$$F(x, B) = 0 \tag{1.1}$$

whose left member satisfies a condition, where $B \in R$ is a parameter, $x, F(x, B) \in R^n$, and F is a C^{k+2} mapping from R^{n+1} to R^n. Here we call a point $(\hat{x}, \hat{B})$ satisfying the equation (1.1) a "bifurcation point of the equation (1.1)" if the conditions

$$\operatorname{rank} F_x(\hat{x}, \hat{B}) = \operatorname{rank}(F_x(\hat{x}, \hat{B}), F_B(\hat{x}, \hat{B})) = n-1 \tag{1.2}$$

are satisfied, where $F_x(x, B)$ denotes the Jacobian matrix of $F(x, B)$ with respect to x and $F_B(x, B)$ denotes the partial derivative of $F(x, B)$ with respect to B.

In this paper, we consider the case where the mapping F satisfies the condition

$$F(Sx, B) = SF(x, B) \qquad \text{for} \quad x \in R^n, \quad B \in R, \tag{1.3}$$

where S is a real $n \times n$ nonsingular matrix such that

$$\begin{cases} S \neq E_n \ (n \times n \text{ unit matrix}), \text{ and either all the eigenvalues of } S \\ \text{are equal to } \pm 1 \text{ or all the real eigenvalues of } S \text{ are equal to} \\ \pm 1 \text{ and the remaining eigenvalues are all imaginary numbers.} \end{cases} \tag{1.4}$$

For this matrix S we set

$$X_1=\{x\in R^n;\ Sx=x\}=\{x\in R^n;\ S^{-1}x=x\} \tag{1.5}$$

and

$$X_{-1}=\{x\in R^n;\ Sx=-x\}=\{x\in R^n;\ S^{-1}x=-x\}. \tag{1.6}$$

In the paper [4], B. Werner and A. Spence have shown that it is sufficient to consider only the equation

$$\tilde{G}(\boldsymbol{x})=\begin{pmatrix} F(x,B) \\ F_x(x,B)h \\ h^Th-1 \end{pmatrix}=0 \qquad (\text{where } \boldsymbol{x}=(x,h,B)^T) \tag{1.7}$$

in order to obtain $\hat{\boldsymbol{x}}=(\hat{x},\hat{h},\hat{B})^T$ (where $(\hat{x},\hat{B})$ is a bifurcation point of the equation (1.1)) when $S^2=E_n$ and $(\hat{x},\hat{h})^T\in X_1\times X_{-1}$ because the mapping $\tilde{G}$ defined by the equality (1.7) is a mapping from $M=X_1\times X_{-1}\times R$ to M and the mapping $\tilde{G}'(\hat{\boldsymbol{x}})$ is an isomorphism from M to M if a specific condition is satisfied, where h^T and $(\cdots)^T$ denote the transposed vectors of h and a vector $(\cdots)$, respectively, and $\hat{h}(\hat{h}^T\hat{h}-1=0)$ is an eigenvector corresponding to the eigenvalue zero of the matrix $F_x(\hat{x},\hat{B})$, and $\tilde{G}'(\boldsymbol{x})$ denotes the Jacobian matrix of $\tilde{G}(\boldsymbol{x})$ with respect to $\boldsymbol{x}$. But, if the specific condition above is not satisfied, then $\tilde{G}'(\hat{\boldsymbol{x}})$ is not an isomorphism from M to M. Hence it seems that they can not compute the bifurcation point $(\hat{x},\hat{B})$ with high accuracy.

In this paper, on the other hand, we show that, in such a case, we can compute the bifurcation point $(\hat{x},\hat{B})$ with high accuracy if we introduce another parameter into the equation (1.7). Moreover we discuss the case $\hat{x}\in X_{-1}$. Concerning this case, in addition to the above-mentioned condition (1.3), we must assume some additional conditions. But these conditions seem to be reasonable for such a case. Then, for this case, we have results similar to those obtained in the case $\hat{x}\in X_1$.

In §2 we discuss the case $\hat{x}\in X_1$ and in §3 the case $\hat{x}\in X_{-1}$. In §4, in order to illustrate our theory and method, we present an example.

§2. The Case $\hat{x}\in X_1$

First we define $n\times n$ matrices $Y^{(p)}$'s and n-dimensional vectors $V^{(q)}$'s by

$$\begin{cases} Y^{(1)}=F_x(x,B), \\ Y^{(2j)}=\sum_{i=1}^{j} {}_{j-1}C_{i-1}Y_x^{(2j-2i+1)}h_{2i-1}, \\ Y^{(2j+1)}=\sum_{i=1}^{j} {}_{j-1}C_{i-1}Y_x^{(2j-2i+1)}h_{2i}+Y_B^{(2j-1)} \end{cases} \qquad (j\geq 1) \tag{2.1}$$

and

$$(2.2)\qquad \begin{cases} V^{(1)}=F_B(x, B), \\ V^{(2j)}=\sum_{i=1}^{j} {}_{j-1}C_{i-1}V_x^{(2j-2i+1)}h_{2i-1}, & (j\geq 1) \\ V^{(2j+1)}=\sum_{i=1}^{j} {}_{j-1}C_{i-1}V_x^{(2j-2i+1)}h_{2i}+V_B^{(2j-1)} \end{cases}$$

respectively, where each ${}_jC_i$ denotes the binomial coefficient, and $Y_x^{(p)}$ and $V_x^{(q)}$ denote the derivatives of $Y^{(p)}$ and $V^{(q)}$ with respect to x, respectively, and $Y_B^{(p)}$ and $V_B^{(q)}$ denote the partial derivatives of $Y^{(p)}$ and $V^{(q)}$ with respect to B, respectively, and each h_i is an arbitrary n-dimensional vector. From (2.1) and (2.2) we have the following lemma.

Lemma 1.

$$(2.3)\quad \text{(i)}\quad V_x^{(m)}=Y_B^{(m)}\quad (m\geq 1).$$

$$(2.4)\quad \text{(ii)}\quad \sum_{j=1}^{m+1} {}_mC_{j-1}Y^{(2m+3-2j)}h_{2j-1} = \sum_{j=1}^{m} {}_{m-1}C_{j-1}(Y^{(2m+2-2j)}h_{2j}+Y^{(2m+1-2j)}h_{2j+1})+V^{(2m)}\qquad (m\geq 1).$$

PROOF. (i) From (2.1) and (2.2) we easily get

$$(2.5)\qquad V_x^{(1)}=F_{Bx}(x, B)=F_{xB}(x, B)=Y_B^{(1)}$$

and

$$(2.6)\qquad \begin{cases} V_x^{(2)}=V_{xx}^{(1)}h_1=F_{Bxx}(x, B)h_1=F_{xxB}(x, B)h_1=Y_B^{(2)}, \\ V_x^{(3)}=V_{xx}^{(1)}h_2+V_{Bx}^{(1)}=F_{Bxx}(x, B)h_2+F_{BBx}(x, B) \\ \qquad =F_{xxB}(x, B)h_2+F_{xBB}(x, B)=Y_B^{(3)}. \end{cases}$$

Assume that the equality (2.3) holds up to $2l-1$, that is,

$$(2.7)\qquad V_x^{(j)}=Y_B^{(j)}\quad (j=1, 2, 3,\dots, 2l-1).$$

It follows from (2.7) that

$$(2.8)\qquad \begin{cases} V_x^{(2l)}=\sum_{i=1}^{l} {}_{l-1}C_{i-1}V_{xx}^{(2l+1-2i)}h_{2i-1} \\ \qquad =\sum_{i=1}^{l} {}_{l-1}C_{i-1}Y_{xB}^{(2l+1-2i)}h_{2i-1}=Y_B^{(2l)}, \\ V_x^{(2l+1)}=\sum_{i=1}^{l} {}_{l-1}C_{i-1}V_{xx}^{(2l+1-2i)}h_{2i}+V_{Bx}^{(2l-1)} \\ \qquad =\sum_{i=1}^{l} {}_{l-1}C_{i-1}Y_{xB}^{(2l+1-2i)}h_{2i}+Y_{BB}^{(2l-1)}=Y_B^{(2l+1)}, \end{cases}$$

These imply that the equality (2.3) holds for $2l$ and $2l+1$. Hence the equality (2.3) holds for all $m \geq 1$.

(ii) $$A_1 = \sum_{j=1}^{m} {}_{m-1}C_{j-1} Y^{(2m+2-2j)} h_{2j}$$
$$= \sum_{j=1}^{m} {}_{m-1}C_{j-1} \Big(\sum_{i=1}^{m-j+1} {}_{m-j}C_{i-1} Y_x^{(2(m-j+1)-2i+1)} h_{2i-1} \Big) h_{2j}$$
$$= \sum_{i=1}^{m} \Big(\sum_{j=1}^{m-i+1} {}_{m-1}C_{j-1} \cdot {}_{m-j}C_{i-1} Y_x^{(2(m-i+1)-2j+1)} h_{2j} \Big) h_{2i-1}$$
$$= \sum_{i=1}^{m} {}_{m-1}C_{i-1} \Big(\sum_{j=1}^{m-i+1} {}_{m-i}C_{j-1} Y_x^{(2(m-i+1)-2j+1)} h_{2j} \Big) h_{2i-1}.$$
$$V^{(2m)} = \sum_{i=1}^{m} {}_{m-1}C_{i-1} V_x^{(2(m-i+1)-1)} h_{2i-1}$$
$$= \sum_{i=1}^{m} {}_{m-1}C_{i-1} Y_B^{(2(m-i+1)-1)} h_{2i-1}.$$

Then we have

$$A_1 + V^{(2m)} = \sum_{i=1}^{m} {}_{m-1}C_{i-1} \Big(\sum_{j=1}^{m-i+1} {}_{m-i}C_{j-1} Y_x^{(2(m-i+1)-2j+1)} h_{2j} + Y_B^{(2(m-i+1)-1)} \Big) h_{2i-1} = \sum_{i=1}^{m} {}_{m-1}C_{i-1} Y^{(2(m-i+1)+1)} h_{2i-1}.$$

On the other hand, we have

$$A_2 = \sum_{j=1}^{m} {}_{m-1}C_{j-1} Y^{(2m+1-2j)} h_{2j+1} = \sum_{l=2}^{m+1} {}_{m-1}C_{l-2} Y^{(2(m-l+1)+1)} h_{2l-1}.$$

Hence we have

the right member of the equality (2.4) $= A_1 + A_2 + V^{(2m)}$
$$= \sum_{j=1}^{m} {}_{m-1}C_{j-1} Y^{(2(m-j+1)+1)} h_{2j-1} + \sum_{j=2}^{m+1} {}_{m-1}C_{j-2} Y^{(2(m-j+1)+1)} h_{2j-1}$$
$$= Y^{(2m+1)} h_1 + \sum_{j=2}^{m} ({}_{m-1}C_{j-1} + {}_{m-1}C_{j-2}) Y^{(2(m-j+1)+1)} h_{2j-1} + Y^{(1)} h_{2m+1}$$
$$= Y^{(2m+1)} h_1 + \sum_{j=2}^{m} {}_mC_{j-1} Y^{(2(m-j+1)+1)} h_{2j-1} + Y^{(1)} h_{2m+1}$$
$$= \sum_{j=1}^{m+1} {}_mC_{j-1} Y^{(2(m-j+1)+1)} h_{2j-1}$$

= the left member of the equality (2.4). Q. E. D.

From the condition (1.3) we have

(2.9) $$\frac{\partial^p F_m}{\partial x^p}(Sx, B) Sr_1 Sr_2 \cdots Sr_p = S \frac{\partial^p F_m}{\partial x^p}(x, B) r_1 r_2 \cdots r_p \quad (m \geq 0,\ p \geq 1)$$

for arbitrary vectors $x, r_1, r_2, \ldots, r_p \in R^n$, where $F_0(x, B) = F(x, B)$, $F_i(x, B) =$

$\partial^i F(x, B)/\partial B^i$ $(i \geq 1)$, and $\partial^p F_m(x, B)/\partial x^p$ denotes the p-th derivative of $F_m(x, B)$ with respect to x. For the sake of simplicity, we set $f_m^p(x, B) = \partial^p F_m(x, B)/\partial x^p$. From (2.9) we have the following lemma.

Lemma 2.
For $x \in X_1$ and r_i (either $\in X_1$ or $\in X_{-1}$) $(i \geq 1)$

(2.10) (i) *$f_m^p(x, B) r_1 r_2 \cdots r_p$ belongs to the set X_{-1} if and only if the number of vectors $r_i \in X_{-1}$ is an odd number,*

(2.11) (ii) *$f_m^p(x, B) r_1 r_2 \cdots r_p$ belongs to the set X_1 if and only if either all r_i's belong to the set X_1 or the number of vectors $r_i \in X_{-1}$ is an even number.*

The proof of Lemma 2 is straightforward and will be omitted. From (2.1), (2.2) and Lemma 2 we have the following lemma.

Lemma 3.
For $x \in X_1$ and $h_{2i-1} \in X_{-1}$, $h_{2i} \in X_1$ $(i \geq 1)$

(2.12) (i) $V^{(2j-1)} \in X_1, \quad V^{(2j)} \in X_{-1} \quad (j \geq 1)$,

(2.13) (ii) $$\left\{ \begin{array}{llll} Y^{(2j-1)}\phi \in X_1, & Y^{(2j)}\phi \in X_{-1} & \text{for} & \phi \in X_1 \\ Y^{(2j-1)}\psi \in X_{-1}, & Y^{(2j)}\psi \in X_1 & \text{for} & \psi \in X_{-1} \end{array} \right. \quad (j \geq 1).$$

Proof. (i) From the definition (2.2), $V^{(2j-1)}$ can be written in the form of a linear combination of vectors $F_q(x, B)$'s and $(f_m^p(x, B) r_1 r_2 \cdots r_p)$'s, and $V^{(2j)}$ can be written in the form of a linear combination of vectors $(f_m^p(x, B) r_1 r_2 \cdots r_p)$'s. Since $x \in X_1$, all $F_q(x, B)$'s belong to X_1. Moreover, in each term $f_m^p(x, B) r_1 r_2 \cdots r_p$ which $V^{(2j-1)}$ contains, all r_i's belong to X_1 because each r_i is equal to h_{2k_i} for a positive integer k_i $(\leq j-1)$ from the definition (2.2). Hence $f_m^p(x, B) r_1 r_2 \cdots r_p \in X_1$ due to Lemma 2–(ii). This implies $V^{(2j-1)} \in X_1$. On the other hand, in each term $f_m^p(x, B) r_1 r_2 \cdots r_p$ which $V^{(2j)}$ contains, one of vectors r_i's is equal to h_{2k_0-1} for a positive integer k_0 $(\leq j)$ and each r_l of the remaining $(p-1)$ vectors is equal to h_{2k_l} for a positive integer k_l $(\leq j-1)$ from the definition (2.2). Since $h_{2k_0-1} \in X_{-1}$ and $h_{2k_l} \in X_1$, we have $f_m^p(x, B) r_1 r_2 \cdots r_p \in X_{-1}$ due to Lemma 2–(i). This implies $V^{(2j)} \in X_{-1}$.

(ii) From the definition (2.1), $Y^{(2j-1)}h$ $(h \in R^n)$ can be written in the form of a linear combination of vectors $(f_m^p(x, B) r_1 \cdots r_{p-1} h)$'s. For $\phi \in X_1$, in each term $f_m^p(x, B) r_1 \cdots r_{p-1}\phi$ which $Y^{(2j-1)}\phi$ contains, all r_i's belong to X_1, so $f_m^p(x, B) r_1 \cdots r_{p-1}\phi \in X_1$ due to Lemma 2–(ii). This implies $Y^{(2j-1)}\phi \in X_1$. On the other hand, for $\psi \in X_{-1}$, every term $f_m^p(x, B) r_1 \cdots r_{p-1}\psi$ which $Y^{(2j-1)}\psi$ contains belongs to X_{-1} due to Lemma 2–(i), because all r_i's belong to X_1. This implies $Y^{(2j-1)}\psi \in X_{-1}$. From the definition (2.1) $Y^{(2j)}h$ $(h \in R^n)$ can also be written in the form of a linear combination of vectors $(f_m^p(x, B) r_1 \cdots r_{p-1} h)$'s. For $\phi \in X_1$, in each term

$f_m^p(x, B)r_1 \cdots r_{p-1}\phi$ which $Y^{(2j)}\phi$ contains, one of vectors r_i's belongs to X_{-1} and the remaining $(p-2)$ vectors all belong to X_1, so $f_m^p(x, B)r_1 \cdots r_{p-1}\phi \in X_{-1}$ due to Lemma 2–(i). This implies $Y^{(2j)}\phi \in X_{-1}$. On the other hand, for $\psi \in X_{-1}$, every term $f_m^p(x, B)r_1 \cdots r_{p-1}\psi$ which $Y^{(2j)}\psi$ contains belongs to X_1 due to Lemma 2–(ii), because one of vectors r_i's belongs to X_{-1} and the remaining $(p-2)$ vectors all belong to X_1. This implies $Y^{(2j)}\psi \in X_1$. Q. E. D.

In order to simplify the following argument, we assume without loss of generality that

$$\operatorname{rank} F_{-1}(\hat{x}, \hat{B}) = \operatorname{rank}(F_{-1}(\hat{x}, \hat{B}), F_B(\hat{x}, \hat{B})) = n-1, \tag{2.14}$$

where $F_{-1}(\hat{x}, \hat{B})$ denotes the $n \times (n-1)$ matrix obtained from $F_x(\hat{x}, \hat{B})$ by deleting the first column vector. Then the equation

$$\begin{cases} F_x(\hat{x}, \hat{B})h_1 = 0, \\ h_1^1 - 1 = 0 \end{cases} \tag{2.15}$$

has only one solution $\hat{h}_1$, where $h_1 = (h_1^1, h_1^2, \dots, h_1^n)^T$. We owe to Brezzi et al. the following lemma concerning this solution $\hat{h}_1$.

Lemma 4 ([1]).
If $\hat{x} \in X_1$, *then either* $\hat{h}_1 \in X_1$ *or* $\hat{h}_1 \in X_{-1}$ *holds.*
First we study the case $\hat{h}_1 \in X_{-1}$. Then, from Lemma 3, we easily get the following lemma.

Lemma 5.

(2.16) (i) $F_x(\hat{x}, \hat{B})$ *is an isomorphism from* X_1 *to* X_1.

(2.17) (ii) $F_x(\hat{x}, \hat{B})$ *is a mapping from* X_{-1} *into* X_{-1}, *so* $F_x(\hat{x}, \hat{B})(X_{-1}) \subsetneqq X_{-1}$, *where*

$$F_x(\hat{x}, \hat{B})(X_{-1}) = \{y;\ y = F_x(\hat{x}, \hat{B})z,\ z \in X_{-1}\}. \tag{2.18}$$

Since $\hat{h}_1 \in X_{-1}$, we have $F_{xx}(\hat{x}, \hat{B})\hat{h}_1\hat{h}_1 \in X_1$ due to Lemma 2–(ii). Hence, from Lemma 5–(i), we have

$$\operatorname{rank}(F_x(\hat{x}, \hat{B}), F_{xx}(\hat{x}, \hat{B})\hat{h}_1\hat{h}_1) = n-1. \tag{2.19}$$

Now, to obtain the bifurcation point $(\hat{x}, \hat{B}) \in X_1 \times R$, we consider the equation

$$G(\boldsymbol{x}) = \begin{pmatrix} F(x, B) \\ F_x(x, B)h_1 \\ h_1^1 - 1 \end{pmatrix} = 0, \tag{2.20}$$

where $x=(x_1, x_2,\dots, x_n)^T$, $h_1=(h_1^1, h_1^2,\dots, h_1^n)^T$, $\boldsymbol{x}=(x, h_1, B)^T$. As is seen from the above argument, the equation (2.20) has a solution $\hat{\boldsymbol{x}}=(\hat{x}, \hat{h}_1, \hat{B})^T\in M$, where $(\hat{x}, \hat{B})\in X_1\times R$ is of course the above-mentioned bifurcation point and $\hat{h}_1\in X_{-1}$ is a solution of the equation (2.15). Due to Lemma 3, the mapping G defined by the equality (2.20) is a mapping from M to M. We denote by $G'(\boldsymbol{x})$ the Jacobian matrix of $G(\boldsymbol{x})$ with respect to $\boldsymbol{x}$. Then, due to Lemma 3, for the solution $\hat{\boldsymbol{x}}$, $G'(\hat{\boldsymbol{x}})$ is also a mapping from M to M. For the mapping $G'(\hat{\boldsymbol{x}})$ we have the following theorem.

Theorem 1.
$G'(\hat{\boldsymbol{x}})$ is an isomorphism from M to M if and only if

$$\hat{l}_1\notin \hat{Y}^{(1)}(X_{-1})=F_x(\hat{x}, \hat{B})(X_{-1}), \tag{2.21}$$

where

$$\hat{l}_1=\hat{Y}^{(3)}\hat{h}_1=\hat{Y}^{(2)}\hat{h}_2+\hat{V}^{(2)}=\{F_{xx}(\hat{x}, \hat{B})\hat{h}_2+F_{xB}(\hat{x}, \hat{B})\}\hat{h}_1. \tag{2.22}$$

Here $\hat{h}_2\in X_1$ is a solution of the equation

$$F_x(\hat{x}, \hat{B})h_2+F_B(\hat{x}, \hat{B})=0 \quad (h_2\in X_1) \tag{2.23}$$

and $\hat{Y}^{(p)}$ $(p=1, 2, 3)$ and $\hat{V}^{(q)}$ $(q=1, 2)$ denote the values of $Y^{(p)}$ and $V^{(q)}$ at $x=\hat{x}$, $h_i=\hat{h}_i$ $(i=1, 2)$ and $B=\hat{B}$, respectively.

PROOF. $G'(\hat{\boldsymbol{x}})$ has the form

$$G'(\hat{\boldsymbol{x}})=\begin{pmatrix} F_x(\hat{x}, \hat{B}) & 0 & F_B(\hat{x}, \hat{B}) \\ F_{xx}(\hat{x}, \hat{B})\hat{h}_1 & F_x(\hat{x}, \hat{B}) & F_{xB}(\hat{x}, \hat{B})\hat{h}_1 \\ 00\cdots0 & 10\cdots0 & 0 \end{pmatrix} \tag{2.24}$$
$$=\begin{pmatrix} \hat{Y}^{(1)} & 0 & \hat{V}^{(1)} \\ \hat{Y}^{(2)} & \hat{Y}^{(1)} & \hat{V}^{(2)} \\ 00\cdots0 & 10\cdots0 & 0 \end{pmatrix}.$$

For $(u_1, u_2, \lambda)^T\in M$ we consider the following equation:

$$\begin{cases} \hat{Y}^{(1)}u_1+\lambda\hat{V}^{(1)}=0, \\ \hat{Y}^{(2)}u_1+\hat{Y}^{(1)}u_2+\lambda\hat{V}^{(2)}=0, \quad u_2^1=0, \end{cases} \tag{2.25}$$

where $u_i=(u_i^1, u_i^2,\dots, u_i^n)^T$ $(i=1, 2)$.

When $\lambda=0$, we have $u_1=0$ from the first of the equation (2.25) because $\hat{Y}^{(1)}=F_x(\hat{x}, \hat{B})$ is an isomorphism from X_1 to X_1. Substituting $\lambda=0$ and $u_1=0$ into the second of (2.25), we have

$$\hat{Y}^{(1)}u_2=0, \quad u_2^1=0. \tag{2.26}$$

This implies $u_2=0$. Thus we obtain a zero solution $(0, 0, 0)^T \in M$ of the equation (2.25).

When $\lambda \neq 0$, we set $\lambda=1$ without loss of generality. Then we have $u_1=\hat{h}_2$ from the first of (2.25). Substituting $\lambda=1$ and $u_1=\hat{h}_2$ into the second of (2.25), we have

$$\hat{Y}^{(1)}u_2+\hat{Y}^{(2)}\hat{h}_2+\hat{V}^{(2)}=\hat{Y}^{(1)}u_2+\hat{l}_1=0, \quad u_2^1=0. \tag{2.27}$$

It follows from (2.27) that if $\hat{l}_1 \notin \hat{Y}^{(1)}(X_{-1})$, then the equation (2.25) has a zero solution only and so $G'(\hat{x})$ is an isomorphism from M to M, and conversely if $G'(\hat{x})$ is an isomorphism from M to M, then the equation (2.25) has a zero solution only and so $\hat{l}_1 \notin \hat{Y}^{(1)}(X_{-1})$. Q. E. D.

Due to Theorem 1, if the condition (2.21) is satisfied, then we can compute the bifurcation point $(\hat{x}, \hat{B}) \in X_1 \times R$ with high accuracy by applying the Newton method to the equation (2.20) when we consider the mapping G defined by the equality (2.20) as a mapping from M to M.

In particular, when $S^2=E_n$, Theorem 1 is the same result that B. Werner and A. Spence [4] obtained.

Next, we consider the case $\hat{l}_1 \in \hat{Y}^{(1)}(X_{-1})$. B. Werner and A. Spence did not describe anything about such a case in the paper [4]. Since $\hat{l}_1 \in \hat{Y}^{(1)}(X_{-1})$, the equation

$$\begin{cases} \hat{Y}^{(1)}h_3+\hat{l}_1=F_x(\hat{x}, \hat{B})h_3+\hat{l}_1=0, \\ h_3^1=0 \quad (\text{where } h_3=(h_3^1, h_3^2,\dots, h_3^n)^T \in X_{-1}) \end{cases} \tag{2.28}$$

has only one solution $\hat{h}_3 \in X_{-1}$. From Lemma 3 we have

$$\hat{Y}^{(3)}\hat{h}_2+\hat{V}^{(3)} \in X_1, \tag{2.29}$$

where $\hat{V}^{(3)}$ denotes the value of $V^{(3)}$ at $x=\hat{x}$, $h_i=\hat{h}_i$ $(i=1, 2)$ and $B=\hat{B}$. Since $\hat{Y}^{(1)}=F_x(\hat{x}, \hat{B})$ is an isomorphism from X_1 to X_1, the equation

$$\hat{Y}^{(1)}h_4+\hat{Y}^{(3)}\hat{h}_2+\hat{V}^{(3)}=0 \quad (h_4 \in X_1) \tag{2.30}$$

has only one solution $\hat{h}_4 \in X_1$. Thus we introduce another parameter β_1 and consider the equation

$$H_1(z_1)=\left(\begin{matrix} \begin{pmatrix} F(x, B) \\ Y^{(1)}h_1-\beta_1 l_2 \end{pmatrix} \\ \begin{pmatrix} Y^{(1)}h_2+V^{(1)} \\ Y^{(1)}h_3+l_1 \end{pmatrix} \end{matrix} \right. =0, \tag{2.31}$$

$$\begin{pmatrix} Y^{(1)}h_4 + Y^{(3)}h_2 + V^{(3)} \\ h_1^1 - 1 \\ h_3^1 \end{pmatrix}$$

where $x=(x_1, x_2,\ldots, x_n)^T$, $h_i=(h_i^1, h_i^2,\ldots, h_i^n)^T$ $(1\le i\le 4)$, $z_1=(x, h_1, h_2, h_3, h_4, B, \beta_1)^T$ and $l_2=2Y^{(3)}h_3+Y^{(5)}h_1=Y^{(4)}h_2+Y^{(3)}h_3+Y^{(2)}h_4+V^{(4)}$. As noted above, the equation (2.31) has a solution $\hat{z}_1=(\hat{x}, \hat{h}_1, \hat{h}_2, \hat{h}_3, \hat{h}_4, \hat{B}, 0)^T\in N_1=W^2\times X_1\times R^2$, where $W=X_1\times X_{-1}$ and $W^2=W\times W$. Due to Lemma 3, the mapping H_1 defined by the equality (2.31) is a mapping from N_1 to N_1. We denote by $H_1'(z_1)$ the Jacobian matrix of $H_1(z_1)$ with respect to z_1. For the solution $\hat{z}_1$, $H_1'(\hat{z}_1)$ has the form

$$(2.32)\qquad H_1'(\hat{z}_1)=\begin{pmatrix} \hat{Y}^{(1)} & 0 & 0 & 0 & 0 & \hat{V}^{(1)} & 0 \\ \hat{Y}^{(2)} & \hat{Y}^{(1)} & 0 & 0 & 0 & \hat{V}^{(2)} & -\hat{l}_2 \\ \hat{Y}^{(3)} & 0 & \hat{Y}^{(1)} & 0 & 0 & \hat{V}^{(3)} & 0 \\ \hat{Y}^{(4)} & \hat{Y}^{(3)} & \hat{Y}^{(2)} & \hat{Y}^{(1)} & 0 & \hat{V}^{(4)} & 0 \\ \hat{Y}^{(5)} & 0 & 2\hat{Y}^{(3)} & 0 & \hat{Y}^{(1)} & \hat{V}^{(5)} & 0 \\ 00\cdots0 & 10\cdots0 & 00\cdots0 & 00\cdots0 & 00\cdots0 & 0 & 0 \\ 00\cdots0 & 00\cdots0 & 00\cdots0 & 10\cdots0 & 00\cdots0 & 0 & 0 \end{pmatrix},$$

where $\hat{Y}^{(p)}$, $\hat{V}^{(p)}$ $(1\le p\le 5)$ and $\hat{l}_2$ denote the values of $Y^{(p)}$, $V^{(p)}$ and l_2 at $x=\hat{x}$, $h_i=\hat{h}_i$ $(1\le i\le 4)$ and $B=\hat{B}$, respectively. Due to Lemma 3, $H_1'(\hat{z}_1)$ is a mapping from N_1 to N_1. For the mapping $H_1'(\hat{z}_1)$ we have the following theorem.

Theorem 2.
$H_1'(\hat{z}_1)$ is an isomorphism from N_1 to N_1 if and only if

$$(2.33)\qquad \hat{l}_2\notin \hat{Y}^{(1)}(X_{-1})=F_x(\hat{x}, \hat{B})(X_{-1}).$$

PROOF. For $(u_1, u_2, u_3, u_4, u_5, \lambda_1, \lambda_2)^T\in N_1$ we consider the following equation:

$$(2.34)\qquad \begin{cases} \hat{Y}^{(1)}u_1+\lambda_1\hat{V}^{(1)}=0, \\ \hat{Y}^{(2)}u_1+\hat{Y}^{(1)}u_2+\lambda_1\hat{V}^{(2)}-\lambda_2\hat{l}_2=0, \quad u_2^1=0, \\ \hat{Y}^{(3)}u_1+\hat{Y}^{(1)}u_3+\lambda_1\hat{V}^{(3)}=0, \\ \hat{Y}^{(4)}u_1+\hat{Y}^{(3)}u_2+\hat{Y}^{(2)}u_3+\hat{Y}^{(1)}u_4+\lambda_1\hat{V}^{(4)}=0, \quad u_4^1=0, \\ \hat{Y}^{(5)}u_1+2\hat{Y}^{(3)}u_3+\hat{Y}^{(1)}u_5+\lambda_1\hat{V}^{(5)}=0, \end{cases}$$

where $u_i=(u_i^1, u_i^2, \dots, u_i^n)^T$ $(1 \le i \le 5)$.

First we prove that if $\hat{l}_2 \notin \hat{Y}^{(1)}(X_{-1})$, then $H_1'(\hat{z}_1)$ is an isomorphism from N_1 to N_1. When $\lambda_1=0$, we have $u_1=0$ from the first of the equation (2.34). Substituting $\lambda_1=0$ and $u_1=0$ into both the second and the third of the equation (2.34), we have

$$\begin{cases} \hat{Y}^{(1)}u_2-\lambda_2\hat{l}_2=0, \quad u_2^1=0, & (2.35) \\ \hat{Y}^{(1)}u_3=0. & (2.36) \end{cases}$$

Since $\hat{l}_2 \notin \hat{Y}^{(1)}(X_{-1})$, we have $\lambda_2=0$ from (2.35). Hence $u_2=0$. We also have $u_3=0$ from (2.36). Substituting $\lambda_1=0$ and $u_i=0$ $(1 \le i \le 3)$ into both the fourth and the fifth of (2.34), we have

$$(2.37) \qquad \begin{cases} \hat{Y}^{(1)}u_4=0, \quad u_4^1=0, \\ \hat{Y}^{(1)}u_5=0, \end{cases}$$

from which follow $u_4=0$ and $u_5=0$. Consequently, we obtain a zero solution of the equation (2.34). When $\lambda_1 \ne 0$, we set $\lambda_1=1$ without loss of generality. Then we have $u_1=\hat{h}_2$ from the first of (2.34). Substituting $\lambda_1=1$ and $u_1=\hat{h}_2$ into both the second and the third of (2.34), we have

$$\begin{cases} \hat{Y}^{(1)}u_2+\hat{Y}^{(2)}\hat{h}_2+\hat{V}^{(2)}-\lambda_2\hat{l}_2=\hat{Y}^{(1)}u_2+\hat{l}_1-\lambda_2\hat{l}_2=0, \quad u_2^1=0, & (2.38) \\ \hat{Y}^{(1)}u_3+\hat{Y}^{(3)}\hat{h}_2+\hat{V}^{(3)}=0. & (2.39) \end{cases}$$

Since $\hat{l}_1=\hat{Y}^{(2)}\hat{h}_2+\hat{V}^{(2)} \in \hat{Y}^{(1)}(X_{-1})$ and $\hat{l}_2 \notin \hat{Y}^{(1)}(X_{-1})$, we have $\lambda_2=0$ from (2.38). Then we have $u_2=\hat{h}_3$ from (2.38). We also have $u_3=\hat{h}_4$ from (2.39). Substituting $\lambda_1=1$ and $u_i=\hat{h}_{i+1}$ $(1 \le i \le 3)$ into both the fourth and the fifth of (2.34), we have

$$\begin{cases} \hat{Y}^{(1)}u_4+\hat{Y}^{(4)}\hat{h}_2+\hat{Y}^{(3)}\hat{h}_3+\hat{Y}^{(2)}\hat{h}_4+\hat{V}^{(4)}=\hat{Y}^{(1)}u_4+\hat{l}_2=0, \quad u_4^1=0, & (2.40) \\ \hat{Y}^{(1)}u_5+2\hat{Y}^{(3)}\hat{h}_4+\hat{Y}^{(5)}\hat{h}_2+\hat{V}^{(5)}=0. & (2.41) \end{cases}$$

Since $\hat{l}_2 \notin \hat{Y}^{(1)}(X_{-1})$, the equation (2.40) has no solution. Therefore the equation (2.34) has a zero solution only. This implies that $H_1'(\hat{z}_1)$ is an isomorphism from N_1 to N_1.

Next, we prove the converse. To do this, we prove its contraposition, that is, if $\hat{l}_2 \in \hat{Y}^{(1)}(X_{-1})$, then $H_1'(\hat{z}_1)$ is not an isomorphism from N_1 to N_1. For λ_1 and λ_2 we have the following three cases: (i) $\lambda_1=\lambda_2=0$, (ii) $\lambda_1 \ne 0$, $\lambda_2=0$, (iii) the others. First we consider the case (i). In this case, similarly to the case $\lambda_1=0$ in the above-mentioned proof, we have a zero solution of the equation (2.34). Secondly, in the case (ii), we set $\lambda_1=1$ without loss of generality. Then, similarly to the case $\lambda_1 \ne 0$ in the above-mentioned proof, we easily have $u_i=\hat{h}_{i+1}$ $(1 \le i \le 3)$ and also the equations (2.40) and (2.41). Since $\hat{l}_2 \in \hat{Y}^{(1)}(X_{-1})$, the equation (2.40) has only one solution $u_4=\hat{h}_5 \in X_{-1}$. Moreover the equation (2.41) has only one solution

$u_5=\hat{h}_6 \in X_1$ because

$$2\hat{Y}^{(3)}\hat{h}_4+\hat{Y}^{(5)}\hat{h}_2+\hat{V}^{(5)} \in X_1 \tag{2.42}$$

due to Lemma 3. Thus the equation (2.34) has a non-zero solution $(\hat{h}_2, \hat{h}_3, \hat{h}_4, \hat{h}_5, \hat{h}_6, 1, 0)^T \in N_1$. Therefore $H_1'(\hat{z}_1)$ is not an isomorphism from N_1 to N_1. Q.E.D.

Next, we consider a more general case which contains the case $\hat{l}_2 \in \hat{Y}^{(1)}(X_{-1})$. We define n-dimensional vectors l_m's by

$$\begin{aligned} l_m &= \sum_{j=1}^{m} {}_mC_{j-1} Y^{(2m+3-2j)} h_{2j-1} \\ &= \sum_{j=1}^{m} {}_{m-1}C_{j-1} Y^{(2m+2-2j)} h_{2j} + \sum_{j=1}^{m-1} {}_{m-1}C_{j-1} Y^{(2m+1-2j)} h_{2j+1} \\ &\quad + V^{(2m)} \quad (m \geq 1), \end{aligned} \tag{2.43}$$

where each h_i is an arbitrary n-dimensional vector.

Assume that there exists a vector $(\hat{x}, \hat{h}_1, \hat{h}_2, \ldots, \hat{h}_{2k-2}, \hat{h}_{2k-1}, \hat{B})^T \in W^k \times R$ such that the following two assumptions (I) and (II) are satisfied $(k \geq 2)$, where $(\hat{x}, \hat{B})$ satisfies both the equation (1.1) and the condition (1.2):

(I) $\hat{z}_{k-1}=(\hat{x}, \hat{h}_1, \hat{h}_2, \ldots, \hat{h}_{2k-1}, \hat{h}_{2k}, \hat{B}, \tilde{0})^T \in N_{k-1}=W^k \times X_1 \times R^k$ ($\tilde{0}$ is the $(k-1)$-dimensional zero vector) is a solution of the equation

$$H_{k-1}(z_{k-1}) = \begin{pmatrix} \begin{pmatrix} F(x, B) \\ Y^{(1)}h_1 - \beta_1 l_k \end{pmatrix} \\ \begin{pmatrix} Y^{(1)}h_2 + V^{(1)} \\ Y^{(1)}h_3 + l_1 - \beta_2 l_k \end{pmatrix} \\ \vdots \\ \begin{pmatrix} \sum_{j=1}^{k-2} {}_{k-3}C_{j-1} Y^{(2k-3-2j)} h_{2j} + V^{(2k-5)} \\ Y^{(1)}h_{2k-3} + l_{k-2} - \beta_{k-1} l_k \end{pmatrix} \\ \begin{pmatrix} \sum_{j=1}^{k-1} {}_{k-2}C_{j-1} Y^{(2k-1-2j)} h_{2j} + V^{(2k-3)} \\ Y^{(1)}h_{2k-1} + l_{k-1} \end{pmatrix} \\ \sum_{j=1}^{k} {}_{k-1}C_{j-1} Y^{(2k+1-2j)} h_{2j} + V^{(2k-1)} \\ \psi_{k-1}(z_{k-1}) \end{pmatrix} = 0, \tag{2.44}$$

where W^k denotes the direct product set $\underbrace{W \times W \times \cdots \times W}_{k \text{ times}}$, and β_i's are parameters,

$x=(x_1, x_2, \ldots, x_n)^T$, $h_j=(h_j^1, h_j^2, \ldots, h_j^n)^T$ $(1 \le j \le 2k)$, $z_{k-1}=(x, h_1, h_2, \ldots, h_{2k-1}, h_{2k}, B, \beta_1, \beta_2, \ldots, \beta_{k-1})^T$, $\psi_{k-1}(z_{k-1})=(h_1^1-1, h_3^1, h_5^1, \ldots, h_{2k-1}^1)^T$, and $\hat{h}_{2k} \in X_1$ is a solution of the equation

$$\hat{Y}^{(1)} h_{2k} + \sum_{j=1}^{k-1} {}_{k-1}C_{j-1} \hat{Y}^{(2k+1-2j)} \hat{h}_{2j} + \hat{V}^{(2k-1)} = 0 \quad (h_{2k} \in X_1). \tag{2.45}$$

Here $\hat{Y}^{(p)}$ and $\hat{V}^{(p)}$ $(1 \le p \le 2k-1)$ denote the values of $Y^{(p)}$ and $V^{(p)}$ at $x=\hat{x}$, $h_i=\hat{h}_i$ $(1 \le i \le 2k-2)$ and $B=\hat{B}$, respectively.

(II) $\hat{l}_k \in \hat{Y}^{(1)}(X_{-1}) = F_x(\hat{x}, \hat{B})(X_{-1})$,

where $\hat{l}_k$ denotes the value of l_k at $x=\hat{x}$, $h_i=\hat{h}_i$ $(1 \le i \le 2k)$ and $B=\hat{B}$, that is,

$$\begin{aligned} \hat{l}_k &= \sum_{j=1}^{k} {}_kC_{j-1} \hat{Y}^{(2k+3-2j)} \hat{h}_{2j-1} \\ &= \sum_{j=1}^{k} {}_{k-1}C_{j-1} \hat{Y}^{(2k+2-2j)} \hat{h}_{2j} + \sum_{j=1}^{k-1} {}_{k-1}C_{j-1} \hat{Y}^{(2k+1-2j)} \hat{h}_{2j+1} \\ &\quad + \hat{V}^{(2k)}. \end{aligned} \tag{2.46}$$

Here $\hat{Y}^{(q)}$ and $\hat{V}^{(q)}$ $(q=2k, 2k+1)$ denote the values of $Y^{(q)}$ and $V^{(q)}$ at $x=\hat{x}$, $h_i=\hat{h}_i$ $(1 \le i \le 2k)$ and $B=\hat{B}$, respectively.

Due to Lemma 3, the mapping H_{k-1} defined by the equality (2.44) is a mapping from N_{k-1} to N_{k-1}. We denote by $H'_{k-1}(z_{k-1})$ the Jacobian matrix of $H_{k-1}(z_{k-1})$ with respect to z_{k-1}. Due to Lemma 3, $H'_{k-1}(\hat{z}_{k-1})$ is also a mapping from N_{k-1} to N_{k-1}. For the mapping $H'_{k-1}(\hat{z}_{k-1})$ we have the following theorem.

Theorem 3.
The mapping $H'_{k-1}(\hat{z}_{k-1})$ is not an isomorphism from N_{k-1} to N_{k-1}.

PROOF. For $(u_1, u_2, \ldots, u_{2k+1}, \lambda_1, \lambda_2, \ldots, \lambda_k)^T \in N_{k-1}$ we consider the following equation:

$$\left\{ \begin{array}{l} \hat{Y}^{(1)} u_1 + \lambda_1 \hat{V}^{(1)} = 0, \\ \hat{Y}^{(2)} u_1 + \hat{Y}^{(1)} u_2 + \lambda_1 \hat{V}^{(2)} - \lambda_2 \hat{l}_k = 0, \quad u_2^1 = 0, \\ \hat{Y}^{(3)} u_1 + \hat{Y}^{(1)} u_3 + \lambda_1 \hat{V}^{(3)} = 0, \\ \hat{Y}^{(4)} u_1 + \hat{Y}^{(3)} u_2 + \hat{Y}^{(2)} u_3 + \hat{Y}^{(1)} u_4 + \lambda_1 \hat{V}^{(4)} - \lambda_3 \hat{l}_k = 0, \quad u_4^1 = 0, \\ \qquad \vdots \\ \sum_{j=1}^{k-1} {}_{k-2}C_{j-1} \hat{Y}^{(2k-1-2j)} u_{2j-1} + \lambda_1 \hat{V}^{(2k-3)} = 0, \\ \sum_{j=1}^{k-1} {}_{k-2}C_{j-1} \{ \hat{Y}^{(2k-2j)} u_{2j-1} + \hat{Y}^{(2k-1-2j)} u_{2j} \} + \lambda_1 \hat{V}^{(2k-2)} \\ \quad - \lambda_k \hat{l}_k = 0, \quad u_{2k-2}^1 = 0, \end{array} \right. \tag{2.47}$$

$$\left\{\begin{array}{l}\sum_{j=1}^{k} {}_{k-1}C_{j-1}\hat{Y}^{(2k+1-2j)}u_{2j-1}+\lambda_1\hat{V}^{(2k-1)}=0,\\ \sum_{j=1}^{k} {}_{k-1}C_{j-1}\{\hat{Y}^{(2k+2-2j)}u_{2j-1}+\hat{Y}^{(2k+1-2j)}u_{2j}\}+\lambda_1\hat{V}^{(2k)}=0, \quad u_{2k}^1=0,\\ \sum_{j=1}^{k+1} {}_{k}C_{j-1}\hat{Y}^{(2k+3-2j)}u_{2j-1}+\lambda_1\hat{V}^{(2k+1)}=0,\end{array}\right.$$

where $u_i=(u_i^1, u_i^2,\ldots, u_i^n)^T$ $(1\leq i\leq 2k+1)$.

For λ_i $(1\leq i\leq k)$ we have the following three cases: (i) $\lambda_i=0\,(1\leq i\leq k)$, (ii) $\lambda_1\neq 0$, $\lambda_i=0$ $(2\leq i\leq k)$, (iii) the others. In the case (i), we have a zero solution of the equation (2.47). In the case (ii), we set $\lambda_1=1$ without loss of generality. Then we have $u_1=\hat{h}_2$ from the first of (2.47). Substituting $\lambda_1=1$, $\lambda_2=0$ and $u_1=\hat{h}_2$ into both the second and the third of (2.47), we have

$$(2.48)\qquad \left\{\begin{array}{l}\hat{Y}^{(1)}u_2+\hat{Y}^{(2)}\hat{h}_2+\hat{V}^{(2)}=\hat{Y}^{(1)}u_2+\hat{l}_1=0, \quad u_2^1=0,\\ \hat{Y}^{(1)}u_3+\hat{Y}^{(3)}\hat{h}_2+\hat{V}^{(3)}=0.\end{array}\right.$$

We readily have $u_2=\hat{h}_3$ and $u_3=\hat{h}_4$ from (2.48). In the same way, we have $u_i=\hat{h}_{i+1}$ $(1\leq i\leq 2k-1)$. Substituting $\lambda_1=1, \lambda_i=0\,(2\leq i\leq k)$ and $u_j=\hat{h}_{j+1}$ $(1\leq j\leq 2k-1)$ into both the $2k$-th and the $(2k+1)$-th of (2.47), we have

$$(2.49)\qquad \begin{array}{l}\hat{Y}^{(1)}u_{2k}+(\sum_{j=1}^{k} {}_{k-1}C_{j-1}\hat{Y}^{(2k+2-2j)}\hat{h}_{2j}\\ \quad +\sum_{j=1}^{k-1} {}_{k-1}C_{j-1}\hat{Y}^{(2k+1-2j)}\hat{h}_{2j+1})+\hat{V}^{(2k)}=\hat{Y}^{(1)}u_{2k}+\hat{l}_k=0, \quad u_{2k}^1=0,\end{array}$$

$$(2.50)\qquad \hat{Y}^{(1)}u_{2k+1}+\sum_{j=1}^{k} {}_{k}C_{j-1}\hat{Y}^{(2k+3-2j)}\hat{h}_{2j}+\hat{V}^{(2k+1)}=0.$$

Since $\hat{l}_k\in\hat{Y}^{(1)}(X_{-1})$ due to the assumption (II), the equation (2.49) has only one solution $u_{2k}=\hat{h}_{2k+1}\in X_{-1}$. Moreover the equation (2.50) has only one solution $u_{2k+1}=\hat{h}_{2k+2}\in X_1$ because

$$(2.51)\qquad \sum_{j=1}^{k} {}_{k}C_{j-1}\hat{Y}^{(2k+3-2j)}\hat{h}_{2j}+\hat{V}^{(2k+1)}\in X_1$$

due to Lemma 3. Thus the equation (2.47) has a non-zero solution $(\hat{h}_2, \hat{h}_3,\ldots, \hat{h}_{2k+1}, \hat{h}_{2k+2}, 1, \tilde{0})^T\in N_{k-1}$. Hence $H'_{k-1}(\hat{z}_{k-1})$ is not an isomorphism from N_{k-1} to N_{k-1}. Q.E.D.

In the case $\hat{l}_k\in\hat{Y}^{(1)}(X_{-1})$, we introduce another parameter β_k and consider the equation

$$
(2.52)\qquad H_k(z_k)=\begin{pmatrix} \begin{pmatrix} F(x,\,B) \\ Y^{(1)}h_1-\beta_1 l_{k+1} \end{pmatrix} \\ \begin{pmatrix} Y^{(1)}h_2+V^{(1)} \\ Y^{(1)}h_3+l_1-\beta_2 l_{k+1} \end{pmatrix} \\ \vdots \\ \begin{pmatrix} \sum_{j=1}^{k-1} {}_{k-2}C_{j-1}Y^{(2k-1-2j)}h_{2j}+V^{(2k-3)} \\ Y^{(1)}h_{2k-1}+l_{k-1}-\beta_k l_{k+1} \end{pmatrix} \\ \begin{pmatrix} \sum_{j=1}^{k} {}_{k-1}C_{j-1}Y^{(2k+1-2j)}h_{2j}+V^{(2k-1)} \\ Y^{(1)}h_{2k+1}+l_k \end{pmatrix} \\ \sum_{j=1}^{k+1} {}_{k}C_{j-1}Y^{(2k+3-2j)}h_{2j}+V^{(2k+1)} \\ \psi_k(z_k) \end{pmatrix}=0,
$$

where β_i's are parameters, $x=(x_1,\, x_2,\dots,\, x_n)^T$, $h_j=(h_j^1,\, h_j^2,\dots,\, h_j^n)^T$ $(1\leq j\leq 2k+2)$, $z_k=(x,\, h_1,\, h_2,\dots,\, h_{2k+1},\, h_{2k+2},\, B,\, \beta_1,\, \beta_2,\dots,\, \beta_k)^T$ and $\psi_k(z_k)=(h_1^1-1,\, h_3^1,\, h_5^1,\dots,\, h_{2k+1}^1)^T$. As noted above, the equation (2.52) has a solution $\hat{z}_k=(\hat{x},\, \hat{h}_1,\, \hat{h}_2,\dots,\, \hat{h}_{2k+1},\, \hat{h}_{2k+2},\, \hat{B},\, \hat{0})^T\in N_k=W^{k+1}\times X_1\times R^{k+1}$, where 0 is the k-dimensional zero vector. Due to Lemma 3, the mapping H_k defined by the equality (2.52) is a mapping from N_k to N_k. We denote by $H_k'(z_k)$ the Jacobian matrix of $H_k(z_k)$ with respect to z_k. Due to Lemma 3, $H_k'(\hat{z}_k)$ is also a mapping from N_k to N_k. For the mapping $H_k'(\hat{z}_k)$ we have the following theorem.

Theorem 4.

$H_k'(\hat{z}_k)$ is an isomorphism from N_k to N_k if and only if

$$(2.53)\qquad \hat{l}_{k+1}\notin \hat{Y}^{(1)}(X_{-1})=F_x(\hat{x},\, \hat{B})(X_{-1}),$$

where $\hat{l}_{k+1}$ denotes the value of l_{k+1} at $x=\hat{x}$, $h_i=\hat{h}_i$ $(1\leq i\leq 2k+2)$ and $B=\hat{B}$, that is,

$$
\begin{aligned}
(2.54)\quad \hat{l}_{k+1}&=\sum_{j=1}^{k+1} {}_{k+1}C_{j-1}\hat{Y}^{(2k+5-2j)}\hat{h}_{2j-1} \\
&=\sum_{j=1}^{k+1} {}_{k}C_{j-1}\hat{Y}^{(2k+4-2j)}\hat{h}_{2j}+\sum_{j=1}^{k} {}_{k}C_{j-1}\hat{Y}^{(2k+3-2j)}\hat{h}_{2j+1}+\hat{V}^{(2k+2)}.
\end{aligned}
$$

Here $\hat{Y}^{(p)}$ and $\hat{V}^{(p)}$ $(1\leq p\leq 2k+3)$ denote the values of $Y^{(p)}$ and $V^{(p)}$ at $x=\hat{x}$, $h_i=\hat{h}_i$ $(1\leq i\leq 2k+2)$ and $B=\hat{B}$, respectively.

PROOF. For $(u_1,\, u_2,\dots,\, u_{2k+2},\, u_{2k+3},\, \lambda_1,\, \lambda_2,\dots,\, \lambda_{k+1})^T\in N_k$ we consider the following equation:

$$(2.55)\quad \begin{cases} \hat{Y}^{(1)}u_1+\lambda_1\hat{V}^{(1)}=0, \\ \hat{Y}^{(2)}u_1+\hat{Y}^{(1)}u_2+\lambda_1\hat{V}^{(2)}-\lambda_2\hat{l}_{k+1}=0, \quad u_2^1=0, \\ \hat{Y}^{(3)}u_1+\hat{Y}^{(1)}u_3+\lambda_1\hat{V}^{(3)}=0, \\ \hat{Y}^{(4)}u_1+\hat{Y}^{(3)}u_2+\hat{Y}^{(2)}u_3+\hat{Y}^{(1)}u_4+\lambda_1\hat{V}^{(4)}-\lambda_3\hat{l}_{k+1}=0, \quad u_4^1=0, \\ \qquad \vdots \\ \sum_{j=1}^{k} {}_{k-1}C_{j-1}\hat{Y}^{(2k+1-2j)}u_{2j-1}+\lambda_1\hat{V}^{(2k-1)}=0, \\ \sum_{j=1}^{k} {}_{k-1}C_{j-1}\{\hat{Y}^{(2k+2-2j)}u_{2j-1}+\hat{Y}^{(2k+1-2j)}u_{2j}\}+\lambda_1\hat{V}^{(2k)} \\ \quad -\lambda_{k+1}\hat{l}_{k+1}=0, \quad u_{2k}^1=0, \\ \sum_{j=1}^{k+1} {}_kC_{j-1}\hat{Y}^{(2k+3-2j)}u_{2j-1}+\lambda_1\hat{V}^{(2k+1)}=0, \\ \sum_{j=1}^{k+1} {}_kC_{j-1}\{\hat{Y}^{(2k+4-2j)}u_{2j-1}+\hat{Y}^{(2k+3-2j)}u_{2j}\}+\lambda_1\hat{V}^{(2k+2)}=0, \quad u_{2k+2}^1=0, \\ \sum_{j=1}^{k+2} {}_{k+1}C_{j-1}\hat{Y}^{(2k+5-2j)}u_{2j-1}+\lambda_1\hat{V}^{(2k+3)}=0, \end{cases}$$

where $u_i=(u_i^1, u_i^2,\dots, u_i^n)^T$ $(1\le i\le 2k+3)$.

First we prove that if $\hat{l}_{k+1}\notin\hat{Y}^{(1)}(X_{-1})$, then $H_k'(\hat{z}_k)$ is an isomorphism from N_k to N_k. When $\lambda_1=0$, we have $u_1=0$ from the first of the equation (2.55). Substituting $\lambda_1=0$ and $u_1=0$ into both the second and the third of (2.55), we have

$$(2.56)\quad \hat{Y}^{(1)}u_2-\lambda_2\hat{l}_{k+1}=0, \quad u_2^1=0,$$

$$(2.57)\quad \hat{Y}^{(1)}u_3=0.$$

Since $\hat{l}_{k+1}\notin\hat{Y}^{(1)}(X_{-1})$, we have $\lambda_2=0$ from (2.56). Hence $u_2=0$. We also have $u_3=0$ from (2.57). In the same way, we have $\lambda_i=0\,(2\le i\le k+1)$ and $u_j=0\,(1\le j\le 2k+1)$. Substituting $\lambda_i=0$ $(1\le i\le k+1)$ and $u_j=0$ $(1\le j\le 2k+1)$ into both the $(2k+2)$-th and the $(2k+3)$-th of (2.55), we have

$$(2.58)\quad \begin{cases} \hat{Y}^{(1)}u_{2k+2}=0, \quad u_{2k+2}^1=0, \\ \hat{Y}^{(1)}u_{2k+3}=0, \end{cases}$$

from which follow $u_{2k+2}=0$ and $u_{2k+3}=0$. Consequently, we obtain a zero solution of the equation (2.55). When $\lambda_1\ne 0$, we set $\lambda_1=1$ without loss of generality. Then we have $u_1=\hat{h}_2$ from the first of (2.55). Substituting $\lambda_1=1$ and $u_1=\hat{h}_2$ into both the second and the third of (2.55), we have

$$(2.59)\quad \hat{Y}^{(1)}u_2+\hat{Y}^{(2)}\hat{h}_2+\hat{V}^{(2)}-\lambda_2\hat{l}_{k+1}=\hat{Y}^{(1)}u_2+\hat{l}_1-\lambda_2\hat{l}_{k+1}=0, \quad u_2^1=0,$$

$$(2.60)\quad \hat{Y}^{(1)}u_3+\hat{Y}^{(3)}\hat{h}_2+\hat{V}^{(3)}=0.$$

Since $\hat{l}_1 \in \hat{Y}^{(1)}(X_{-1})$ and $\hat{l}_{k+1} \notin \hat{Y}^{(1)}(X_{-1})$, we have $\lambda_2=0$ from (2.59). Then we have $u_2=\hat{h}_3$ from (2.59). We also have $u_3=\hat{h}_4$ from (2.60). In the same way, we have $\lambda_i=0$ $(2\le i\le k+1)$ and $u_j=\hat{h}_{j+1}$ $(1\le j\le 2k+1)$. Substituting $\lambda_1=1$, $\lambda_i=0$ $(2\le i\le k+1)$ and $u_j=\hat{h}_{j+1}$ $(1\le j\le 2k+1)$ into both the $(2k+2)$-th and the $(2k+3)$-th of (2.55), we have

$$\begin{cases} \begin{cases} \hat{Y}^{(1)}u_{2k+2}+\sum_{j=1}^{k+1} {}_kC_{j-1}\hat{Y}^{(2k+4-2j)}\hat{h}_{2j}+\sum_{j=1}^{k} {}_kC_{j-1}\hat{Y}^{(2k+3-2j)}\hat{h}_{2j+1} \\ \quad +\hat{V}^{(2k+2)}=\hat{Y}^{(1)}u_{2k+2}+\hat{l}_{k+1}=0, \quad u^1_{2k+2}=0, \end{cases} & (2.61) \\ \hat{Y}^{(1)}u_{2k+3}+\sum_{j=1}^{k+1} {}_{k+1}C_{j-1}\hat{Y}^{(2k+5-2j)}\hat{h}_{2j}+\hat{V}^{(2k+3)}=0. & (2.62) \end{cases}$$

Since $\hat{l}_{k+1} \notin \hat{Y}^{(1)}(X_{-1})$, the equation (2.61) has no solution. Therefore the equation (2.55) has a zero solution only. This implies that $H_k'(\hat{z}_k)$ is an isomorphism from N_k to N_k.

Next, we prove the converse. To do this, we prove its contraposition, that is, if $\hat{l}_{k+1} \in \hat{Y}^{(1)}(X_{-1})$, then $H_k'(\hat{z}_k)$ is not an isomorphism from N_k to N_k. For λ_i $(1\le i\le k+1)$ we have the following three cases: (i) $\lambda_i=0$ $(1\le i\le k+1)$, (ii) $\lambda_1\neq 0$, $\lambda_i=0$ $(2\le i\le k+1)$, (iii) the others. First we consider the case (i). In this case, similarly to the case $\lambda_1=0$ in the above-mentioned proof, we have again a zero solution of the equation (2.55). Secondly, in the case (ii), we set $\lambda_1=1$ without loss of generality. Then, similarly to the case $\lambda_1\neq 0$ in the above-mentioned proof, we easily have $u_i=\hat{h}_{i+1}$ $(1\le i\le 2k+1)$ and also the equations (2.61) and (2.62). Since $\hat{l}_{k+1} \in \hat{Y}^{(1)}(X_{-1})$, the equation (2.61) has only one solution $u_{2k+2}=\hat{h}_{2k+3}\in X_{-1}$. Moreover the equation (2.62) has only one solution $u_{2k+3}=\hat{h}_{2k+4}\in X_1$ because

$$\sum_{j=1}^{k+1} {}_{k+1}C_{j-1}\hat{Y}^{(2k+5-2j)}\hat{h}_{2j}+\hat{V}^{(2k+3)}\in X_1 \tag{2.63}$$

due to Lemma 3. Thus the equation (2.55) has a non-zero solution $(\hat{h}_2, \hat{h}_3,\dots, \hat{h}_{2k+3}, \hat{h}_{2k+4}, 1, \hat{0})^T\in N_k$. Hence $H_k'(\hat{z}_k)$ is not an isomorphism from N_k to N_k. Q. E. D.

In particular, we consider the case where

$$S\neq E_n \quad \text{and} \quad S^{2p}=E_n \quad \text{for a positive integer } p. \tag{2.64}$$

We set

$$X_\omega=\{x\in R^n;\ h(S)x=0\}, \tag{2.65}$$

where

$$h(S)=\sum_{i=0}^{p-1} S^{2i}=E_n+S^2+S^4+\cdots+S^{2p-2}. \tag{2.66}$$

Then we have the following lemma.

Lemma 6.

$$R^n = X_1 \oplus X_{-1} \oplus X_\omega, \tag{2.67}$$

where $X_1 \oplus X_{-1} \oplus X_\omega$ *denotes the direct sum of* X_1, X_{-1} *and* X_ω.

PROOF. It is clear that $R^n \supset X_1 \oplus X_{-1} \oplus X_\omega$. For an arbitrary $x \in R^n$ we can write x in the form

$$x = \frac{f(S)}{2p}x - \frac{g(S)}{2p}x + \left\{E_n - \frac{h(S)}{p}\right\}x, \tag{2.68}$$

where

$$\begin{cases} f(S) = \sum_{i=0}^{2p-1} S^i = E_n + S + S^2 + \cdots + S^{2p-1}, \\ g(S) = \sum_{i=0}^{2p-1} (-1)^{i+1} S^i = -E_n + S - S^2 + \cdots - S^{2p-2} + S^{2p-1}. \end{cases} \tag{2.69}$$

Since $f(S)x/2p \in X_1$, $-g(S)x/2p \in X_{-1}$ and $\{E_n - h(S)/p\}x \in X_\omega$, we have $x \in X_1 \oplus X_{-1} \oplus X_\omega$. This implies $R^n \subset X_1 \oplus X_{-1} \oplus X_\omega$. Q. E. D.

As an immediate consequence of Lemma 6, we have the following lemma.

Lemma 7.

There exists a positive integer j' $(1 \le j' \le n)$ *such that*

$$b = -\frac{g(S)}{2p}e_{j'} \notin \hat{Y}^{(1)}(X_{-1}) = F_x(\hat{x}, \hat{B})(X_{-1}) \quad \textit{and} \quad b \in X_{-1}, \tag{2.70}$$

where $e_{j'} = (a^1_{j'}, a^2_{j'}, \ldots, a^n_{j'})^T \in R^n$ *is a unit vector such that* $a^{j'}_{j'} = 1$ *and* $a^i_{j'} = 0$ $(i \ne j')$.

PROOF. From Lemma 5–(ii), there exists a vector $d = (d_1, d_2, \ldots, d_n)^T \in X_{-1}$ such that $d \notin \hat{Y}^{(1)}(X_{-1})$. Setting $x = e_i$ in (2.68), we have

$$e_i = \frac{f(S)}{2p}e_i - \frac{g(S)}{2p}e_i + \left\{E_n - \frac{h(S)}{p}\right\}e_i \qquad (1 \le i \le n), \tag{2.71}$$

where each $e_i = (a^1_i, a^2_i, \ldots, a^n_i)^T \in R^n$ is a unit vector such that $a^i_i = 1$ and $a^m_i = 0$ $(m \ne i)$. Then we have

$$d = \sum_{i=1}^n d_i e_i = \sum_{i=1}^n d_i \frac{f(S)}{2p}e_i - \sum_{i=1}^n d_i \frac{g(S)}{2p}e_i + \sum_{i=1}^n d_i \left\{E_n - \frac{h(S)}{p}\right\}e_i. \tag{2.72}$$

Since $d \in X_{-1}$, $\sum_{i=1}^n d_i f(S)e_i/2p \in X_1$ and $\sum_{i=1}^n d_i\{E_n - h(S)/p\}e_i \in X_\omega$, we have

$$\sum_{i=1}^n d_i \frac{f(S)}{2p}e_i = 0 \quad \text{and} \quad \sum_{i=1}^n d_i \left\{E_n - \frac{h(S)}{p}\right\}e_i = 0. \tag{2.73}$$

Hence

(2.74) $$d=-\sum_{i=1}^{n} d_i \frac{g(S)}{2p} e_i \in X_{-1}.$$

It follows from (2.74) that there exists a positive integer j' $(1 \leq j' \leq n)$ such that

(2.75) $$-\frac{g(S)}{2p} e_{j'} \notin \hat{Y}^{(1)}(X_{-1})$$

because $d \notin \hat{Y}^{(1)}(X_{-1})$. Q. E. D.

From Lemma 7, in the case $\hat{l}_1 \in \hat{Y}^{(1)}(X_{-1})$, we may consider the equation

(2.76) $$G_1(\boldsymbol{x}_1)=\begin{pmatrix} \begin{pmatrix} F(x, B) \\ Y^{(1)}h_1-\beta_1 b \end{pmatrix} \\ \begin{pmatrix} Y^{(1)}h_2+V^{(1)} \\ Y^{(1)}h_3+l_1 \end{pmatrix} \\ h_1^1-1 \\ h_3^1 \end{pmatrix}=0$$

in place of the equation (2.31), where β_1 is a parameter, $x=(x_1, x_2, \ldots, x_n)^T$, $h_i=(h_i^1, h_i^2, \ldots, h_i^n)^T$ $(1 \leq i \leq 3)$ and $\boldsymbol{x}_1=(x, h_1, h_2, h_3, B, \beta_1)^T$. Then the equation (2.76) has a solution $\hat{\boldsymbol{x}}_1=(\hat{x}, \hat{h}_1, \hat{h}_2, \hat{h}_3, \hat{B}, 0)^T \in M_1=W^2 \times R^2$. Due to Lemma 3, the mapping G_1 defined by the equality (2.76) is a mapping from M_1 to M_1. We denote by $G_1'(\boldsymbol{x}_1)$ the Jacobian matrix of $G_1(\boldsymbol{x}_1)$ with respect to $\boldsymbol{x}_1$. Due to Lemma 3, for the solution $\hat{\boldsymbol{x}}_1$, $G_1'(\hat{\boldsymbol{x}}_1)$ is also a mapping from M_1 to M_1. For the mapping $G_1'(\hat{\boldsymbol{x}}_1)$ we easily get the following theorem.

Theorem 5.
$G_1'(\hat{\boldsymbol{x}}_1)$ is an isomorphism from M_1 to M_1 if and only if

(2.77) $$\hat{l}_2 \notin \hat{Y}^{(1)}(X_{-1})=F_x(\hat{x}, \hat{B})(X_{-1}),$$

where $\hat{l}_2$ is the vector referred to in Theorem 2.

Next, in the case where there exists a vector $(\hat{x}, \hat{h}_1, \hat{h}_2, \ldots, \hat{h}_{2k-1}, \hat{B})^T \in W^k \times R$ satisfying the assumptions (I) and (II), we may consider the equation

$$\begin{pmatrix} \begin{pmatrix} F(x, B) \\ Y^{(1)}h_1-\beta_1 b \end{pmatrix} \\ \begin{pmatrix} Y^{(1)}h_2+V^{(1)} \\ Y^{(1)}h_3+l_1-\beta_2 b \end{pmatrix} \\ \vdots \end{pmatrix}$$

(2.78)
$$G_k(\boldsymbol{x}_k)=\begin{pmatrix} \begin{pmatrix} \sum_{j=1}^{k-1} {}_{k-2}C_{j-1}Y^{(2k-1-2j)}h_{2j}+V^{(2k-3)} \\ Y^{(1)}h_{2k-1}+l_{k-1}-\beta_k b \end{pmatrix} \\ \begin{pmatrix} \sum_{j=1}^{k} {}_{k-1}C_{j-1}Y^{(2k+1-2j)}h_{2j}+V^{(2k-1)} \\ Y^{(1)}h_{2k+1}+l_k \end{pmatrix} \\ \phi_k(\boldsymbol{x}_k) \end{pmatrix}=0$$

instead of the equation (2.52), where β_i's are parameters, $x=(x_1, x_2,\dots, x_n)^T$, $h_j=(h_j^1, h_j^2,\dots, h_j^n)^T$ $(1\leq j\leq 2k+1)$, $\boldsymbol{x}_k=(x, h_1, h_2,\dots, h_{2k+1}, B, \beta_1, \beta_2,\dots, \beta_k)^T$ and $\phi_k(\boldsymbol{x}_k)=(h_1^1-1, h_3^1, h_5^1,\dots, h_{2k+1}^1)^T$. Then the equation (2.78) has a solution $\hat{\boldsymbol{x}}_k=(\hat{x}, \hat{h}_1, \hat{h}_2,\dots, \hat{h}_{2k+1}, \hat{B}, \hat{0})^T\in M_k=W^{k+1}\times R^{k+1}$, where $\hat{h}_{2k}\in X_1$ is a solution of the equation (2.45) and $\hat{h}_{2k+1}\in X_{-1}$ is a solution of the equation (2.49). Due to Lemma 3, the mapping G_k defined by the equality (2.78) is a mapping from M_k to M_k. We denote by $G_k'(\boldsymbol{x}_k)$ the Jacobian matrix of $G_k(\boldsymbol{x}_k)$ with respect to $\boldsymbol{x}_k$. Due to Lemma 3, $G_k'(\hat{\boldsymbol{x}}_k)$ is also a mapping from M_k to M_k. For the mapping $G_k'(\hat{\boldsymbol{x}}_k)$ we easily get the following theorem.

Theorem 6.
$G_k'(\hat{\boldsymbol{x}}_k)$ is an isomorphism from M_k to M_k if and only if

(2.79) $$\hat{l}_{k+1}\notin \hat{Y}^{(1)}(X_{-1})=F_x(\hat{x}, \hat{B})(X_{-1}),$$

where $\hat{l}_{k+1}$ is the vector referred to in Theorem 4.

Next, we consider the case where

(2.80) $$S\neq E_n \quad \text{and} \quad S^m=E_n \quad \text{for a positive integer } m$$

and the solution $\hat{h}_1$ of the equation (2.15) belongs to X_1. In this case, it is clear that

(2.81) $$R^n=X_1\oplus X_a,$$

where

(2.82) $$X_a=\{x\in R^n;\ K(S)x=0\}$$

and $X_1\oplus X_a$ denotes the direct sum of X_1 and X_a. Here

$$K(S)=\sum_{i=0}^{m-1} S^i=E_n+S+S^2+\cdots+S^{m-1}.$$

Moreover we have the following lemmas.

Lemma 8.
For $x\in X_1$ and $h_i\in X_1$ $(i\geq 1)$

(2.83) (i) $V^{(j)}\in X_1$ $(j\geq 1)$, (2.84) (ii) $Y^{(j)}\phi \in X_1$ *for* $\phi \in X_1$ $(j \geq 1)$.

Lemma 9.

(2.85) $\hat{Y}^{(1)} = F_x(\hat{x}, \hat{B})$ *is a mapping from* X_1 *into* X_1, *so* $F_x(\hat{x}, \hat{B})(X_1) \subsetneqq X_1$,

where

$$F_x(\hat{x}, \hat{B})(X_1) = \{y;\ y = F_x(\hat{x}, \hat{B})z,\ z \in X_1\}. \tag{2.86}$$

Lemma 10.

There exists a positive integer j_0 $(1 \leq j_0 \leq n)$ *such that*

$$v = \frac{K(S)}{m} e_{j_0} \notin \hat{Y}^{(1)}(X_1) = F_x(\hat{x}, \hat{B})(X_1) \quad and \quad v \in X_1. \tag{2.87}$$

PROOF. From Lemma 9, there exists a vector $w = (w_1, w_2, \ldots, w_n)^T \in X_1$ such that $w \notin \hat{Y}^{(1)}(X_1)$. Then

$$w = \sum_{i=1}^{n} w_i e_i = \sum_{i=1}^{n} w_i \frac{K(S)}{m} e_i + \sum_{i=1}^{n} w_i L(S) e_i, \tag{2.88}$$

where $L(S) = E_n - K(S)/m = \{(m-1)E_n - (S + S^2 + \cdots + S^{m-1})\}/m$. Since $K(S)e_i/m \in X_1$ and $L(S)e_i \in X_a$ $(1 \leq i \leq n)$, we have

$$\sum_{i=1}^{n} w_i L(S) e_i = 0. \tag{2.89}$$

Hence

$$w = \sum_{i=1}^{n} w_i \frac{K(S)}{m} e_i \in X_1. \tag{2.90}$$

Since $w \notin \hat{Y}^{(1)}(X_1)$, there exists a positive integer j_0 $(1 \leq j_0 \leq n)$ such that

$$\frac{K(S)}{m} e_{j_0} \notin \hat{Y}^{(1)}(X_1). \tag{2.91}$$

Q.E.D.

In this case, the equation (2.20) has a solution $\hat{\boldsymbol{x}} = (\hat{x}, \hat{h}_1, \hat{B})^T \in Q = X_1 \times X_1 \times R$, and the mapping G defined by the equality (2.20) is a mapping from Q to Q due to Lemma 8. For the solution $\hat{\boldsymbol{x}}$, due to Lemma 8, $G'(\hat{\boldsymbol{x}})$ is also a mapping from Q to Q. But $G'(\hat{\boldsymbol{x}})$ is not an isomorphism from Q to Q. Hence we need to introduce another parameter β and consider the equation which contains the equation (2.20) and some additional equations. That is, we consider the equation

$$(2.92)\qquad J(y)=\begin{pmatrix} F(x, B)-\beta v \\ F_x(x, B)h_1 \\ F_x(x, B)h_2+F_B(x, B) \\ h_1^1-1 \\ h_2^1 \end{pmatrix}=0,$$

where $x=(x_1, x_2,\dots, x_n)^T$, $h_i=(h_i^1, h_i^2,\dots, h_i^n)^T$ $(i=1, 2)$ and $y=(x, h_1, h_2, B, \beta)^T$. From (2.14) the equation

$$(2.93)\qquad \begin{cases} F_x(\hat{x}, \hat{B})h_2+F_B(\hat{x}, \hat{B})=0, \\ h_2^1=0 \qquad (\text{where } h_2=(h_2^1, h_2^2,\dots, h_2^n)^T\in X_1) \end{cases}$$

has only one solution $\hat{h}_2\in X_1$, so the equation (2.92) has a solution $\hat{y}=(\hat{x}, \hat{h}_1, \hat{h}_2, \hat{B}, 0)^T\in Q_1=X_1\times X_1\times X_1\times R^2$. Obviously, the mapping J defined by the equality (2.92) is a mapping from Q_1 to Q_1. We denote by $J'(y)$ the Jacobian matrix of $J(y)$ with respect to y. For the solution $\hat{y}$, $J'(\hat{y})$ has the form

$$(2.94)\quad J'(\hat{y})=\begin{pmatrix} \hat{Y}^{(1)} & 0 & 0 & \hat{V}^{(1)} & -v \\ \hat{Y}^{(2)} & \hat{Y}^{(1)} & 0 & \hat{V}^{(2)} & 0 \\ \hat{Y}^{(3)} & 0 & \hat{Y}^{(1)} & \hat{V}^{(3)} & 0 \\ 00\cdots0 & 10\cdots0 & 00\cdots0 & 0 & 0 \\ 00\cdots0 & 00\cdots0 & 10\cdots0 & 0 & 0 \end{pmatrix},$$

where $\hat{Y}^{(p)}$ and $\hat{V}^{(p)}$ $(1\leq p\leq 3)$ denote the values of $Y^{(p)}$ and $V^{(p)}$ at $x=\hat{x}$, $h_i=\hat{h}_i$ $(i=1, 2)$ and $B=\hat{B}$, respectively. Due to Lemma 8, $J'(\hat{y})$ is also a mapping from Q_1 to Q_1.

Now, concerning the mapping $J'(\hat{y})$, we discuss whether $J'(\hat{y})$ is an isomorphism from Q_1 to Q_1 or not. To do this, for $(u_1, u_2, u_3, \lambda_1, \lambda_2)^T\in Q_1$ we consider the equation

$$(2.95)\qquad \begin{cases} \hat{Y}^{(1)}u_1+\lambda_1\hat{V}^{(1)}-\lambda_2 v=0, \\ \hat{Y}^{(2)}u_1+\hat{Y}^{(1)}u_2+\lambda_1\hat{V}^{(2)}=0, \quad u_2^1=0, \\ \hat{Y}^{(3)}u_1+\hat{Y}^{(1)}u_3+\lambda_1\hat{V}^{(3)}=0, \quad u_3^1=0, \end{cases}$$

where $u_i=(u_i^1, u_i^2,\dots, u_i^n)^T$ $(1\leq i\leq 3)$. Since $\hat{V}^{(1)}=F_B(\hat{x}, \hat{B})\in \hat{Y}^{(1)}(X_1)$ and $v\notin \hat{Y}^{(1)}(X_1)$, we have $\lambda_2=0$ from the first of the equation (2.95), so $u_1=\lambda_1\hat{h}_2+c\hat{h}_1$, where c is an arbitrary constant. Substituting $u_1=\lambda_1\hat{h}_2+c\hat{h}_1$ into both the second and the third of (2.95), we have

$$\begin{cases} \hat{Y}^{(1)}u_2+\lambda_1\hat{\mu}_1+c\hat{\mu}_2=0, & u_2^1=0, \qquad (2.96)\\ \hat{Y}^{(1)}u_3+\lambda_1\hat{\mu}_3+c\hat{\mu}_1=0, & u_3^1=0, \qquad (2.97)\end{cases}$$

where

$$(2.98)\qquad \begin{cases} \hat{\mu}_1=\hat{Y}^{(2)}\hat{h}_2+\hat{V}^{(2)}=\hat{Y}^{(3)}\hat{h}_1,\\ \hat{\mu}_2=\hat{Y}^{(2)}\hat{h}_1,\\ \hat{\mu}_3=\hat{Y}^{(3)}\hat{h}_2+\hat{V}^{(3)}.\end{cases}$$

Then, for the mapping $J'(\hat{y})$, we have the following theorems from (2.96) and (2.97).

Theorem 7.

If $\hat{\mu}_1\notin\hat{Y}^{(1)}(X_1)$ and $\hat{\mu}_2\in\hat{Y}^{(1)}(X_1)$, then $J'(\hat{y})$ is an isomorphism from Q_1 to Q_1.

PROOF. Since $\hat{\mu}_1\notin\hat{Y}^{(1)}(X_1)$, we have $\lambda_1=0$ from (2.96). Hence the equation (2.97) becomes

$$(2.99)\qquad \hat{Y}^{(1)}u_3+c\hat{\mu}_1=0, \quad u_3^1=0.$$

Since $\hat{\mu}_1\notin\hat{Y}^{(1)}(X_1)$, we have $c=0$ from (2.99). Then $u_3=0$, and the equation (2.96) becomes

$$(2.100)\qquad \hat{Y}^{(1)}u_2=0, \quad u_2^1=0,$$

from which follows $u_2=0$. Moreover, $u_1=\lambda_1\hat{h}_2+c\hat{h}_1=0$. Thus the equation (2.95) has a zero solution only. This implies that $J'(\hat{y})$ is an isomorphism from Q_1 to Q_1. Q.E.D.

Theorem 8.

If $\hat{\mu}_1\in\hat{Y}^{(1)}(X_1)$ and $\hat{\mu}_2\notin\hat{Y}^{(1)}(X_1)$, then $J'(\hat{y})$ is an isomorphism from Q_1 to Q_1 if and only if

$$(2.101)\qquad \hat{\mu}_3\notin\hat{Y}^{(1)}(X_1).$$

PROOF. Since $\hat{\mu}_2\notin\hat{Y}^{(1)}(X_1)$, we have $c=0$ from (2.96). Then the equation (2.97) becomes

$$(2.102)\qquad \hat{Y}^{(1)}u_3+\lambda_1\hat{\mu}_3=0, \quad u_3^1=0.$$

This shows that if $\hat{\mu}_3\notin\hat{Y}^{(1)}(X_1)$, then we have $\lambda_1=0$, and so $u_3=0$. Since $\lambda_1=c=0$, we have $u_1=0$ and the equation (2.96) becomes

$$(2.103)\qquad \hat{Y}^{(1)}u_2=0, \quad u_2^1=0,$$

from which follows $u_2=0$. Thus the equation (2.95) has a zero solution only. This implies that $J'(\hat{y})$ is an isomorphism from Q_1 to Q_1. Conversely, if $J'(\hat{y})$ is an isomorphism from Q_1 to Q_1, then the equation (2.95) has a zero solution only.

It follows from (2.102) that $\hat{\mu}_3 \notin \hat{Y}^{(1)}(X_1)$. Q. E. D.

Theorem 9.

If $\hat{\mu}_1 \notin \hat{Y}^{(1)}(X_1)$ *and* $\hat{\mu}_2 \notin \hat{Y}^{(1)}(X_1)$, *then* $J'(\hat{\mathbf{y}})$ *is an isomorphism from* Q_1 *to* Q_1 *if and only if*

(2.104) $$\hat{\delta} \notin \hat{Y}^{(1)}(X_1),$$

where

(2.105) $$\hat{\delta} = \hat{\mu}_3 + \hat{\eta}\hat{\mu}_1 \quad (\in X_1).$$

Here $\hat{\eta}$ *is the* η*-component of the solution* $(\hat{\zeta}, \hat{\eta}) \in X_1 \times R$ *of the equation*

(2.106) $$\begin{cases} F_x(\hat{x}, \hat{B})\zeta + \hat{\mu}_1 + \eta\hat{\mu}_2 = 0, \\ \zeta_1 = 0 \qquad (\textit{where } \zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n)^T \in X_1). \end{cases}$$

PROOF. Since $1 + \dim \hat{Y}^{(1)}(X_1) = \dim X_1$ and $\hat{\mu}_1, \hat{\mu}_2 \in X_1$, the equation (2.106) certainly has only one solution $(\hat{\zeta}, \hat{\eta}) \in X_1 \times R$, where $\dim X$ denotes the dimension of a linear space X. Then the solution (u_2, c) of the equation (2.96) can be written in the form

(2.107) $$u_2 = \lambda_1 \hat{\zeta} \quad \text{and} \quad c = \lambda_1 \hat{\eta}.$$

Then the equation (2.97) becomes

(2.108) $$\hat{Y}^{(1)}u_3 + \lambda_1(\hat{\mu}_3 + \hat{\eta}\hat{\mu}_1) = \hat{Y}^{(1)}u_3 + \lambda_1\hat{\delta} = 0, \quad u_3^1 = 0.$$

Since $\hat{\mu}_1, \hat{\mu}_3 \in X_1$ due to Lemma 8, we have $\hat{\delta} \in X_1$. Therefore, if $\hat{\delta} \notin \hat{Y}^{(1)}(X_1)$, then we have $\lambda_1 = 0$ from (2.108), and so $u_3 = 0$. Then we also have $u_2 = 0$ and $c = 0$ from (2.107). Since $\lambda_1 = c = 0$, we have $u_1 = \lambda_1 \hat{h}_2 + c\hat{h}_1 = 0$. Thus the equation (2.95) has a zero solution only. This implies that $J'(\hat{\mathbf{y}})$ is an isomorphism from Q_1 to Q_1. Conversely, if $J'(\hat{\mathbf{y}})$ is an isomorphism from Q_1 to Q_1, then the equation (2.95) has a zero solution only. Hence, by (2.108), we have $\hat{\delta} \notin \hat{Y}^{(1)}(X_1)$. Q. E. D.

Remark 1.

Theorems 7–9 are essentially the same as Theorems 4–6 stated in the paper [7]. In the case where $\hat{\mu}_1 \in \hat{Y}^{(1)}(X_1)$ and $\hat{\mu}_2 \in \hat{Y}^{(1)}(X_1)$, $J'(\hat{\mathbf{y}})$ is not an isomorphism from Q_1 to Q_1. Hence we need to introduce another parameter and consider another equation which contains the equation (2.92).

Remark 2.

In the case $\hat{\mu}_1 \notin \hat{Y}^{(1)}(X_1)$, when we consider the equation

$$W(y)=\begin{pmatrix} F(x, B)-\beta\mu_1 \\ F_x(x, B)h_1 \\ F_x(x, B)h_2+F_B(x, B) \\ h_1^1-1 \\ h_2^1 \end{pmatrix}=0 \tag{2.109}$$

instead of the equation (2.92), we need not look for the vector $v=K(S)e_{j_0}/m$ satisfying the condition (2.87), where $\mu_1=Y^{(2)}h_2+V^{(2)}=\{F_{xx}(x, B)h_2+F_{Bx}(x, B)\}h_1$. Similarly, in the case $\hat{\mu}_2\notin \hat{Y}^{(1)}(X_1)$, we may consider the equation

$$\tilde{W}(y)=\begin{pmatrix} F(x, B)-\beta\mu_2 \\ F_x(x, B)h_1 \\ F_x(x, B)h_2+F_B(x, B) \\ h_1^1-1 \\ h_2^1 \end{pmatrix}=0 \tag{2.110}$$

instead of the equation (2.92), where $\mu_2=Y^{(2)}h_1=F_{xx}(x, B)h_1h_1$. Then we have results similar to Theorems 7–9.

§3. The Case $\hat{x}\in X_{-1}$

As is mentioned in §1, in addition to the conditions (1.3) and (1.4), we assume that for $x\in X_{-1}$ and $w_i\in X_{-1}$ $(i\geq 1)$

$$\begin{cases} F_m(-x, B)=-F_m(x, B),\ \dfrac{\partial F_m}{\partial x}(-x, B)=\dfrac{\partial F_m}{\partial x}(x, B) \\ \text{and} \\ \dfrac{\partial^p F_m}{\partial x^p}(-x, B)w_1w_2\cdots w_{p-1}=(-1)^{p-1}\dfrac{\partial^p F_m}{\partial x^p}(x, B)w_1w_2\cdots w_{p-1} \end{cases} \tag{3.1}$$

$$(m\geq 0,\ p\geq 2),$$

where $F_0(x, B)=F(x, B)$ and $F_i(x, B)=\partial^i F(x, B)/\partial B^i$ $(i\geq 1)$. From (2.9) and (3.1) we have the following lemma.

Lemma 11.

For $x, \psi\in X_{-1}$, $w_i\in X_{-1}$ $(i\geq 1)$ *and* $\phi\in X_1$

(3.2) (i) $f_m^1(x, B)\phi\in X_1$, $f_m^1(x, B)\psi\in X_{-1}$,

$$(3.3)\quad \text{(ii)}\quad \begin{cases} f_m^p(x, B)w_1w_2\cdots w_{p-1}\phi \in X_1 \\ f_m^p(x, B)w_1w_2\cdots w_{p-1}\psi \in X_{-1} \end{cases} \quad (p \geq 2),$$

where $f_m^j(x, B) = \partial^j F_m(x, B)/\partial x^j$ $(j \geq 1)$.

PROOF. (i) $S^{-1}f_m^1(x, B)\phi = S^{-1}f_m^1(S(-x), B)S\phi = f_m^1(-x, B)\phi = f_m^1(x, B)\phi$. Hence $f_m^1(x, B)\phi \in X_1$. On the other hand,

$$S^{-1}f_m^1(x, B)\psi = S^{-1}f_m^1(S(-x), B)(-S\psi) = -S^{-1}f_m^1(S(-x), B)S\psi$$
$$= -f_m^1(-x, B)\psi = -f_m^1(x, B)\psi.$$

Hence $f_m^1(x, B)\psi \in X_{-1}$.

(ii) $S^{-1}f_m^p(x, B)w_1w_2\cdots w_{p-1}\phi$

$$= S^{-1}f_m^p(S(-x), B)(-Sw_1)(-Sw_2)\cdots(-Sw_{p-1})(S\phi)$$
$$= (-1)^{p-1}S^{-1}f_m^p(S(-x), B)Sw_1Sw_2\cdots Sw_{p-1}S\phi$$
$$= (-1)^{p-1}f_m^p(-x, B)w_1w_2\cdots w_{p-1}\phi$$
$$= (-1)^{p-1}\cdot(-1)^{p-1}f_m^p(x, B)w_1w_2\cdots w_{p-1}\phi$$
$$= f_m^p(x, B)w_1w_2\cdots w_{p-1}\phi.$$

Hence $f_m^p(x, B)w_1w_2\cdots w_{p-1}\phi \in X_1$. On the other hand,

$$S^{-1}f_m^p(x, B)w_1w_2\cdots w_{p-1}\psi$$
$$= S^{-1}f_m^p(S(-x), B)(-Sw_1)(-Sw_2)\cdots(-Sw_{p-1})(-S\psi)$$
$$= (-1)^p S^{-1}f_m^p(S(-x), B)Sw_1Sw_2\cdots Sw_{p-1}S\psi$$
$$= (-1)^p f_m^p(-x, B)w_1w_2\cdots w_{p-1}\psi$$
$$= (-1)^p\cdot(-1)^{p-1}f_m^p(x, B)w_1w_2\cdots w_{p-1}\psi$$
$$= -f_m^p(x, B)w_1w_2\cdots w_{p-1}\psi.$$

Hence $f_m^p(x, B)w_1w_2\cdots w_{p-1}\psi \in X_{-1}$. Q.E.D.

From (2.1), (2.2) and Lemma 11, we have the following lemma similar to Lemma 3.

Lemma 12.

For $x \in X_{-1}$ *and* $h_{2i-1} \in X_1$, $h_{2i} \in X_{-1}$ $(i \geq 1)$

$$(3.4)\quad \text{(i)}\quad V^{(2j-1)} \in X_{-1}, \quad V^{(2j)} \in X_1 \quad (j \geq 1),$$

$$(3.5)\quad \text{(ii)}\quad \begin{cases} Y^{(2j-1)}\phi \in X_1 & \textit{for}\quad \phi \in X_1 \\ Y^{(2j-1)}\psi \in X_{-1},\quad Y^{(2j)}\psi \in X_1 & \textit{for}\quad \psi \in X_{-1} \end{cases} \qquad (j \geq 1).$$

PROOF. We prove that $Y^{(2j)}\psi \in X_1$ for $\psi \in X_{-1}$. From the definition (2.1) $Y^{(2j)}\psi$ can be written in the form of a linear combination of vectors $(f_m^p(x, B)r_1 \cdots r_{p-1}\psi)$'s. In each term $f_m^p(x, B)r_1 \cdots r_{p-1}\psi$ which $Y^{(2j)}\psi$ contains, one of vectors r_i's belongs to X_1 and the remaining $(p-2)$ vectors all belong to X_{-1}. Since $\psi \in X_{-1}$, we have $f_m^p(x, B)r_1 \cdots r_{p-1}\psi \in X_1$ due to Lemma 11. This implies $Y^{(2j)}\psi \in X_1$. The proofs for the other cases are similar. Thus we leave them to the reader.

Q. E. D.

Analogously to Lemma 4, we have the following lemma.

Lemma 13.

For the bifurcation point $(\hat{x}, \hat{B}) \in X_{-1} \times R$ *the only one solution* $\hat{h}_1$ *of the equation* (2.15) *belongs to either* X_1 *or* X_{-1}.

First we consider the case $\hat{h}_1 \in X_1$. Then we readily get the following lemma from Lemma 12.

Lemma 14.

(3.6) (i) $F_x(\hat{x}, \hat{B})$ *is an isomorphism from* X_{-1} *to* X_{-1}.

(3.7) (ii) $F_x(\hat{x}, \hat{B})$ *is a mapping from* X_1 *into* X_1, *so* $F_x(\hat{x}, \hat{B})(X_1) \subsetneqq X_1$,

where

$$(3.8)\qquad F_x(\hat{x}, \hat{B})(X_1) = \{y;\ y = F_x(\hat{x}, \hat{B})z,\ z \in X_1\}.$$

In this case, when we consider the equation (2.20) in §2, the mapping G defined by the equality (2.20) is a mapping from $L = X_{-1} \times X_1 \times R$ to L due to Lemma 12. As noted above, the equation (2.20) has a solution $\hat{\boldsymbol{x}} = (\hat{x}, \hat{h}_1, \hat{B})^T \in L$ and for the solution $\hat{\boldsymbol{x}}$, $G'(\hat{\boldsymbol{x}})$ is also a mapping from L to L due to Lemma 12. For the mapping $G'(\hat{\boldsymbol{x}})$ we have the following theorem similar to Theorem 1.

Theorem 10.

Under the assumption (3.1), $G'(\hat{\boldsymbol{x}})$ *is an isomorphism from L to L if and only if*

$$(3.9)\qquad \hat{l}_1 \notin \hat{Y}^{(1)}(X_1) = F_x(\hat{x}, \hat{B})(X_1),$$

where

$$(3.10)\qquad \hat{l}_1 = \hat{Y}^{(2)}\hat{h}_2 + \hat{V}^{(2)} = \{F_{xx}(\hat{x}, \hat{B})\hat{h}_2 + F_{Bx}(\hat{x}, \hat{B})\}\hat{h}_1.$$

Here $\hat{h}_2 \in X_{-1}$ *is a solution of the equation*

$$F_x(\hat{x}, \hat{B})h_2+F_B(\hat{x}, \hat{B})=0 \qquad (h_2 \in X_{-1}) \tag{3.11}$$

and $\hat{Y}^{(p)}$ *and* $\hat{V}^{(p)}$ $(p=1, 2)$ *denote the values of* $Y^{(p)}$ *and* $V^{(p)}$ *at* $x=\hat{x}$, $h_1=\hat{h}_1$ *and* $B=\hat{B}$, *respectively.*

When $\hat{l}_1 \in \hat{Y}^{(1)}(X_1)$, the equation

$$\begin{cases} \hat{Y}^{(1)}h_3+\hat{l}_1=F_x(\hat{x}, \hat{B})h_3+\hat{l}_1=0, \\ h_3^1=0 \qquad (\text{where } h_3=(h_3^1, h_3^2,\ldots, h_3^n)^T \in X_1) \end{cases} \tag{3.12}$$

has only one solution $\hat{h}_3 \in X_1$. Moreover, since

$$\hat{Y}^{(3)}\hat{h}_2+\hat{V}^{(3)} \in X_{-1} \tag{3.13}$$

due to Lemma 12, the equation

$$\hat{Y}^{(1)}h_4+\hat{Y}^{(3)}\hat{h}_2+\hat{V}^{(3)}=0 \qquad (h_4 \in X_{-1}) \tag{3.14}$$

has only one solution $\hat{h}_4 \in X_{-1}$, where $\hat{Y}^{(3)}$ and $\hat{V}^{(3)}$ denote the values of $Y^{(3)}$ and $V^{(3)}$ at $x=\hat{x}$, $h_i=\hat{h}_i$ $(i=1, 2)$ and $B=\hat{B}$, respectively. Hence we consider the equation (2.31). The mapping H_1 defined by the equality (2.31) is a mapping from $L_1=Z^2 \times X_{-1} \times R^2$ to L_1 due to Lemma 12, where $Z=X_{-1} \times X_1$ and $Z^2=Z \times Z$. As noted above, the equation (2.31) has a solution $\hat{z}_1=(\hat{x}, \hat{h}_1, \hat{h}_2, \hat{h}_3, \hat{h}_4, \hat{B}, 0)^T \in L_1$. For the solution $\hat{z}_1$, $H_1'(\hat{z}_1)$ is also a mapping from L_1 to L_1 due to Lemma 12. For the mapping $H_1'(\hat{z}_1)$ we have the following theorem.

Theorem 11.

Under the assumption (3.1), $H_1'(\hat{z}_1)$ *is an isomorphism from* L_1 *to* L_1 *if and only if*

$$\hat{l}_2 \notin \hat{Y}^{(1)}(X_1)=F_x(\hat{x}, \hat{B})(X_1), \tag{3.15}$$

where

$$\hat{l}_2=2\hat{Y}^{(3)}\hat{h}_3+\hat{Y}^{(5)}\hat{h}_1=\hat{Y}^{(4)}\hat{h}_2+\hat{Y}^{(3)}\hat{h}_3+\hat{Y}^{(2)}\hat{h}_4+\hat{V}^{(4)}. \tag{3.16}$$

Here $\hat{Y}^{(p)}$ *and* $\hat{V}^{(p)}$ $(1 \le p \le 5)$ *denote the values of* $Y^{(p)}$ *and* $V^{(p)}$ *at* $x=\hat{x}$, $h_i=\hat{h}_i$ $(1 \le i \le 4)$ *and* $B=\hat{B}$, *respectively.*

Next, we consider a more general case which contains the case $\hat{l}_2 \in \hat{Y}^{(1)}(X_1)$. Assume that there exists a vector $(\hat{x}, \hat{h}_1, \hat{h}_2,\ldots, \hat{h}_{2k-2}, \hat{h}_{2k-1}, \hat{B})^T \in Z^k \times R$ such that the following two assumptions (1) and (2) are satisfied $(k \ge 2)$, where $(\hat{x}, \hat{B})$ satisfies both the equation (1.1) and the condition (1.2):

(1) $\hat{z}_{k-1}=(\hat{x}, \hat{h}_1, \hat{h}_2,\ldots, \hat{h}_{2k-1}, \hat{h}_{2k}, \hat{B}, \tilde{0})^T \in L_{k-1}=Z^k \times X_{-1} \times R^k$ is a solution of the equation (2.44), where Z^k denotes the direct product set $\underbrace{Z \times Z \times \cdots \times Z}_{k \text{ times}}$, and $\hat{h}_{2k} \in X_{-1}$ is a solution of the equation

(3.17) $$\hat{Y}^{(1)}h_{2k}+\sum_{j=1}^{k-1}{}_{k-1}C_{j-1}\hat{Y}^{(2k+1-2j)}\hat{h}_{2j}+\hat{V}^{(2k-1)}=0 \quad (h_{2k}\in X_{-1}).$$

Here $\hat{Y}^{(p)}$ and $\hat{V}^{(p)}$ $(1\leq p\leq 2k-1)$ denote the values of $Y^{(p)}$ and $V^{(p)}$ at $x=\hat{x}$, $h_i=\hat{h}_i$ $(1\leq i\leq 2k-2)$ and $B=\hat{B}$, respectively.

(3.18) (2) $$\hat{l}_k\in\hat{Y}^{(1)}(X_1)=F_x(\hat{x},\hat{B})(X_1),$$

where $\hat{l}_k$ denotes the value of l_k at $x=\hat{x}$, $h_i=\hat{h}_i$ $(1\leq i\leq 2k)$ and $B=\hat{B}$, that is,

(3.19) $$\hat{l}_k=\sum_{j=1}^{k}{}_kC_{j-1}\hat{Y}^{(2k+3-2j)}\hat{h}_{2j-1}$$
$$=\sum_{j=1}^{k}{}_{k-1}C_{j-1}\hat{Y}^{(2k+2-2j)}\hat{h}_{2j}+\sum_{j=1}^{k-1}{}_{k-1}C_{j-1}\hat{Y}^{(2k+1-2j)}\hat{h}_{2j+1}+\hat{V}^{(2k)}.$$

Here $\hat{Y}^{(q)}$ and $\hat{V}^{(q)}$ $(q=2k, 2k+1)$ denote the values of $Y^{(q)}$ and $V^{(q)}$ at $x=\hat{x}$, $h_i=\hat{h}_i$ $(1\leq i\leq 2k)$ and $B=\hat{B}$, respectively.

Due to Lemma 12, both H_{k-1} and $H'_{k-1}(\hat{z}_{k-1})$ are mappings from L_{k-1} to L_{k-1}. But $H'_{k-1}(\hat{z}_{k-1})$ is not an isomorphism from L_{k-1} to L_{k-1} because $\hat{l}_k\in\hat{Y}^{(1)}(X_1)$. In the case $\hat{l}_k\in\hat{Y}^{(1)}(X_1)$, the equation

(3.20) $$\begin{cases}\hat{Y}^{(1)}h_{2k+1}+\hat{l}_k=0,\\ h^1_{2k+1}=0 \qquad (\text{where } h_{2k+1}=(h^1_{2k+1}, h^2_{2k+1},\dots, h^n_{2k+1})^T\in X_1)\end{cases}$$

has only one solution $\hat{h}_{2k+1}\in X_1$. Moreover the equation

(3.21) $$\hat{Y}^{(1)}h_{2k+2}+\sum_{j=1}^{k}{}_kC_{j-1}\hat{Y}^{(2k+3-2j)}\hat{h}_{2j}+\hat{V}^{(2k+1)}=0 \qquad (h_{2k+2}\in X_{-1})$$

has only one solution $\hat{h}_{2k+2}\in X_{-1}$ because

(3.22) $$\sum_{j=1}^{k}{}_kC_{j-1}\hat{Y}^{(2k+3-2j)}\hat{h}_{2j}+\hat{V}^{(2k+1)}\in X_{-1}$$

due to Lemma 12. Therefore, to obtain the bifurcation point $(\hat{x},\hat{B})\in X_{-1}\times R$ with high accuracy, we consider the equation (2.52). As noted above, the equation (2.52) has a solution $\hat{z}_k=(\hat{x}, \hat{h}_1, \hat{h}_2,\dots, \hat{h}_{2k+1}, \hat{h}_{2k+2}, \hat{B}, \hat{0})^T\in L_k=Z^{k+1}\times X_{-1}\times R^{k+1}$. Both H_k and $H'_k(\hat{z}_k)$ are mappings from L_k to L_k due to Lemma 12. For the mapping $H'_k(\hat{z}_k)$ we have the following theorem.

Theorem 12.

Under the assumption (3.1), $H'_k(\hat{z}_k)$ *is an isomorphism from* L_k *to* L_k *if and only if*

(3.23) $$\hat{l}_{k+1}\notin\hat{Y}^{(1)}(X_1)=F_x(\hat{x},\hat{B})(X_1),$$

where $\hat{l}_{k+1}$ *denotes the value of* l_{k+1} *at* $x=\hat{x}$, $h_i=\hat{h}_i$ $(1\leq i\leq 2k+2)$ *and* $B=\hat{B}$.

Next, in particular, we consider the case where S is a matrix satisfying the condition (2.64). From Lemma 14–(ii) we have the following lemma.

Lemma 15.

There exists a positive integer k_0 $(1\leq k_0\leq n)$ *such that*

$$\tilde{b}=\frac{f(S)}{2p}e_{k_0}\notin\hat{Y}^{(1)}(X_1)=F_x(\hat{x},\hat{B})(X_1) \quad \text{and} \quad \tilde{b}\in X_1. \tag{3.24}$$

In this case, when $\hat{l}_1\in\hat{Y}^{(1)}(X_1)$, we may consider the equation

$$A_1(\boldsymbol{x}_1)=\begin{pmatrix}\begin{pmatrix}F(x,B)\\ Y^{(1)}h_1-\beta_1\tilde{b}\end{pmatrix}\\ \begin{pmatrix}Y^{(1)}h_2+V^{(1)}\\ Y^{(1)}h_3+l_1\end{pmatrix}\\ h_1^1-1\\ h_3^1\end{pmatrix}=0 \tag{3.25}$$

in place of the equation (2.31), where β_1 is a parameter, $x=(x_1,x_2,\dots,x_n)^T$, $h_i=(h_i^1,h_i^2,\dots,h_i^n)^T$ $(1\leq i\leq 3)$ and $\boldsymbol{x}_1=(x,h_1,h_2,h_3,B,\beta_1)^T$. Obviously, the equation (3.25) has a solution $\hat{\boldsymbol{x}}_1=(\hat{x},\hat{h}_1,\hat{h}_2,\hat{h}_3,\hat{B},0)^T\in U_1=Z^2\times R^2$. Due to Lemma 12, the mapping A_1 defined by the equality (3.25) is a mapping from U_1 to U_1. We denote by $A_1'(\boldsymbol{x}_1)$ the Jacobian matrix of $A_1(\boldsymbol{x}_1)$ with respect to $\boldsymbol{x}_1$. For the solution $\hat{\boldsymbol{x}}_1$, $A_1'(\hat{\boldsymbol{x}}_1)$ is also a mapping from U_1 to U_1 due to Lemma 12. For the mapping $A_1'(\hat{\boldsymbol{x}}_1)$ we easily get the following theorem.

Theorem 13.

Under the assumption (3.1), $A_1'(\hat{\boldsymbol{x}}_1)$ *is an isomorphism from* U_1 *to* U_1 *if and only if*

$$\hat{l}_2\notin\hat{Y}^{(1)}(X_1)=F_x(\hat{x},\hat{B})(X_1), \tag{3.26}$$

where $\hat{l}_2$ *is the vector referred to in Theorem* 11.

Next, in the case where there exists a vector $(\hat{x},\hat{h}_1,\hat{h}_2,\dots,\hat{h}_{2k-1},\hat{B})^T\in Z^k\times R$ satisfying the assumptions (1) and (2), we may consider the equation

$$(3.27)\qquad A_k(\boldsymbol{x}_k)=\begin{pmatrix} \begin{pmatrix} F(x, B) \\ Y^{(1)}h_1-\beta_1\tilde{b} \end{pmatrix} \\ \begin{pmatrix} Y^{(1)}h_2+V^{(1)} \\ Y^{(1)}h_3+l_1-\beta_2\tilde{b} \end{pmatrix} \\ \vdots \\ \begin{pmatrix} \sum_{j=1}^{k-1} {}_{k-2}C_{j-1}Y^{(2k-1-2j)}h_{2j}+V^{(2k-3)} \\ Y^{(1)}h_{2k-1}+l_{k-1}-\beta_k\tilde{b} \end{pmatrix} \\ \begin{pmatrix} \sum_{j=1}^{k} {}_{k-1}C_{j-1}Y^{(2k+1-2j)}h_{2j}+V^{(2k-1)} \\ Y^{(1)}h_{2k+1}+l_k \end{pmatrix} \\ \phi_k(\boldsymbol{x}_k) \end{pmatrix}=0$$

instead of the equation (2.52), where β_i's are parameters, $x=(x_1, x_2,\dots, x_n)^T$, $h_j=(h_j^1, h_j^2,\dots, h_j^n)^T$ $(1\leq j\leq 2k+1)$, $\boldsymbol{x}_k=(x, h_1, h_2,\dots, h_{2k+1}, B, \beta_1, \beta_2,\dots, \beta_k)^T$ and $\phi_k(\boldsymbol{x}_k)=(h_1^1-1, h_3^1, h_5^1,\dots, h_{2k+1}^1)^T$. Then the equation (3.27) has a solution $\hat{\boldsymbol{x}}_k=(\hat{x}, \hat{h}_1, \hat{h}_2,\dots, \hat{h}_{2k+1}, \hat{B}, \hat{0})^T\in U_k=Z^{k+1}\times R^{k+1}$, where $\hat{h}_{2k}\in X_{-1}$ is a solution of the equation (3.17) and $\hat{h}_{2k+1}\in X_1$ is a solution of the equation (3.20). Due to Lemma 12, the mapping A_k defined by the equality (3.27) is a mapping from U_k to U_k. We denote by $A_k'(\boldsymbol{x}_k)$ the Jacobian matrix of $A_k(\boldsymbol{x}_k)$ with respect to $\boldsymbol{x}_k$. For the solution $\hat{\boldsymbol{x}}_k$, $A_k'(\hat{\boldsymbol{x}}_k)$ is also a mapping from U_k to U_k due to Lemma 12. For the mapping $A_k'(\hat{\boldsymbol{x}}_k)$ we easily have the following theorem.

Theorem 14.

Under the assumption (3.1), $A_k'(\hat{\boldsymbol{x}}_k)$ *is an isomorphism from* U_k *to* U_k *if and only if*

$$(3.28)\qquad \hat{l}_{k+1}\notin \hat{Y}^{(1)}(X_1)=F_x(\hat{x}, \hat{B})(X_1),$$

where $\hat{l}_{k+1}$ *is the vector referred to in Theorem* 12.

Next, we consider the case where S is a matrix satisfying the condition (2.64) and the solution $\hat{h}_1$ of the equation (2.15) belongs to X_{-1}. In this case, we have the following lemmas.

Lemma 16.

For $x\in X_{-1}$ *and* $h_i\in X_{-1}$ $(i\geq 1)$

$$(3.29)\quad \text{(i)}\quad V^{(j)}\in X_{-1}\qquad (j\geq 1),$$

$$(3.30)\quad \text{(ii)}\quad Y^{(j)}\psi\in X_{-1}\quad \textit{for}\quad \psi\in X_{-1}\qquad (j\geq 1).$$

Lemma 17.

(3.31) $\hat{Y}^{(1)}=F_x(\hat{x}, \hat{B})$ *is a mapping from* X_{-1} *into* X_{-1}, *so*

$$\hat{Y}^{(1)}(X_{-1})=F_x(\hat{x}, \hat{B})(X_{-1})\subsetneqq X_{-1},$$

where

$$\hat{Y}^{(1)}(X_{-1})=F_x(\hat{x}, \hat{B})(X_{-1})=\{y;\ y=F_x(\hat{x}, \hat{B})z,\ z\in X_{-1}\}. \tag{3.32}$$

Lemma 18.
There exists a positive integer i_0 $(1\leq i_0\leq n)$ *such that*

$$\tilde{v}=-\frac{g(S)}{2p}e_{i_0}\notin \hat{Y}^{(1)}(X_{-1})=F_x(\hat{x}, \hat{B})(X_{-1}) \quad \textit{and} \quad \tilde{v}\in X_{-1}. \tag{3.33}$$

In this case, as noted above, the equation (2.20) certainly has a solution $\hat{\boldsymbol{x}}=(\hat{x}, \hat{h}_1, \hat{B})^T\in C=X_{-1}\times X_{-1}\times R$. Both G and $G'(\hat{\boldsymbol{x}})$ are mappings from C to C due to Lemma 16. But $G'(\hat{\boldsymbol{x}})$ is not an isomorphism from C to C. Hence we need to introduce another parameter β and consider the following equation

$$P(\boldsymbol{y})=\begin{pmatrix} F(x, B)-\beta\tilde{v} \\ F_x(x, B)h_1 \\ F_x(x, B)h_2+F_B(x, B) \\ h_1^1-1 \\ h_2^1 \end{pmatrix}=0, \tag{3.34}$$

where $x=(x_1, x_2,\dots, x_n)^T$, $h_i=(h_i^1, h_i^2,\dots, h_i^n)^T$ $(i=1, 2)$ and $\boldsymbol{y}=(x, h_1, h_2, B, \beta)^T$. Since the equation

$$\begin{cases} F_x(x, B)h_2+F_B(x, B)=0, \\ h_2^1=0 \qquad (\text{where } h_2=(h_2^1, h_2^2,\dots, h_2^n)^T\in X_{-1}) \end{cases} \tag{3.35}$$

has only one solution $\hat{h}_2\in X_{-1}$ due to (1.2) (or (2.14)), the equation (3.34) has a solution $\hat{\boldsymbol{y}}=(\hat{x}, \hat{h}_1, \hat{h}_2, \hat{B}, 0)^T\in C_1=X_{-1}\times X_{-1}\times X_{-1}\times R^2$. Due to Lemma 16, the mapping P defined by the equality (3.34) is a mapping from C_1 to C_1. We denote by $P'(\boldsymbol{y})$ the Jacobian matrix of $P(\boldsymbol{y})$ with respect to $\boldsymbol{y}$. For the solution $\hat{\boldsymbol{y}}$, $P'(\hat{\boldsymbol{y}})$ has the form

$$(3.36)\qquad P'(\hat{y})=\begin{pmatrix} \hat{Y}^{(1)} & 0 & 0 & \hat{V}^{(1)} & -\tilde{v} \\ \hat{Y}^{(2)} & \hat{Y}^{(1)} & 0 & \hat{V}^{(2)} & 0 \\ \hat{Y}^{(3)} & 0 & \hat{Y}^{(1)} & \hat{V}^{(3)} & 0 \\ 00\cdots0 & 10\cdots0 & 00\cdots0 & 0 & 0 \\ 00\cdots0 & 00\cdots0 & 10\cdots0 & 0 & 0 \end{pmatrix},$$

where $\hat{Y}^{(q)}$ and $\hat{V}^{(q)}$ $(1\le q\le 3)$ denote the values of $Y^{(q)}$ and $V^{(q)}$ at $x=\hat{x}$, $h_i=\hat{h}_i$ $(i=1, 2)$ and $B=\hat{B}$, respectively. Due to Lemma 16, $P'(\hat{y})$ is a mapping from C_1 to C_1.

Now let us discuss whether the mapping $P'(\hat{y})$ is an isomorphism from C_1 to C_1 or not. To do this, for $(u_1, u_2, u_3, \lambda_1, \lambda_2)^T\in C_1$ we consider the following equation:

$$(3.37)\qquad \begin{cases} \hat{Y}^{(1)}u_1+\lambda_1\hat{V}^{(1)}-\lambda_2\tilde{v}=0, \\ \hat{Y}^{(2)}u_1+\hat{Y}^{(1)}u_2+\lambda_1\hat{V}^{(2)}=0, \quad u_2^1=0, \\ \hat{Y}^{(3)}u_1+\hat{Y}^{(1)}u_3+\lambda_1\hat{V}^{(3)}=0, \quad u_3^1=0, \end{cases}$$

where $u_i=(u_i^1, u_i^2,\dots, u_i^n)^T$ $(1\le i\le 3)$. Since $\hat{V}^{(1)}=F_B(\hat{x}, \hat{B})\in\hat{Y}^{(1)}(X_{-1})$ and $\tilde{v}\notin\hat{Y}^{(1)}(X_{-1})$, we have $\lambda_2=0$ from the first of (3.37), so $u_1=\lambda_1\hat{h}_2+c\hat{h}_1$, where c is an arbitrary constant. Substituting $u_1=\lambda_1\hat{h}_2+c\hat{h}_1$ into both the second and the third of (3.37), we have

$$\begin{cases} \hat{Y}^{(1)}u_2+\lambda_1\hat{p}_1+c\hat{p}_2=0, \quad u_2^1=0, & (3.38)\\ \hat{Y}^{(1)}u_3+\lambda_1\hat{p}_3+c\hat{p}_1=0, \quad u_3^1=0, & (3.39) \end{cases}$$

where

$$(3.40)\qquad \begin{cases} \hat{p}_1=\hat{Y}^{(2)}\hat{h}_2+\hat{V}^{(2)}=\hat{Y}^{(3)}\hat{h}_1, \\ \hat{p}_2=\hat{Y}^{(2)}\hat{h}_1, \\ \hat{p}_3=\hat{Y}^{(3)}\hat{h}_2+\hat{V}^{(3)}. \end{cases}$$

Then, for the mapping $P'(\hat{y})$, we have the following theorems from (3.38) and (3.39).

Theorem 15.

If $\hat{p}_1\notin\hat{Y}^{(1)}(X_{-1})$ *and* $\hat{p}_2\in\hat{Y}^{(1)}(X_{-1})$, *then* $P'(\hat{y})$ *is an isomorphism from* C_1 *to* C_1.

Theorem 16.

If $\hat{p}_1\in\hat{Y}^{(1)}(X_{-1})$ *and* $\hat{p}_2\notin\hat{Y}^{(1)}(X_{-1})$, *then* $P'(\hat{y})$ *is an isomorphism from* C_1 *to* C_1 *if and only if*

$$(3.41)\qquad \hat{p}_3\notin\hat{Y}^{(1)}(X_{-1}).$$

Theorem 17.

If $\hat{\rho}_1 \notin \hat{Y}^{(1)}(X_{-1})$ *and* $\hat{\rho}_2 \notin \hat{Y}^{(1)}(X_{-1})$, *then* $P'(\hat{y})$ *is an isomorphism from* C_1 *to* C_1 *if and only if*

(3.42) $$\hat{\sigma} \notin \hat{Y}^{(1)}(X_{-1}),$$

where

(3.43) $$\hat{\sigma} = \hat{\rho}_3 + \hat{v}\hat{\rho}_1 \quad (\in X_{-1}).$$

Here $\hat{v}$ *is the v-component of the solution* $(\xi, \hat{v}) \in X_{-1} \times R$ *of the equation*

(3.44) $$\begin{cases} F_x(\hat{x}, \hat{B})\xi + \hat{\rho}_1 + v\hat{\rho}_2 = 0, \\ \xi_1 = 0 \qquad (\textit{where } \xi = (\xi_1, \xi_2, \dots, \xi_n)^T \in X_{-1}). \end{cases}$$

Remark 3.

In the case where

(3.45) $$S \neq E_n \quad \text{and} \quad S^{2q+1} = -E_n \quad \text{for a positive integer } q,$$

it is clear that

(3.46) $$R^n = X_{-1} \oplus X_\theta,$$

where

(3.47) $$X_\theta = \{x \in R^n;\ \gamma(S)x = 0\}.$$

Here $\gamma(S) = \sum_{i=0}^{2q} (-1)^i S^i = E_n - S + S^2 - \cdots - S^{2q-1} + S^{2q}$. Then there exists a positive integer m_0 $(1 \leq m_0 \leq n)$ such that

(3.48) $$\alpha = \frac{\gamma(S)}{2q+1} e_{m_0} \notin \hat{Y}^{(1)}(X_{-1}) = F_x(\hat{x}, \hat{B})(X_{-1}) \quad \text{and} \quad \alpha \in X_{-1}.$$

Therefore, in this case, we consider the equation

(3.49) $$D(y) = \begin{pmatrix} F(x, B) - \beta\alpha \\ F_x(x, B)h_1 \\ F_x(x, B)h_2 + F_B(x, B) \\ h_1^1 - 1 \\ h_2^1 \end{pmatrix} = 0$$

instead of the equation (3.34). Then the equation (3.49) has a solution $\hat{y} = (\hat{x}, \hat{h}_1, \hat{h}_2, \hat{B}, 0)^T \in C_1$. We denote by $D'(y)$ the Jacobian matrix of $D(y)$ with respect to y. Both D and $D'(\hat{y})$ are mappings from C_1 to C_1 due to Lemma 16. For the mapping

$D'(\hat{y})$ we have results similar to Theorems 15–17.

Remark 4.

In the case $\hat{\rho}_1 \notin \hat{Y}^{(1)}(X_{-1})$, when we consider the equation

$$(3.50) \qquad U(y) = \begin{pmatrix} F(x, B) - \beta\rho_1 \\ F_x(x, B)h_1 \\ F_x(x, B)h_2 + F_B(x, B) \\ h_1^1 - 1 \\ h_2^1 \end{pmatrix} = 0$$

instead of the equation (3.34), we need not look for the vector $\tilde{v} = -g(S)e_{i_0}/2p$ satisfying the condition (3.33), where $\rho_1 = Y^{(2)}h_2 + V^{(2)}$. Similarly, in the case $\hat{\rho}_2 \notin \hat{Y}^{(1)}(X_{-1})$, we may consider the equation

$$(3.51) \qquad \tilde{U}(y) = \begin{pmatrix} F(x, B) - \beta\rho_2 \\ F_x(x, B)h_1 \\ F_x(x, B)h_2 + F_B(x, B) \\ h_1^1 - 1 \\ h_2^1 \end{pmatrix} = 0$$

instead of the equation (3.34), where $\rho_2 = Y^{(2)}h_1$. Then we have results similar to Theorems 15–17.

§4. An Example

To illustrate our theory and method, we present an example.

Example ([3]).

We consider the equation

$$(4.1) \qquad F(x, B) = \begin{pmatrix} x_1 + B(x_1^3 - x_1 + x_1x_2^2) \\ 10x_2 - B(x_2 + 2x_1^2x_2 + x_2^3) \end{pmatrix} = 0,$$

where $x = (x_1, x_2)^T$ and B is a parameter. The mapping F defined by the equality (4.1) satisfies the condition

$$(4.2) \qquad F(Sx, B) = SF(x, B) \quad \text{for} \quad x \in R^2, \quad B \in R$$

where

$$S=\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.3}$$

It is clear that $S^2=E_2=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. From (4.3) we have

$$X_1=\{x\in R^2;\ Sx=x\}=\text{span}\,\{(0,\ 1)^T\} \tag{4.4}$$

and

$$X_{-1}=\{x\in R^2;\ Sx=-x\}=\text{span}\,\{(1,\ 0)^T\}, \tag{4.5}$$

where span $\{\alpha\}$ $(\alpha\in R^2)$ denotes a linear space spanned by a vector α.

The equation (4.1) has some bifurcation points. A point $(\hat{x},\ \hat{B})^T=(\hat{x}_1,\ \hat{x}_2,\ \hat{B})^T=(0,\ 9/10,\ 100/19)^T\in X_1\times R$ is one of them. In fact, for $(\hat{x},\ \hat{B})^T=(0,\ 9/10,\ 100/19)^T$ we have

$$\text{rank}\,F_x(\hat{x},\ \hat{B})=\text{rank}\begin{pmatrix} 0 & 0 \\ 0 & -\frac{90}{19} \end{pmatrix}=1 \tag{4.6}$$

and

$$\text{rank}\,(F_x(\hat{x},\ \hat{B}),\ F_B(\hat{x},\ \hat{B}))=\text{rank}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{90}{19} & -\frac{171}{100} \end{pmatrix}=1. \tag{4.7}$$

By (4.6) the equation

$$\begin{cases} F_x(\hat{x},\ \hat{B})h_1=0, \\ h_1^1-1=0 \qquad (\text{where } h_1=(h_1^1,\ h_1^2)^T) \end{cases} \tag{4.8}$$

has only one solution $\hat{h}_1=(1,\ 0)^T$, so $\hat{h}_1\in X_{-1}$. Then, as stated in §2, we consider the equation

$$G(\boldsymbol{x})=\begin{pmatrix} F(x,\ B) \\ F_x(x,\ B)h_1 \\ h_1^1-1 \end{pmatrix}=0, \tag{4.9}$$

where $x=(x_1,\ x_2)^T$, $h_1=(h_1^1,\ h_1^2)^T$ and $\boldsymbol{x}=(x,\ h_1,\ B)^T$. As noted above, the equation (4.9) certainly has a solution $\hat{\boldsymbol{x}}=(\hat{x},\ \hat{h}_1,\ \hat{B})^T=(0,\ 9/10,\ 1,\ 0,\ 100/19)^T\in M=X_1\times X_{-1}\times R$. We denote by $G'(\boldsymbol{x})$ the Jacobian matrix of $G(\boldsymbol{x})$ with respect to $\boldsymbol{x}$. Both G and $G'(\hat{\boldsymbol{x}})$ are mappings from M to M. For the solution $\hat{\boldsymbol{x}}$ we have

$$\hat{l}_1 = F_{xx}(\hat{x}, \hat{B})\hat{h}_1\hat{h}_2 + F_{Bx}(\hat{x}, \hat{B})\hat{h}_1 = \left(-\frac{361}{100}, 0\right)^T \quad (\in X_{-1}), \tag{4.10}$$

where $\hat{h}_2 \in X_1$ is a solution of the equation

$$F_x(\hat{x}, \hat{B})h_2 + F_B(\hat{x}, \hat{B}) = 0 \qquad (h_2 \in X_1). \tag{4.11}$$

In fact, $\hat{h}_2 = (0, -361/1000)^T$. Since $F_x(\hat{x}, \hat{B})(X_{-1}) = \{0\}$, we have $\hat{l}_1 \notin F_x(\hat{x}, \hat{B})(X_{-1})$. Hence $G'(\hat{\boldsymbol{x}})$ is an isomorphism from M to M due to Theorem 1 in §2.

On the other hand, a point $(\bar{x}, \bar{B})^T = (\bar{x}_1, \bar{x}_2, \bar{B})^T = (\sqrt{3}/2, 0, 4)^T \in X_{-1} \times R$ is also a bifurcation point of the equation (4.1). In fact, for $(\bar{x}, \bar{B})^T = (\sqrt{3}/2, 0, 4)^T$ we have

$$\operatorname{rank} F_x(\bar{x}, \bar{B}) = \operatorname{rank}\begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix} = 1 \tag{4.12}$$

and

$$\operatorname{rank}(F_x(\bar{x}, \bar{B}), F_B(\bar{x}, \bar{B})) = \operatorname{rank}\begin{pmatrix} 6 & 0 & -\frac{\sqrt{3}}{8} \\ 0 & 0 & 0 \end{pmatrix} = 1. \tag{4.13}$$

Moreover the mapping F satisfies the condition (3.1) in §3. By (4.12) the equation

$$\begin{cases} F_x(\bar{x}, \bar{B})h_1 = 0, \\ h_1^2 - 1 = 0 \end{cases} \tag{4.14}$$

has only one solution $\bar{h}_1 = (0, 1)^T$, so $\bar{h}_1 \in X_1$. Then, in this case, we consider the equation

$$H(\boldsymbol{x}) = \begin{pmatrix} F(x, B) \\ F_x(x, B)h_1 \\ h_1^2 - 1 \end{pmatrix} = 0. \tag{4.15}$$

As noted above, the equation (4.15) certainly has a solution $\bar{\boldsymbol{x}} = (\bar{x}, \bar{h}_1, \bar{B})^T = (\sqrt{3}/2, 0, 0, 1, 4)^T \in L = X_{-1} \times X_1 \times R$. We denote by $H'(\boldsymbol{x})$ the Jacobian matrix of $H(\boldsymbol{x})$ with respect to $\boldsymbol{x}$. Then both H and $H'(\bar{\boldsymbol{x}})$ are mappings from L to L. For the solution $\bar{\boldsymbol{x}}$ we have

$$\bar{l}_1 = F_{xx}(\bar{x}, \bar{B})\bar{h}_1\bar{h}_2 + F_{Bx}(\bar{x}, \bar{B})\bar{h}_1 = (0, -3)^T \qquad (\in X_1), \tag{4.16}$$

where $\bar{h}_2 \in X_{-1}$ is a solution of the equation

$$F_x(\bar{x}, \bar{B})h_2 + F_B(\bar{x}, \bar{B}) = 0 \qquad (h_2 \in X_{-1}). \tag{4.17}$$

In fact, $\bar{h}_2=(\sqrt{3}/48, 0)^T$. Since $F_x(\bar{x}, \bar{B})(X_1)=\{0\}$, we have $\bar{l}_1 \notin F_x(\bar{x}, \bar{B})(X_1)$. Hence $H'(\bar{\boldsymbol{x}})$ is an isomorphism from L to L due to Theorem 10 in §3.

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