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Theory of (Vector Valued) Fourier Hyperfunctions. Their Realization as Boundary Values of (Vector Valued) Slowly Increasing Holomorphic Functions, (I)

By

Yoshifumi ITO

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Introduction

The purpose of this paper is that we will realize the six kinds of Fourier hyperfunctions and vector valued Fourier hyperfunctions which were defined in Ito [11] as "boundary values" of holomorphic functions of corresponding types or that we will reconstruct the Sato-Kawai's theory of hyperfunctions. We will prove vanishing theorems and duality theorems of (relative) cohomology groups which are necessary for that purpose and will realize the above goal as their consequences.

We will prove Theorem B due to Oka [27] and Cartan [1], here, for several kinds of sheaves of germs of slowly increasing or rapidly decreasing holomorphic functions by the way of Kawai [18], [19] and Hörmander [3] for scalar valued cases and by the tensor product method for vector valued cases.

The Dolbeault-Grothendieck resolutions with coefficients in sheaves of germs of slowly increasing or rapidly decreasing locally square integrable functions and in sheaves of germs of slowly increasing or rapidly decreasing C^∞ functions play important roles in this paper. In several cases, by using those resolutions we can prove some results for the smoothness of solutions of $\bar{\partial}$ equation without using Sobolev's lemma.

Malgrange's Theorems are modifications of Malgrange [23]. Serre's duality theorems are those of Serre [35]. Martineau-Harvey's Theorems are those of Martineau [24] and Harvey [2]. Sato's Theorems are modifications of Sato [32], [33].

For the outline of this paper, see "Contents". But Chapters 5 to 12 will be published in the second and third parts of this paper which were submitted to the following volumes of this Journal.

Contents

Chapter 1. Cases of sheaves $\tilde{\mathcal{O}}$, $\tilde{\mathcal{A}}$, $\underset{\sim}{\mathcal{O}}$ and $\underset{\sim}{\mathcal{A}}$

1.1. The Oka-Cartan-Kawai Theorem B

In this section we will prove the Oka-Cartan-Kawai Theorem B for the sheaves $\tilde{\mathcal{O}}$ and $\underset{\sim}{\mathcal{O}}$.

We denote by $\boldsymbol{D}^n$ the radial compactification of $\boldsymbol{R}^n$ in the sense of Kawai (see Kawai [19], Definition 1.1.1.). We denote by $\tilde{\boldsymbol{C}}^n$ the space $\boldsymbol{D}^n \times \sqrt{-1}\boldsymbol{R}^n$ endowed with the direct product topology.

Definition 1.1.1 (The sheaf $\tilde{\mathcal{O}}$ of germs of slowly increasing holomorphic functions). *We define $\tilde{\mathcal{O}}$ to be the sheafification of the presheaf $\{\tilde{\mathcal{O}}(\Omega);\ \Omega\subset\tilde{\boldsymbol{C}}^n \text{ open}\}$, where the section module $\tilde{\mathcal{O}}(\Omega)$ on an open set Ω in $\tilde{\boldsymbol{C}}^n$ is the space of all holomorphic functions $f(z)$ on $\Omega\cap\boldsymbol{C}^n$ such that, for any positive number ε and for any compact set K in Ω, the estimate $\sup\{|f(z)e(-\varepsilon|z|)|;\ z\in K\cap\boldsymbol{C}^n\}<\infty$ holds. Here $e(z)$ denotes the function $e^z=\exp(z)$ and we put $|z|=(|z_1|^2+\cdots+|z_n|^2)^{1/2}$.*

Definition 1.1.2 (The sheaf $\underset{\sim}{\mathcal{O}}$ of germs of rapidly decreasing holomorphic functions). *We define $\underset{\sim}{\mathcal{O}}$ to be the sheafification of the presheaf $\{\underset{\sim}{\mathcal{O}}(\Omega);\ \Omega\subset\tilde{\boldsymbol{C}}^n \text{ open}\}$, where the section module $\underset{\sim}{\mathcal{O}}(\Omega)$ on an open set Ω in $\tilde{\boldsymbol{C}}^n$ is the space of all holomorphic functions $f(z)$ on $\Omega\cap\boldsymbol{C}^n$ such that, for any compact set K in Ω, there exists some* *positive constant* δ *so that the estimate* $\sup\{|f(z)e(\delta|z|)|;\ z\in K\cap \boldsymbol{C}^n\}<\infty$ *holds.*

Definition 1.1.3. *An open set* V *in* $\tilde{\boldsymbol{C}}^n$ *is said to be an* $\tilde{\mathcal{O}}$*-pseudoconvex open set if it satisfies the conditions*:

(1) $\sup\{|\mathrm{Im}\, z|;\ z\in V\cap \boldsymbol{C}^n\}<\infty$, *where we put* $\mathrm{Im}\, z=(\mathrm{Im}\, z_1,\dots,\mathrm{Im}\, z_n)$ *and* $|\mathrm{Im}\, z|$ *denotes the Euclidean norm of* $\mathrm{Im}\, z$.

(2) *There exists a* C^∞*-plurisubharmonic function* $\varphi(z)$ *on* $V\cap \boldsymbol{C}^n$ *having the following two properties*:

(i) *The closure of* $V_c=\{z\in V\cap \boldsymbol{C}^n;\ \varphi(z)<c\}$ *in* $\tilde{\boldsymbol{C}}^n$ *is a compact subset of* V *for any real* c.

(ii) $\varphi(z)$ *is bounded on* $L\cap \boldsymbol{C}^n$ *for any compact subset* L *of* V.

Then we can prove the Oka-Cartan-Kawai Theorem B by the same method as that of Kawai [19] with a slight modification.

Theorem 1.1.4 (The Oka-Cartan-Kawai Theorem B). *For any* $\tilde{\mathcal{O}}$*-pseudoconvex open set* V *in* $\tilde{\boldsymbol{C}}^n$, *we have* $H^s(V, \tilde{\mathcal{O}}^p)=0$, $(p\geqq 0,\ s\geqq 1)$. *Here we denote by* $\mathscr{F}^{p,q}$ *the sheaf of germs of differential forms of type* (p, q) *with coefficients in a sheaf* $\mathscr{F}$ *and* $\mathscr{F}^p=\mathscr{F}^{p,0}$.

Proof. Since V is paracompact, $H^s(V, \tilde{\mathcal{O}}^p)$ coincides with the Čech cohomology group. So we have only to prove $\varinjlim_{\mathfrak{U}} H^s(\mathfrak{U}, \tilde{\mathcal{O}}^p)=0$, where $\mathfrak{U}=\{U_j\}_{j\geqq 1}$ is a locally finite open covering of V so that $V_j=U_j\cap \boldsymbol{C}^n$ is pseudoconvex. We can choose such a covering of V because V is an $\tilde{\mathcal{O}}$-pseudoconvex open set.

Now we define $C^s(\tilde{Z}^{\mathrm{loc}}_{(p,q)}(\{V_j\}))$ to be the set of all cochains $c=\{c_J;\ J=\{j_0, j_1,\dots,j_s\}\in \boldsymbol{N}^{s+1}\}$ of forms of type (p, q) satisfying the two conditions.

(i) $\bar\partial c_J=0$ in $V_J=V_{j_0}\cap V_{j_1}\cap\cdots\cap V_{j_s}$.

(ii) For any positive ε and any finite subset M of $\boldsymbol{N}^{s+1}$, the estimate

$$\sum_{J\in M}\int_{V_J}|c_J|^2 e(-\varepsilon\|z\|)d\lambda<\infty$$

holds, where $d\lambda$ is the Lebesgue measure on $\boldsymbol{C}^n$ and $\|z\|$ denotes the modification of $\sum_{j=1}^n |z_j|$ so as to become C^∞ and convex.

Now we will prove the following

Lemma 1.1.5. *If* $c\in C^s(\tilde{Z}^{\mathrm{loc}}_{(p,q)}(\{V_j\}))$ *satisfies the conditions* $\delta c=0$, *then we can find some* $c'\in C^{s-1}(\tilde{Z}^{\mathrm{loc}}_{(p,q)}(\{V_j\}))$ *such that* $\delta c'=c$. *Here* δ *means the coboundary operator.*

If this Lemma is proved, the theorem will follow from this Lemma as the special case where $q=0$ because we can use Cauchy's integral formula to change the L_2-norm to the sup-norm for holomorphic functions.

Proof of Lemma 1.1.5. We denote by $\{\chi_j\}$ the partition of unity subordinate to $\{V_j\}$ and define $b_I=\sum_j \chi_j c_{jI}$ for $I\in N^s$. Since $\delta c=0$, we have $\delta b=c$. So $\delta\bar{\partial}b=0$ because $\bar{\partial}c=0$. Since $\sum\chi_j=1$ and $\chi_j\geqq 0$, we have

$$\int_{V_I}|b_I|^2e(-\varepsilon\|z\|)d\lambda\leqq\sum_j\int_{V_I}\chi_j|c_{jI}|^2e(-\varepsilon\|z\|)d\lambda$$

for any positive number ε by virtue of Cauchy-Schwarz's inequality.

By the assumption of the existence of C^∞ plurisubharmonic function $\varphi(z)$ in Definition 1.1.3, we can find some plurisubharmonic function $\psi(z)$ on $W=V\cap C^n$ which satisfies the following two conditions;

(1) $\sum|\bar{\partial}\chi_j|\leqq e(\psi(z))$,

(2) $\sup\{\psi(z);\ z\in K\cap C^n\}\leqq C_K$ for any $K\Subset W$.

Thus it follows from the condition on c that

$$\sum_{I\in N}\int_{V_I}|\bar{\partial}b_I|^2e(-\varepsilon\|z\|-\psi(z))d\lambda<\infty$$

for any positive number ε and any finite subset N of N^s.

Now we consider the case $s=1$. By the fact that $\delta(\bar{\partial}b)=0$, $\bar{\partial}b$ defines a global section f on $W=V\cap C^n$. Then, by Hörmander [4], Theorem 4.4.2, p. 94, we can prove the existence of u such that $\bar{\partial}u=f$ and the estimate

$$\int_{K\cap C^n}|u|^2e(-\varepsilon\|z\|)(1+|z|^2)^{-2}d\lambda<\infty$$

holds for any positive number ε and any $K\Subset V$.

If we define $c'_I=b_I-u\,|\,V_I$, then $\bar{\partial}c'_I=0$ and $\delta c'=\delta b=c$. Clearly $c'\in C^{s-1}\cdot(\tilde{Z}^{\mathrm{loc}}_{(p,q)}(\{V_j\}))$.

Now we go on to the case $s>1$. In this case we use the induction on s. By the induction hypotheses there exists $b'\in C^{s-2}(\tilde{Z}^{\mathrm{loc}}_{(p,q+1)}(\{V_j\}))$ such that $\delta b'=\bar{\partial}b$. By virtue of Hörmander [4], Theorem 4.4.2, p. 94, we can also find $b''=\{b''_H\}_{H\in N^{s-1}}$ such that $b'_H=\bar{\partial}b''_H$ and the estimate

$$\sum_{H\in L}\int_{V_H}|b''_H|^2e(-\varepsilon\|z\|-\psi(z))(1+|z|^2)^{-2}d\lambda<\infty$$

holds for any positive number ε and any finite subset L of N^{s-1}. Therefore $c'=b-\delta b''$ satisfies all the required conditions. Q.E.D.

This completes the proof of the theorem. Q.E.D.

Remark. This method of proof is essentially due to Hörmander [3], §2.4.

Now we will prove the Malgrange Theorem for the sheaf $\tilde{\mathscr{A}}$ of germs of slowly increasing real analytic functions. Here we define the sheaf $\tilde{\mathscr{A}}$ to be the restriction of $\tilde{\mathcal{O}}$ to $\boldsymbol{D}^n$: $\tilde{\mathscr{A}}=\tilde{\mathcal{O}}|\boldsymbol{D}^n$. Then we have the following

Theorem 1.1.6 (Malgrange). *For an arbitrary set Ω in $\boldsymbol{D}^n$, we have*

$$H^s(\Omega, \tilde{\mathscr{A}}^p)=0, \ (p\geqq 0, \ s\geqq 1).$$

Proof. We know, by virtue of Theorem 2.1.6 of Kawai [19], that Ω has a fundamental system $\{\tilde{\Omega}\}$ of $\tilde{\mathcal{O}}$-pseudoconvex open neighborhoods. Then, it follows from the Oka-Cartan-Kawai Theorem B (cf. Theorem 1.1.4) and Theorem B42 of Schapira [34], p. 38 that, for $p\geqq 0$ and $s>0$, we have

$$H^s(\Omega, \tilde{\mathscr{A}}^p)=\varinjlim_{\tilde{\Omega}\cap\boldsymbol{D}^n=\Omega} H^s(\tilde{\Omega}, \tilde{\mathcal{O}}^p)=0.$$

Q.E.D.

Next we will prove the Oka-Cartan-Kawai Theorem B for the sheaf $\underset{\sim}{\mathcal{O}}$. This can be proved by the same method as Theorem 1.1.4. Thus we have the following

Theorem 1.1.7 (The Oka-Cartan-Kawai Theorem B). *For any $\tilde{\mathcal{O}}$-pseudoconvex open set V in $\tilde{\boldsymbol{C}}^n$, we have $H^s(V, \underset{\sim}{\mathcal{O}}^p)=0$ for $p\geqq 0$ and $s\geqq 1$.*

Proof. Since V is paracompact, $H^s(V, \underset{\sim}{\mathcal{O}}^p)$ coincides with the Čech cohomology group. So we have only to prove $\varinjlim_{\mathfrak{U}} H^s(\mathfrak{U}, \underset{\sim}{\mathcal{O}}^p)=0$, where $\mathfrak{U}=\{U_j\}_{j\geqq 1}$ is a locally finite open covering of V so that $V_j=U_j\cap\boldsymbol{C}^n$ is pseudoconvex. We can choose such a covering of V because V is an $\tilde{\mathcal{O}}$-pseudoconvex open set.

Here we use the notations in the proof of Theorem 1.1.4.

For any cocycle $d=\{d_J\}$ representing an element in $H^s(\mathfrak{U}, \underset{\sim}{\mathcal{O}}^p)$, we can define an element $c=\{c_J\}$ in $C^s(\tilde{Z}^{\mathrm{loc}}_{(p,0)}(\{V_j\}))$ such as $\delta c=0$ by putting $c_J=d_J\cdot h_\varepsilon(z)$, $h_\varepsilon(z)=\prod_{j=1}^{n}\cosh(\varepsilon z_j)$ for some positive ε, where δ denotes the coboundary operator. Then we can find some $c'\in C^{s-1}(\tilde{Z}^{\mathrm{loc}}_{(p,0)}(\{V_j\}))$ such that $\delta c'=c$. If we put $d'_I=c'_I\cdot(h_\varepsilon(z))^{-1}$, then $d'=\{d'_I\}$ is a cochain with values in $\underset{\sim}{\mathcal{O}}$ such that $\delta d'=d$. Thus the element in $H^s(\mathfrak{U}, \underset{\sim}{\mathcal{O}}^p)$ represented by d is zero. Since a class $[d]$ with a representative d is an arbitraty element in $H^s(\mathfrak{U}, \underset{\sim}{\mathcal{O}}^p)$, we have $H^s(\mathfrak{U}, \underset{\sim}{\mathcal{O}}^p)=0$. This completes the proof.

Q.E.D.

At last we will prove the Malgrange theorem for the sheaf $\underset{\sim}{\mathscr{A}}$ of germs of rapidly decreasing real analytic functions. Here we define the sheaf $\underset{\sim}{\mathscr{A}}$ to be the restriction of $\underset{\sim}{\mathcal{O}}$ to $\boldsymbol{D}^n$: $\underset{\sim}{\mathscr{A}}=\underset{\sim}{\mathcal{O}}\,|\,\boldsymbol{D}^n$. Then we have the following

Theorem 1.1.8 (Malgrange). *For an arbitrary set Ω in $\boldsymbol{D}^n$, we have $H^s(\Omega,$*

$\underset{\sim}{\mathscr{A}}^p)=0$ *for* $p\geqq 0$ *and* $s\geqq 1$.

Proof. We can prove this by the method similar to that of Theorem 1.1.6.

Q. E. D.

1.2. The Dolbeault-Grothendieck resolutions of $\tilde{\mathcal{O}}$ and $\underset{\sim}{\mathcal{O}}$

In this section we will recall the soft resolution of $\tilde{\mathcal{O}}$ and prove some of its consequences and the similar results for the sheaf $\underset{\sim}{\mathcal{O}}$.

At first we will recall the definition of the sheaf $\tilde{\mathscr{E}}$ of germs of slowly increasing C^∞-functions over $\tilde{\boldsymbol{C}}^n$ following Ito [10] and Junker [14].

Definition 1.2.1. *We define the sheaf $\tilde{\mathscr{E}}$ to be the sheafification of the presheaf $\{\tilde{\mathscr{E}}(\Omega);\ \Omega\subset\tilde{\boldsymbol{C}}^n$ open$\}$, where, for an open set Ω in $\tilde{\boldsymbol{C}}^n$, the module $\tilde{\mathscr{E}}(\Omega)$ is defined as follows:*

$$\tilde{\mathscr{E}}(\Omega)=\{f\in\mathscr{E}(\Omega\cap\boldsymbol{C}^n);\ \text{for any positive } \varepsilon \text{ and any compact set } K \text{ in } \Omega \text{ and any } \alpha\in\bar{\boldsymbol{N}}^{2n}, \text{ the estimate } \sup\{|f^{(\alpha)}(z)|e(-\varepsilon|z|);\ z\in K\cap\boldsymbol{C}^n\}<\infty \text{ holds}\}.$$

Here $\bar{\boldsymbol{N}}=\boldsymbol{N}\cup\{0\}$ and $\mathscr{E}(\Omega\cap\boldsymbol{C}^n)$ is the module of C^∞-functions on the open set $\Omega\cap\boldsymbol{C}^n$ in $\boldsymbol{C}^n$.

Then it is easy to see that $\tilde{\mathscr{E}}$ is a soft nuclear Fréchet sheaf. Then we define the sheaf $\tilde{\mathscr{E}}^{p,q}$ to be the sheaf of germs of differential forms of type (p, q) with coefficients in $\tilde{\mathscr{E}}$ and denote the Cauchy-Riemann operator by $\bar{\partial}$. We also define the sheaf $\tilde{\mathcal{O}}^{p,q}$ in the similar way and denote $\tilde{\mathcal{O}}^p=\tilde{\mathcal{O}}^{p,0}$. Then we have the following

Theorem 1.2.2 (The Dolbeault-Grothendieck resolution). *The sequence of sheaves over $\tilde{\boldsymbol{C}}^n$*

$$0\longrightarrow\tilde{\mathcal{O}}^p\longrightarrow\tilde{\mathscr{E}}^{p,0}\xrightarrow{\bar{\partial}}\tilde{\mathscr{E}}^{p,1}\xrightarrow{\bar{\partial}}\cdots\cdots\xrightarrow{\bar{\partial}}\tilde{\mathscr{E}}^{p,n}\longrightarrow 0$$

is exact.

Proof. See Ito [10], Corollary to Theorem 3.1. See also Junker [15].

Q. E. D.

Corollary 1. *For an open set Ω in $\tilde{\boldsymbol{C}}^n$, we have the following isomorphism:*

$$H^q(\Omega,\tilde{\mathcal{O}}^p)\cong\{f\in\tilde{\mathscr{E}}^{p,q}(\Omega);\ \bar{\partial}f=0\}/\{\bar{\partial}g;\ g\in\tilde{\mathscr{E}}^{p,q-1}(\Omega)\},\ (p\geqq 0,\ q\geqq 1).$$

Proof. It follows from Theorem 1.2.2 and Komatsu [21], Theorems II.2.9 and II.2.19.

Q. E. D.

Corollary 2. *Let Ω be an $\tilde{\mathcal{O}}$-pseudoconvex open set. Then the equation $\bar{\partial}u=f$ has a solution $u\in\tilde{\mathscr{E}}^{p,q}(\Omega)$ for every $f\in\tilde{\mathscr{E}}^{p,q+1}(\Omega)$ such that $\bar{\partial}f=0$. Here p, q are* *nonnegative integers.*

Proof. It follows from Theorem 1.1.4 and Corollary 1 to Theorem 1.2.2. Q. E. D.

Now we will define the sheaf $\underset{\sim}{\mathscr{E}}$ of germs of rapidly decreasing C^∞-functions over $\tilde{\boldsymbol{C}}^n$.

Definition 1.2.3. *We define the sheaf $\underset{\sim}{\mathscr{E}}$ to be the sheafification of the presheaf $\{\underset{\sim}{\mathscr{E}}(\Omega);\ \Omega\subset\tilde{\boldsymbol{C}}^n$ open$\}$, where the section module $\underset{\sim}{\mathscr{E}}(\Omega)$ on an open set Ω in $\tilde{\boldsymbol{C}}^n$ is the space of all C^∞-functions on $\Omega\cap\boldsymbol{C}^n$ such that, for any compact set K in Ω and any $\alpha\in\bar{\boldsymbol{N}}^{2n}$, there exists some positive constant δ so that the estimate*

$$\sup\{|f^{(\alpha)}(z)|e(\delta|z|);\ z\in K\cap\boldsymbol{C}^n\}<\infty$$

holds.

Then it is easy to see that $\underset{\sim}{\mathscr{E}}$ is a soft nuclear Fréchet sheaf. Then we define the sheaf $\underset{\sim}{\mathscr{E}}^{p,q}$ to be the sheaf of germs of differential forms of type (p, q) with coefficients in $\underset{\sim}{\mathscr{E}}$ and denote the Cauchy-Riemann operator by $\bar{\partial}$. We also define the sheaf $\underset{\sim}{\mathcal{O}}^{p,q}$ in the similar way and denote $\underset{\sim}{\mathcal{O}}^p=\underset{\sim}{\mathcal{O}}^{p,0}$. Then we have the following

Theorem 1.2.4 (The Dolbeault-Grothendieck resolution). *The sequence of sheaves over $\tilde{\boldsymbol{C}}^n$*

$$0\longrightarrow\underset{\sim}{\mathcal{O}}^p\longrightarrow\underset{\sim}{\mathscr{E}}^{p,0}\xrightarrow{\bar{\partial}}\underset{\sim}{\mathscr{E}}^{p,1}\xrightarrow{\bar{\partial}}\cdots\cdots\xrightarrow{\bar{\partial}}\underset{\sim}{\mathscr{E}}^{p,n}\longrightarrow 0$$

is exact.

Proof. It follows in the same way as Ito [10], §3. Q. E. D.

Corollary 1. *For an open set Ω in $\tilde{\boldsymbol{C}}^n$, we have the following isomorphism:*

$$H^q(\Omega,\underset{\sim}{\mathcal{O}}^p)\cong\{f\in\underset{\sim}{\mathscr{E}}^{p,q}(\Omega);\ \bar{\partial}f=0\}/\{\bar{\partial}g;\ g\in\underset{\sim}{\mathscr{E}}^{p,q-1}(\Omega)\},\quad(p\geqq 0,\ q\geqq 1).$$

Corollary 2. *Let Ω be an $\tilde{\mathcal{O}}$-pseudoconvex open set. Then the equation $\bar{\partial}u=f$ has a solution $u\in\underset{\sim}{\mathscr{E}}^{p,q}(\Omega)$ for every $f\in\underset{\sim}{\mathscr{E}}^{p,q+1}(\Omega)$ such that $\bar{\partial}f=0$. Here p, q are nonnegative integers.*

Proof. It follows from Theorem 1.1.7 and Corollary 1 to Theorem 1.2.4. Q. E. D.

Now, for later applications, we will prove another soft resolutions of $\tilde{\mathcal{O}}$ and $\underset{\sim}{\mathcal{O}}$ following Kaneko [17], p. 175.

We will now recall the definition of the sheaf $\tilde{L}=\tilde{L}_{2,\mathrm{loc}}$ of germs of slowly increasing locally L_2-functions.

Definition 1.2.5. *We define the sheaf $\tilde{L}$ to be the sheafification of the presheaf $\{\tilde{L}(\Omega); \Omega \subset \tilde{C}^n$ open$\}$, where, for an open set Ω in $\tilde{C}^n$, the section module $\tilde{L}(\Omega)$ is the space of all $f \in L_{2,\mathrm{loc}}(\Omega \cap C^n)$ such as, for any $\varepsilon > 0$ and any relatively compact open subset ω of Ω, $e(-\varepsilon\|z\|)f(z)|\omega$ belongs to L_2 $(\omega \cap C^n)$. Here $e(-\varepsilon\|z\|)f(z)|\omega$ denotes the restriction of $e(-\varepsilon|z|)f(z)$ to ω and $\|z\|$ denotes the modification of $\sum_{j=1}^{n} |z_j|$ so as to become C^∞ and convex.*

Then it is easy to see that $\tilde{L}$ is a soft FS* sheaf. Then we define the sheaf $\tilde{L}^{p,q}$ to be the sheaf of germs of differential forms of type (p, q) with coefficients in $\tilde{L}$.

Definition 1.2.6 (The sheaf $\tilde{\mathscr{L}}^{p,q}$). *We define the sheaf $\tilde{\mathscr{L}}^{p,q} = \tilde{\mathscr{L}}^{p,q}_{2,\mathrm{loc}}$ to be the sheafification of the presheaf $\{\tilde{\mathscr{L}}^{p,q}(\Omega); \Omega \subset \tilde{C}^n$ open$\}$, where, for an open set Ω in $\tilde{C}^n$, the section module $\tilde{\mathscr{L}}^{p,q}(\Omega)$ is the space of all $f \in \tilde{L}^{p,q}(\Omega) = \tilde{L}^{p,q}_{2,\mathrm{loc}}(\Omega)$ such that $\bar{\partial} f \in \tilde{L}^{p,q+1}(\Omega) = \tilde{L}^{p,q+1}_{2,\mathrm{loc}}(\Omega)$. We put $\tilde{\mathscr{L}} = \tilde{\mathscr{L}}^{0,0}$.*

Then $\tilde{\mathscr{L}}^{p,q}$ is a soft FS* sheaf. Then we have the following

Theorem 1.2.7 (The Dolbeault-Grothendieck resolution). *The sequence of sheaves over $\tilde{C}^n$*

$$0 \longrightarrow \tilde{\mathcal{O}}^p \longrightarrow \tilde{\mathscr{L}}^{p,0} \xrightarrow{\bar{\partial}} \tilde{\mathscr{L}}^{p,1} \xrightarrow{\bar{\partial}} \cdots\cdots \xrightarrow{\bar{\partial}} \tilde{\mathscr{L}}^{p,n} \longrightarrow 0$$

is exact.

Proof. The exactness of the sequence

$$0 \longrightarrow \tilde{\mathcal{O}}^p \longrightarrow \tilde{\mathscr{L}}^{p,0} \xrightarrow{\bar{\partial}} \tilde{\mathscr{L}}^{p,1}$$

is evident. In fact, let Ω be a relatively compact open set in $\tilde{C}^n$. Let $u \in \tilde{\mathscr{L}}^{p,0}(\Omega)$ such that $\bar{\partial} u = 0$. Then, if we write u in the form

$$u = \sum_{|I|=p} u_I dz^I,$$

we have

$$\partial u_I / \partial \bar{z}_j = 0, \quad j = 1, 2, \ldots, n.$$

But this is the Cauchy-Riemann equation. Thus u_I is holomorphic in $\Omega \cap C^n$. The condition that u_I is slowly increasing is already satisfied by the fact that we can interchange the sup-norm and L_2-norm for a holomorphic function. Thus the exactness of the above sequence was proved.

Next we have to prove the exactness of the sequence

$$\tilde{\mathscr{L}}^{p,0} \xrightarrow{\bar{\partial}} \tilde{\mathscr{L}}^{p,1} \xrightarrow{\bar{\partial}} \cdots\cdots \xrightarrow{\bar{\partial}} \tilde{\mathscr{L}}^{p,n} \longrightarrow 0.$$

We will reason as in Hörmander [4], p. 32. Thus it follows from the following

Lemma 1.2.8. *Let Ω be a relatively compact open set in $\tilde{C}^n$. Let $f \in \tilde{\mathscr{L}}^{p,q+1}(\Omega)$ $(p, q \geqq 0)$ satisfy the condition $\bar{\partial} f=0$. If Ω' is a relatively compact open subset of Ω, we can find $u \in \tilde{\mathscr{L}}^{p,q}(\Omega')$ such that $\bar{\partial} u=f$ in Ω'.*

Proof of Lemma 1.2.8. We shall prove inductively that Lemma 1.2.8 is true if f does not involve $d\bar{z}_{k+1},\dots, d\bar{z}_n$. This is trivial if $k=0$, for f must then be zero since every term in f is of degree $q+1>0$ with respect to $d\bar{z}$. For $k=n$, the statement is identical to Lemma 1.2.8. Assuming that it has already been proved when k is replaced by $k-1$, we write

$$f=d\bar{z}_k \wedge g+h,$$

where $g \in \tilde{\mathscr{L}}^{p,q}(\Omega)$, $h \in \tilde{\mathscr{L}}^{p,q+1}(\Omega)$ and g and h are independent of $d\bar{z}_k,\dots, d\bar{z}_n$. Write

$$g= \sum_{|I|=p}{}' \sum_{|J|=q}{}' g_{I,J} dz^I \wedge d\bar{z}^J,$$

where Σ' means that we sum only over increasing multi-indices. Since $\bar{\partial} f=0$, we obtain

$$\partial g_{I,J}/\partial \bar{z}_j=0, \quad j>k.$$

Thus $g_{I,J}$ is holomorphic in these variables.

We now choose a solution $G_{I,J}$ of the equation

$$\partial G_{I,J}/\partial \bar{z}_k=g_{I,J}.$$

To do so, we choose a bounded function $\psi \in C_0^\infty(\Omega)$ with bounded derivatives of any degree so that $\psi(z)=1$ in a neighborhood $\Omega'' \Subset \Omega$ of $\bar{\Omega}'$, and set

$$\begin{aligned} G_{I,J}=&(2i\pi)^{-1} \iint (\tau-z_k)^{-1} e(-(\tau-z_k)^2) \\ &\times \psi(z_1,\dots, z_{k-1}, \tau, z_{k+1},\dots, z_n) \\ &\times g_{I,J}(z_1,\dots, z_{k-1}, \tau, z_{k+1},\dots, z_n) d\tau \wedge d\bar{\tau}. \end{aligned}$$

Then it is easy to see that $G_{I,J} \in \tilde{\mathscr{L}}(\Omega)$. Here we can consider $G_{I,J}$ to be a convolution product of $\psi(z)g_{I,J}(z)$ and the fundamental solution $(\pi z_k)^{-1}e(-z_k^2)$ of the Cauchy-Riemann operator $\partial/\partial \bar{z}_k$. So that, by Theorem 3.4.11 of Ito [11], we have

$$\partial G_{I,J}/\partial \bar{z}_k=g_{I,J} \quad \text{in} \quad \Omega'', \ \partial G_{I,J}/\partial \bar{z}_j=0, j>k.$$

If we set

$$G= \sum_{I,J}{}' G_{I,J} dz^I \wedge d\bar{z}^J,$$

it follows that in Ω''

$$\bar{\partial}G = d\bar{z}_k \wedge g + h_1,$$

where h_1 is independent of $d\bar{z}_k, \ldots, d\bar{z}_n$. Hence $h - h_1 = f - \bar{\partial}G$ does not involve $d\bar{z}_k, \ldots, d\bar{z}_n$, so by the inductive hypothesis we can find $v \in \tilde{\mathscr{L}}^{p,q}(\Omega')$ so that $\bar{\partial}v = f - \bar{\partial}G$ there. But then $u = v + G$ satisfies the equation $\bar{\partial}u = f$, which completes the proof. Q.E.D.

Corollary 1. *For an open set Ω in $\tilde{C}^n$, we have the following isomorphism:*

$$H^q(\Omega, \tilde{\mathcal{O}}^p) \cong \{f \in \tilde{\mathscr{L}}^{p,q}_{2,\mathrm{loc}}(\Omega);\ \bar{\partial}f = 0\}/\{\bar{\partial}g;\ g \in \tilde{\mathscr{L}}^{p,q-1}_{2,\mathrm{loc}}(\Omega)\}, \quad (p \geq 0,\ q \geq 1).$$

Corollary 2. *Let Ω be an $\tilde{\mathcal{O}}$-pseudoconvex open set in $\tilde{C}^n$. Then the equation $\bar{\partial}u = f$ has a solution $u \in \tilde{\mathscr{L}}^{p,q}_{2,\mathrm{loc}}(\Omega)$ for every $f \in \tilde{\mathscr{L}}^{p,q+1}_{2,\mathrm{loc}}(\Omega)$ such that $\bar{\partial}f = 0$. Here p, q are nonnegative integers.*

Proof. It follows from Theorem 1.1.4 and Corollary 1 to Theorem 1.2.7. Q.E.D.

We will now recall the definition of the sheaf $\underset{\sim}{L} = \underset{\sim}{L}_{2,\mathrm{loc}}$ of germs of rapidly decreasing locally L_2-functions.

Definition 1.2.9. *We define the sheaf $\underset{\sim}{L}$ to be the sheafification of the presheaf $\{\underset{\sim}{L}(\Omega);\ \Omega \subset \tilde{C}^n \text{ open}\}$, where, for an open set Ω in $\tilde{C}^n$, the section module $\underset{\sim}{L}(\Omega)$ is the space of all $f \in L_{2,\mathrm{loc}}(\Omega \cap C^n)$ such as, for any relatively compact open subset ω of Ω, there exists some positive δ such that $e(\delta|z|)f(z)|\omega \in L_2(\omega \cap C^n)$.*

Then it is easy to see that $\underset{\sim}{L}$ is a soft FS* sheaf. Then we define the sheaf $\underset{\sim}{L}^{p,q}$ to be the sheaf of germs of differential forms of type (p, q) with coefficients in $\underset{\sim}{L}$.

Definition 1.2.10 (The sheaf $\underset{\sim}{\mathscr{L}}^{p,q}$). *We define the sheaf $\underset{\sim}{\mathscr{L}}^{p,q} = \underset{\sim}{\mathscr{L}}^{p,q}_{2,\mathrm{loc}}$ to be the sheafification of the presheaf $\{\underset{\sim}{\mathscr{L}}^{p,q}(\Omega);\ \Omega \subset \tilde{C}^n \text{ open}\}$, where, for an open set Ω in $\tilde{C}^n$, the section module $\underset{\sim}{\mathscr{L}}^{p,q}(\Omega)$ is the space of all $f \in \underset{\sim}{L}^{p,q}(\Omega) = \underset{\sim}{L}^{p,q}_{2,\mathrm{loc}}(\Omega)$ such that $\bar{\partial}f \in \underset{\sim}{L}^{p,q+1}(\Omega) = \underset{\sim}{L}^{p,q+1}_{2,\mathrm{loc}}(\Omega)$. We put $\underset{\sim}{\mathscr{L}} = \underset{\sim}{\mathscr{L}}^{0,0}$.*

Then $\underset{\sim}{\mathscr{L}}^{p,q}$ is a soft FS* sheaf. Then we have the following

Theorem 1.2.11 (The Dolbeault-Grothendieck resolution). *The sequence of sheaves over $\tilde{C}^n$*

$$0 \longrightarrow \underset{\sim}{\mathcal{O}}^p \longrightarrow \underset{\sim}{\mathscr{L}}^{p,0} \xrightarrow{\bar{\partial}} \underset{\sim}{\mathscr{L}}^{p,1} \xrightarrow{\bar{\partial}} \cdots\cdots \xrightarrow{\bar{\partial}} \underset{\sim}{\mathscr{L}}^{p,n} \longrightarrow 0$$

is exact.

Proof. It follows in the same way as Theorem 1.2.7. Q.E.D.

Corollary 1. *For an open set Ω in $\tilde{C}^n$, we have the following isomorphism:*

$$H^q(\Omega, \underset{\sim}{\mathcal{O}}^p)\cong\{f\in \underset{\sim}{\mathscr{L}}^{p,q}_{2,\mathrm{loc}}(\Omega);\ \bar{\partial}f=0\}/\{\bar{\partial}g;\ g\in \underset{\sim}{\mathscr{L}}^{p,q-1}_{2,\mathrm{loc}}(\Omega)\},\quad (p\geqq 0,\ q\geqq 1).$$

Corollary 2. *Let Ω be an $\tilde{\mathcal{O}}$-pseudoconvex open set in $\tilde{\boldsymbol{C}}^n$. Then the equation $\bar{\partial}u=f$ has a solution $u\in \underset{\sim}{\mathscr{L}}^{p,q}_{2,\mathrm{loc}}(\Omega)$ for every $f\in \underset{\sim}{\mathscr{L}}^{p,q+1}_{2,\mathrm{loc}}(\Omega)$ such that $\bar{\partial}f=0$. Here p, q are nonnegative integers.*

Proof. It follows from Theorem 1.1.7 and Corollary 1 to Theorem 1.2.11.

Q. E. D.

1.3. Malgrange's Theorem

Theorem 1.3.1. *Let Ω be an open set in $\tilde{\boldsymbol{C}}^n$. Then we have $H^n(\Omega, \tilde{\mathcal{O}})=0$.*

Proof. By virtue of Corollary 1 to Theorem 1.2.7, we have only to prove the exactness of the sequence

$$\tilde{\mathscr{L}}^{0,n-1}(\Omega) \xrightarrow{\bar{\partial}} \tilde{\mathscr{L}}^{0,n}(\Omega) \longrightarrow 0$$

in the notations of Theorem 1.2.7. But, in order to do so, we have only to prove the injectiveness and the closedness of the range of $\vartheta=(\bar{\partial})'$ in the dual sequence

$$\underset{\sim}{\mathscr{L}}^{0,1}_c(\Omega) \xleftarrow{\vartheta} \underset{\sim}{\mathscr{L}}^{0,0}_c(\Omega) \longleftarrow 0$$

in the notations of Theorem 1.2.11 by virtue of the Serre-Komatsu duality theorem for FS*-spaces. Here $\underset{\sim}{\mathscr{L}}^{p,q}_c(\Omega)$ denotes the space of sections with compact support of $\underset{\sim}{\mathscr{L}}^{p,q}$ on Ω. This has already been proved by Kawai [19], p. 479. Q. E. D.

Corollary. Flabby dim $\tilde{\mathcal{O}}\leqq n$.

1.4. Serre's duality theorem

In this section we will prove Serre's duality theorem.

Theorem 1.4.1. *Let Ω be an open set in $\tilde{\boldsymbol{C}}^n$ such that* $\dim H^p(\Omega, \tilde{\mathcal{O}})<\infty$ *holds ($p\geqq 1$). Then we have the isomorphism $[H^p(\Omega,\tilde{\mathcal{O}})]'\cong H^{n-p}_c(\Omega, \underset{\sim}{\mathcal{O}})$ $(0\leqq p\leqq n)$.*

Proof. By virtue of Corollary 1 to Theorem 1.2.7 and Corollary 1 to Theorem 1.2.11, cohomology groups $H^p(\Omega, \tilde{\mathcal{O}})$ and $H^{n-p}_c(\Omega, \underset{\sim}{\mathcal{O}})$ are cohomology groups respectively of the complexes

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \tilde{\mathscr{L}}^{0,0}(\Omega) & \xrightarrow{\bar{\partial}} & \tilde{\mathscr{L}}^{0,1}(\Omega) & \xrightarrow{\bar{\partial}} & \cdots\cdots & \xrightarrow{\bar{\partial}} & \tilde{\mathscr{L}}^{0,n}(\Omega) & \longrightarrow & 0\\
 & & \updownarrow & & \updownarrow & & & & \updownarrow & & \\
0 & \longleftarrow & \underset{\sim}{\mathscr{L}}^{0,n}_c(\Omega) & \xleftarrow{-\bar{\partial}} & \underset{\sim}{\mathscr{L}}^{0,n-1}_c(\Omega) & \xleftarrow{-\bar{\partial}} & \cdots\cdots & \xleftarrow{-\bar{\partial}} & \underset{\sim}{\mathscr{L}}^{0,0}_c(\Omega) & \longleftarrow & 0.
\end{array}$$

Here the upper complex is composed of FS* spaces and the lower complex is composed of DFS* spaces. Since the ranges of operators $\bar{\partial}$ in the upper complex are all closed by virtue of Schwartz's Lemma (cf. Komatsu [20]), the ranges of operators $-\bar{\partial}=(\bar{\partial})'$ in the lower complex are also all closed. Hence we have the isomorphism

$$[H^p(\Omega, \tilde{\mathcal{O}})]' \cong H_c^{n-p}(\Omega, \underset{\sim}{\mathcal{O}})$$

by virtue of Serre's Lemma (cf. Komatsu [20]). Q.E.D.

1.5. Martineau-Harvey's Theorem

In this section we will prove Martineau-Harvey's Theorem.

Theorem 1.5.1. *Let K be a compact set in $\tilde{\boldsymbol{C}}^n$ such that it has an $\tilde{\mathcal{O}}$-pseudoconvex open neighborhood Ω and satisfies the conditions $H^p(K, \underset{\sim}{\mathcal{O}})=0$ $(p\geqq 1)$. Then we have $H_K^p(\Omega, \tilde{\mathcal{O}})=0$ for $p\neq n$ and isomorphisms $H_K^n(\Omega, \tilde{\mathcal{O}})\cong H^{n-1}(\Omega\backslash K, \tilde{\mathcal{O}})\cong \underset{\sim}{\mathcal{O}}(K)'$.*

Remark. If a compact set K in $\tilde{\boldsymbol{C}}^n$ has a fundamental system of $\tilde{\mathcal{O}}$-pseudoconvex open neighborhoods, it satisfies the assumptions in Theorem 1.5.1.

Proof. It goes in the same way as Ito and Nagamachi [13].

By the excision theorem, $H_K^p(\Omega, \tilde{\mathcal{O}})$ is independent of an open neighborhood Ω of K. So, we may assume that Ω is an $\tilde{\mathcal{O}}$-pseudoconvex open neighborhood in the assumptions in this theorem. Then in the long exact sequence of cohomology groups (cf. Komatsu [21], Theorem II.3.2):

$$\begin{aligned}
&0 \longrightarrow H_K^0(\Omega, \tilde{\mathcal{O}}) \longrightarrow H^0(\Omega, \tilde{\mathcal{O}}) \longrightarrow H^0(\Omega\backslash K, \tilde{\mathcal{O}})\\
&\longrightarrow H_K^1(\Omega, \tilde{\mathcal{O}}) \longrightarrow H^1(\Omega, \tilde{\mathcal{O}}) \longrightarrow H^1(\Omega\backslash K, \tilde{\mathcal{O}})\\
&\longrightarrow \cdots\cdots\\
&\longrightarrow H_K^n(\Omega, \tilde{\mathcal{O}}) \longrightarrow H^n(\Omega, \tilde{\mathcal{O}}) \longrightarrow H^n(\Omega\backslash K, \tilde{\mathcal{O}}) \longrightarrow \cdots\cdots,
\end{aligned}$$

we have $H^p(\Omega, \tilde{\mathcal{O}})=0$ for $p\geqq 1$ and $H_K^0(\Omega, \tilde{\mathcal{O}})=0$ by the unique continuation theorem. Hence we have isomorphisms

$$\begin{aligned}
&H_K^1(\Omega, \tilde{\mathcal{O}})\cong \tilde{\mathcal{O}}(\Omega\backslash K)/\tilde{\mathcal{O}}(\Omega),\\
&H_K^p(\Omega, \tilde{\mathcal{O}})\cong H^{p-1}(\Omega\backslash K, \tilde{\mathcal{O}}),\ p\geqq 2.
\end{aligned}$$

We also have the long exact sequence of cohomology groups with compact support (cf. Komatsu [21], Theorem II.3.15):

$$\begin{aligned}
&0 \longrightarrow H_c^0(\Omega\backslash K, \underset{\sim}{\mathcal{O}}) \longrightarrow H_c^0(\Omega, \underset{\sim}{\mathcal{O}}) \longrightarrow H^0(K, \underset{\sim}{\mathcal{O}})\\
&\longrightarrow H_c^1(\Omega\backslash K, \underset{\sim}{\mathcal{O}}) \longrightarrow H_c^1(\Omega, \underset{\sim}{\mathcal{O}}) \longrightarrow H^1(K, \underset{\sim}{\mathcal{O}})\\
&\longrightarrow \cdots\cdots\cdots
\end{aligned}$$

$$\longrightarrow H_c^p(\Omega\backslash K, \underset{\sim}{\mathcal{O}}) \longrightarrow H_c^p(\Omega, \underset{\sim}{\mathcal{O}}) \longrightarrow H^p(K, \underset{\sim}{\mathcal{O}}) \longrightarrow \cdots\cdots.$$

Here $H^p(K, \underset{\sim}{\mathcal{O}})=0$ $(p\geqq 1)$ by the assumption on K. Therefore we obtain the isomorphisms

$$\underset{\sim}{\mathcal{O}}(K)\cong H_c^1(\Omega\backslash K, \underset{\sim}{\mathcal{O}}),$$

$$H_c^p(\Omega, \underset{\sim}{\mathcal{O}})\cong H_c^p(\Omega\backslash K, \underset{\sim}{\mathcal{O}}),\ p\geqq 2.$$

By the theorem 1.4.1, we have $H_c^p(\Omega, \underset{\sim}{\mathcal{O}})=0$ $(p\neq n)$. Thus we have the following isomorphisms

$$H_c^p(\Omega\backslash K, \underset{\sim}{\mathcal{O}})=0, \quad p\neq 1,\ n,$$

$$H_c^n(\Omega\backslash K, \underset{\sim}{\mathcal{O}})\cong \tilde{\mathcal{O}}(\Omega)'.$$

Now we consider the following dual complexes:

$$\begin{array}{ccccccccccc}
0 \longrightarrow & \tilde{\mathscr{L}}^{0,0}(\Omega\backslash K) & \xrightarrow{\bar\partial_0} & \tilde{\mathscr{L}}^{0,1}(\Omega\backslash K) & \xrightarrow{\bar\partial_1} & \cdots\cdots & \xrightarrow{\bar\partial_{n-2}} & \tilde{\mathscr{L}}^{0,n-1}(\Omega\backslash K) & \longrightarrow & (*) \\
& \updownarrow & & \updownarrow & & & & \updownarrow & & \\
0 \longleftarrow & \underset{\sim}{\mathscr{L}}_c^{0,n}(\Omega\backslash K) & \xrightarrow{-\bar\partial_{n-1}} & \underset{\sim}{\mathscr{L}}_c^{0,n-1}(\Omega\backslash K) & \xleftarrow{-\bar\partial_{n-2}} & \cdots\cdots & \xleftarrow{-\bar\partial_1} & \underset{\sim}{\mathscr{L}}_c^{0,1}(\Omega\backslash K) & \longleftarrow & (**)
\end{array}$$

$$\begin{array}{cccc}
(*) & \xrightarrow{\bar\partial_{n-1}} & \tilde{\mathscr{L}}^{0,n}(\Omega\backslash K) & \longrightarrow 0 \\
& & \updownarrow & \\
(**) & \xleftarrow{-\bar\partial_0} & \underset{\sim}{\mathscr{L}}_c^{0,0}(\Omega\backslash K) & \longleftarrow 0.
\end{array}$$

Then, since $H_c^p(\Omega\backslash K, \underset{\sim}{\mathcal{O}})=0$ $(p\neq 1,\ n)$, the range of $-\bar\partial_j=(\bar\partial_{n-j-1})'$ is closed except for $j=0,\ n-1$. However $\bar\partial_{n-1}$ is of closed range by Malgrange's Theorem. Hence, by the closed range theorem, $-\bar\partial_0$ is of closed range (cf. Komatsu [20], Theorem 19, p. 381).

In order to prove the closedness of the range of $-\bar\partial_{n-1}$, we consider the following diagram:

$$\begin{array}{ccccc}
0 \longleftarrow & \underset{\sim}{\mathscr{L}}_c^{0,n}(\Omega\backslash K) & \xleftarrow{-\bar\partial_{n-1}^{\Omega\backslash K}} & \underset{\sim}{\mathscr{L}}_c^{0,n-1}(\Omega\backslash K) \\
& \cap\ \big\downarrow i & & \cap\ \big\downarrow \\
0 \longleftarrow & \underset{\sim}{\mathscr{L}}_c^{0,n}(\Omega) & \xleftarrow{-\bar\partial_{n-1}^{\Omega}} & \underset{\sim}{\mathscr{L}}_c^{0,n-1}(\Omega),
\end{array}$$

where the map i is the natural injection. However, in the dual complexes for Ω, $\bar\partial_0^\Omega$ is of closed range since $H^1(\Omega, \tilde{\mathcal{O}})=0$. Thus, by the closed range theorem, $\mathrm{Im}\,(-\bar\partial_{n-1}^{\Omega\backslash K})=i^{-1}(\mathrm{Im}\,(-\bar\partial_{n-1}^{\Omega}))$ is closed. Therefore all $-\bar\partial_j^{\Omega\backslash K}$ are of closed range. Hence by the Serre-Komatsu duality theorem, we have the isomorphisms $[H^p(\Omega\backslash K, \tilde{\mathcal{O}})]'\cong H_c^{n-p}(\Omega\backslash K, \underset{\sim}{\mathcal{O}})$, for $0\leqq p\leqq n$. Hence we have $\tilde{\mathcal{O}}(\Omega\backslash K)'\cong H_c^n(\Omega\backslash K, \underset{\sim}{\mathcal{O}})\cong H_c^n(\Omega, \underset{\sim}{\mathcal{O}})\cong\tilde{\mathcal{O}}(\Omega)'$. Here $\tilde{\mathcal{O}}(\Omega\backslash K)$ and $\tilde{\mathcal{O}}(\Omega)$ are both FS spaces, a posteriori, reflexive. Hence we have the isomorphism $\tilde{\mathcal{O}}(\Omega)\cong\tilde{\mathcal{O}}(\Omega\backslash K)$. Thus $H_K^1(\Omega, \tilde{\mathcal{O}})\cong\tilde{\mathcal{O}}(\Omega\backslash K)/\tilde{\mathcal{O}}(\Omega)=0$. Hence, for $p\geqq 2$, $p\neq n$, we have $0=H_c^{n-p+1}(\Omega, \underset{\sim}{\mathcal{O}})\cong H_c^{n-p+1}(\Omega\backslash K, \underset{\sim}{\mathcal{O}})\cong$

$[H^{p-1}(\Omega\backslash K, \tilde{\mathcal{O}})]' \cong [H^p_K(\Omega, \tilde{\mathcal{O}})]'$. Thus $H^p_K(\Omega, \tilde{\mathcal{O}})=0$. In the case $p=n$, we have the isomorphisms $[H^n_K(\Omega, \tilde{\mathcal{O}})]' \cong [H^{n-1}(\Omega\backslash K, \tilde{\mathcal{O}})]' \cong H^1_c(\Omega\backslash K, \underset{\sim}{\mathcal{O}}) \cong H^0(K, \underset{\sim}{\mathcal{O}}) \cong \underset{\sim}{\mathcal{O}}(K)$. Since $\underset{\sim}{\mathcal{O}}(K)$ is a DFS space, it follows from the Serre-Komatsu duality theorem that the above isomorphisms are topological isomorphisms. Hence we have the isomorphism $H^n_K(\Omega, \tilde{\mathcal{O}}) \cong \underset{\sim}{\mathcal{O}}(K)'$. Q. E. D.

1.6. Sato's Theorem

In this section we will prove the pure-codimensionality of $\boldsymbol{D}^n$ with respect to $\tilde{\mathcal{O}}$. Then we will realize Fourier hyperfunctions as "boundary values" of slowly increasing holomorphic functions or as (relative) cohomology classes of slowly increasing holomorphic functions.

Theorem 1.6.1 (Sato's Theorem). *Let Ω be an open set in $\boldsymbol{D}^n$ and V an open set in $\tilde{\boldsymbol{C}}^n$ which contains Ω as its closed subsets. Then we have the following*

(1) *The relative cohomology groups $H^p_\Omega(V, \tilde{\mathcal{O}})$ are zero for $p \neq n$.*

(2) *The presheaf over $\boldsymbol{D}^n$*

$$\Omega \longrightarrow H^n_\Omega(V, \tilde{\mathcal{O}})$$

is a flabby sheaf.

(3) *This sheaf (2) is isomorphic to the sheaf $\mathcal{R}$ of Fourier hyperfunctions.*

Proof. (1) It goes in the same way as Kawai [19], p. 482.

(2) By Malgrange's Theorem, we can conclude that flabby $\dim \tilde{\mathcal{O}} \leqq n$. Thus, by (1) and by the theorem II.3.24 of Komatsu [21], we have the conclusion.

(3) Consider the following exact sequence of relative cohomology groups

$$\begin{aligned} 0 &\longrightarrow H^0_{\partial\Omega}(V, \tilde{\mathcal{O}}) \longrightarrow H^0_{\Omega^a}(V, \tilde{\mathcal{O}}) \longrightarrow H^0_\Omega(V, \tilde{\mathcal{O}}) \\ &\longrightarrow H^1_{\partial\Omega}(V, \tilde{\mathcal{O}}) \longrightarrow \cdots\cdots \longrightarrow H^{n-1}_\Omega(V, \tilde{\mathcal{O}}) \\ &\longrightarrow H^n_{\partial\Omega}(V, \tilde{\mathcal{O}}) \longrightarrow H^n_{\Omega^a}(V, \tilde{\mathcal{O}}) \longrightarrow H^n_\Omega(V, \tilde{\mathcal{O}}) \\ &\longrightarrow H^{n+1}_{\partial\Omega}(V, \tilde{\mathcal{O}}) \longrightarrow \cdots\cdots . \end{aligned}$$

(Here Ω^a denotes the closure of Ω). Then, by (1) and by Martineau-Harvey's Theorem, we have $H^{n-1}_\Omega(V, \tilde{\mathcal{O}})=0$, $H^{n+1}_{\partial\Omega}(V, \tilde{\mathcal{O}})=0$. Thus we have the exact sequence

$$0 \longrightarrow H^n_{\partial\Omega}(V, \tilde{\mathcal{O}}) \longrightarrow H^n_{\Omega^a}(V, \tilde{\mathcal{O}}) \longrightarrow H^n_\Omega(V, \tilde{\mathcal{O}}) \longrightarrow 0.$$

Since, by Martineau-Harvey's Theorem, we have isomorphisms

$$H^n_{\partial\Omega}(V, \tilde{\mathcal{O}}) \cong \underset{\sim}{\mathcal{A}}(\partial\Omega)', \quad H^n_{\Omega^a}(V, \tilde{\mathcal{O}}) \cong \underset{\sim}{\mathcal{A}}(\Omega^a)',$$

we obtain the isomorphism

$$H_{\Omega}^{n}(V, \tilde{\mathcal{O}}) \cong \mathscr{A}(\Omega^{a})'/\mathscr{A}(\partial\Omega)' = \mathscr{R}(\Omega).$$

Thus the sheaf $\Omega \rightarrow H_{\Omega}^{n}(V, \tilde{\mathcal{O}})$ is isomorphic to the sheaf $\mathscr{R}$ of Fourier hyperfunctions over $\boldsymbol{D}^{n}$. Q.E.D.

Let Ω be an open set in $\boldsymbol{D}^{n}$. Then there exists an $\tilde{\mathcal{O}}$-pseudoconvex open neighborhood V of Ω such that $V \cap \boldsymbol{D}^{n} = \Omega$(cf. Kawai [19], Theorem 2.1.6.). We put $V_0 = V$ and $V_j = V \backslash \{z \in V;\ \mathrm{Im}\, z_j = 0\}^{a}$, $j = 1, 2, \ldots, n$. Then $\mathfrak{V} = \{V_0, V_1, \ldots, V_n\}$ and $\mathfrak{V}' = \{V_1, \ldots, V_n\}$ cover V and $V \backslash \Omega$ respectively. Since V_j and their intersections are also $\tilde{\mathcal{O}}$-pseudoconvex open sets, the covering $(\mathfrak{V}, \mathfrak{V}')$ satisfies the conditions of Leray's Theorem (cf. Komatsu [23]). Thus, by Leray's Theorem, we obtain the isomorphism $H_{\Omega}^{n}(V, \tilde{\mathcal{O}}) \cong H^{n}(\mathfrak{V}, \mathfrak{V}', \tilde{\mathcal{O}})$. Since the covering $\mathfrak{V}$ is composed of only $n+1$ open sets V_j $(j = 0, 1, \ldots, n)$, we easily obtain the isomorphisms

$$Z^{n}(\mathfrak{V}, \mathfrak{V}', \tilde{\mathcal{O}}) \cong \tilde{\mathcal{O}}(\cap_j V_j),$$

$$C^{n-1}(\mathfrak{V}, \mathfrak{V}', \tilde{\mathcal{O}}) \cong \bigoplus_{j=1}^{n} (\bigcap_{i \neq j} V_i).$$

Hence we have

$$\delta C^{n-1}(\mathfrak{V}, \mathfrak{V}', \tilde{\mathcal{O}}) \cong \sum_{j=1}^{n} \tilde{\mathcal{O}}(\bigcap_{i \neq j} V_i) | V_1 \cap \cdots\cdots \cap V_n.$$

Thus we have the isomorphisms

$$\begin{aligned} H_{\Omega}^{n}(V, \tilde{\mathcal{O}}) &\cong H^{n}(\mathfrak{V}, \mathfrak{V}', \tilde{\mathcal{O}}) \cong Z^{n}(\mathfrak{V}, \mathfrak{V}', \tilde{\mathcal{O}}) / \delta C^{n-1}(\mathfrak{V}, \mathfrak{V}', \tilde{\mathcal{O}}) \\ &\cong \tilde{\mathcal{O}}(\cap_j V_j) / \sum_{j=1}^{n} \tilde{\mathcal{O}}(\cap_{i \neq j} V_i). \end{aligned}$$

Thus we have the following

Theorem 1.6.2. *We use notations as above. Then we have the isomorphisms*

$$H_{\Omega}^{n}(V, \tilde{\mathcal{O}}) \cong H^{n}(\mathfrak{V}, \mathfrak{V}', \tilde{\mathcal{O}}) \cong \tilde{\mathcal{O}}(\cap_j V_j) / \sum_{j=1}^{n} \tilde{\mathcal{O}}(\cap_{i \neq j} V_i).$$

At last we will realize Fourier analytic functionals with certain compact carrier as (relative) cohomology classes with coefficients in $\tilde{\mathcal{O}}$.

Let K be a compact set in $\tilde{\boldsymbol{C}}^{n}$ of the form $K = K_1 \times \cdots \times K_n$ with compact sets K_j in $\tilde{\boldsymbol{C}}$ $(j = 1, 2, \ldots, n)$. Assume that K admits a fundamental system of $\tilde{\mathcal{O}}$-pseudoconvex open neighborhoods. Then we have

$$H^{p}(K, \underset{\sim}{\mathcal{O}}) = 0 \quad \text{for} \quad p > 0.$$

By virtue of Martineau-Harvey's Theorem, there exists the isomorphism

$$\underset{\sim}{\mathcal{Q}}(K)' \cong H_K^n(\tilde{\boldsymbol{C}}^{\iota}, \tilde{\mathcal{O}}).$$

Further assume that there exists an $\tilde{\mathcal{O}}$-pseudoconvex open neighborhood Ω of K such that

$$\Omega_j = \Omega \cap \{z \in \tilde{\boldsymbol{C}}^n;\ z_j \notin K_j\}$$

is also an $\tilde{\mathcal{O}}$-pseudoconvex open set for $j=1, 2,.., n$. Then $\mathfrak{V} = \{\Omega_0 = \Omega, \Omega_1, .., \Omega_n\}$ and $\mathfrak{V}' = \{\Omega_1, \Omega_2, .., \Omega_n\}$ form acyclic coverings of Ω and $\Omega \backslash K$. Set

$$\Omega \sharp K = \bigcap_{j=1}^{n} \Omega_j,$$

$$\Omega^j = \bigcap_{i \neq j} \Omega_i.$$

Let $\sum_j \tilde{\mathcal{O}}(\Omega^j)$ be the image in $\tilde{\mathcal{O}}(\Omega \sharp K)$ of $\prod_{j=1}^{n} \tilde{\mathcal{O}}(\Omega^j)$ by the mapping

$$(f_j)_{j=1}^n \longrightarrow \sum_{j=1}^{n} (-1)^{j+1} f'_j,$$

where f'_j denotes the restriction of f_j to $\Omega \sharp K$.

Then, by the same method as that of Theorem 1.6.2, we have the isomorphisms

$$\underset{\sim}{\mathcal{Q}}(K)' \cong H_K^n(\tilde{\boldsymbol{C}}^n, \tilde{\mathcal{O}}) \cong H^n(\mathfrak{V}, \mathfrak{V}', \tilde{\mathcal{O}}) \cong \tilde{\mathcal{O}}(\Omega \sharp K) / \sum_j \tilde{\mathcal{O}}(\Omega^j).$$

By the above theorem, we can define the canonical mapping

$$b \colon \tilde{\mathcal{O}}(\Omega \sharp K) \longrightarrow \underset{\sim}{\mathcal{Q}}(K)'$$

whose kernel is $\sum_j \tilde{\mathcal{O}}(\Omega^j)$.

Then we have the following

Theorem 1.6.3. *We use the above notations.* (i) *Let $u \in \underset{\sim}{\mathcal{Q}}(K)'$ and put*

$$\tilde{u}(z) = (2i\pi)^{-n} u_\xi\Big(\prod_{j=1}^{n} (\xi_j - z_j)^{-1} \exp(-(\xi_j - z_j)^2)\Big).$$

Then $\tilde{u} \in \tilde{\mathcal{O}}(\Omega \sharp K)$ and $b(\tilde{u}) = u$ holds.

(ii) *Let $f \in \tilde{\mathcal{O}}(\Omega \sharp K)$ and $g \in \underset{\sim}{\mathcal{Q}}(K)$. Let $\omega = \omega_1 \times \cdots \times \omega_n \subset \Omega$ with open neighborhoods ω_j of K_j in $\tilde{\boldsymbol{C}}$ and $g \in \underset{\sim}{\mathcal{Q}}(\tilde{\omega})$ where $\tilde{\omega}$ is an open neighborhood of ω with $\tilde{\omega} \subset \Omega$. Let Γ_j $(j=1, 2,..., n)$ be regular contours in $\omega_j \cap \boldsymbol{C}$ enclosing once $K_j \cap \boldsymbol{C}$ and oriented in the usual way. Then we have*

$$b(f)(g) = (-1)^n \int_{\Gamma_1} \cdots \int_{\Gamma_n} f(z) g(z) dz_1 \cdots dz_n.$$

Proof. The integral

$$(-1)^n \int_{\Gamma_1} \cdots \int_{\Gamma_n} f(z) g(z) dz_1 \cdots dz_n$$

does not depend on the chosen contours and defines a linear mapping

$$b' : \tilde{\mathcal{O}}(\Omega \sharp K) \longrightarrow \underset{\sim}{\mathcal{O}}(K)' ,$$

which is zero on $\sum_j \tilde{\mathcal{O}}(\Omega^j)$. Hence, in order to prove (ii), it is sufficient to prove that, if $u \in \underset{\sim}{\mathcal{O}}(K)'$, we have

$$b'(\tilde{u}) = u .$$

But

$$b'(\tilde{u})(g) = (-1)^n (2i\pi)^{-n} \int_{\Gamma_1} \cdots \int_{\Gamma_n} u_\xi((\xi - z)^{-1} \exp(-(\xi - z)^2)) g(z) dz$$

$$= u_\xi((2i\pi)^{-n} \int_{\Gamma_1} \cdots \int_{\Gamma_n} g(z)(z - \xi)^{-1} \exp(-(z - \xi)^2) dz) = u(g) ,$$

where we write

$$(\xi - z)^{-1} \exp(-(\xi - z)^2) = \prod_{j=1}^{n} (\xi_j - z_j)^{-1} \exp(-(\xi_j - z_j)^2) .$$

This proves (i) and completes the proof. Q. E. D.

Chapter 2. The case of the sheaf ${}^E\tilde{\mathcal{O}}$

2.1. The Dolbeault-Grothendieck resolution of ${}^E\tilde{\mathcal{O}}$

In this section we will recall the soft resolution of ${}^E\tilde{\mathcal{O}}$. Here E denotes a quasi-complete locally convex topological vector space (LCTVS) (always assumed to be Hausdorff) unless the contrary is explicitly mentioned and $\mathcal{T} = \mathcal{T}_E$ denotes the family of continuous seminorms of E defining a locally convex topology on E.

At first we will recall the definition of sheaves ${}^E\tilde{\mathcal{O}}$ and ${}^E\tilde{\mathcal{E}}$ following Ito [10] and Junker [14].

Definition 2.1.1 (The sheaf ${}^E\tilde{\mathcal{O}}$ of germs of slowly increasing E-valued holomorphic functions over $\tilde{\boldsymbol{C}}^n$). *We define the sheaf ${}^E\tilde{\mathcal{O}}$ to be the sheafification of the presheaf $\{\tilde{\mathcal{O}}(\Omega; E)\}$, where for an open set Ω in $\tilde{\boldsymbol{C}}^n$, the module $\tilde{\mathcal{O}}(\Omega; E)$ is defined as follows:*

$\tilde{\mathcal{O}}(\Omega; E) = \{f \in \mathcal{O}(\Omega \cap \boldsymbol{C}^n; E)$; *for any positive* ε *and any compact set* K *in* Ω *and any* $q \in \mathcal{T}$, $\sup\{q(f(z)) e(-\varepsilon|z|); z \in K \cap \boldsymbol{C}^n\} < \infty$ *holds*$\}$.

Here we denote by $\mathcal{O}(\Omega \cap \boldsymbol{C}^n; E)$ *the space of all E-valued holomorphic functions on the open set* $\Omega \cap \boldsymbol{C}^n$ *in* $\boldsymbol{C}^n$.

We call this sheaf ${}^E\tilde{\mathcal{O}}$ *the sheaf of germs of slowly increasing E-valued holomorphic functions.*

Definition 2.1.2 (The sheaf ${}^E\tilde{\mathscr{E}}$ of germs of slowly increasing E-valued C^∞-functions). *We define* ${}^E\tilde{\mathscr{E}}$ *to be the sheafification of the presheaf* $\{\tilde{\mathscr{E}}(\Omega; E)\}$, *where, for an open set* Ω *in* $\tilde{\boldsymbol{C}}^n$, *the module* $\tilde{\mathscr{E}}(\Omega; E)$ *is defined as follows*:

$\tilde{\mathscr{E}}(\Omega; E)=\{f \in \mathscr{E}(\Omega \cap \boldsymbol{C}^n; E)$; *for any positive* ε *and any compact set* K *in* Ω *and any* $\alpha \in \bar{\boldsymbol{N}}^{2n}$ *and any* $q \in \mathscr{T}$, $\sup\{q(f^{(\alpha)}(z))e(-\varepsilon|z|); z \in K \cap \boldsymbol{C}^n\} < \infty$ *holds*$\}$.

Here $\bar{\boldsymbol{N}}=\boldsymbol{N} \cup \{0\}$ *and* $\mathscr{E}(\Omega \cap \boldsymbol{C}^n; E)$ *is the space of E-valued* C^∞*-functions on the open set* $\Omega \cap \boldsymbol{C}^n$.

Then the sheaf ${}^E\tilde{\mathscr{E}}$ is soft, and we have the following

Theorem 2.1.3 (The Dolbeault-Grothendieck resolution of ${}^E\tilde{\mathcal{O}}^p$). *Let E be a quasi-complete* LCTVS. *Then the sequence of sheaves over* $\tilde{\boldsymbol{C}}^n$

$$0 \longrightarrow {}^E\tilde{\mathcal{O}}^p \longrightarrow {}^E\tilde{\mathscr{E}}^{p,0} \xrightarrow{\bar{\partial}} {}^E\tilde{\mathscr{E}}^{p,1} \xrightarrow{\bar{\partial}} \cdots\cdots \xrightarrow{\bar{\partial}} {}^E\tilde{\mathscr{E}}^{p,n} \longrightarrow 0$$

is exact. Here we denote by $\mathscr{F}^{p,q}$ *the sheaf of germs of differential forms of type* (p, q) *with coefficients in a sheaf* $\mathscr{F}$ *and put* $\mathscr{F}^p=\mathscr{F}^{p,0}$.

Proof. See Ito [10], Theorem 3.1, p. 989. Q.E.D.

Corollary. *For an open set* Ω *in* $\tilde{\boldsymbol{C}}^n$, *we have the following isomorphism*:

$$H^q(\Omega, {}^E\tilde{\mathcal{O}}^p) \cong \{f \in \tilde{\mathscr{E}}^{p,q}(\Omega; E); \bar{\partial}f=0\}/\{\bar{\partial}g; g \in \tilde{\mathscr{E}}^{p,q-1}(\Omega; E)\}, \quad (p \geqq 0, q \geqq 1).$$

Proof. It follows from Theorem 2.1.3 and Komatsu [21], Theorems II.2.9 and II.2.19. Q.E.D.

2.2. The Oka-Cartan-Kawai Theorem B

We will recall the Oka-Cartan-Kawai Theorem B for the sheaf ${}^E\tilde{\mathcal{O}}$.

Theorem 2.2.1 (The Oka-Cartan-Kawai Theorem B). *Let E be a Fréchet space. For any* $\tilde{\mathcal{O}}$*-pseudoconvex open set* Ω *in* $\tilde{\boldsymbol{C}}^n$, *we have* $H^q(\Omega, {}^E\tilde{\mathcal{O}}^p)=0$ *for* $p \geqq 0$ *and* $q \geqq 1$.

Proof. See Ito [10], p. 992 or Junker [15], p. 33. Q.E.D.

Corollary. *Let E be a Fréchet space and* Ω *an* $\tilde{\mathcal{O}}$*-pseudoconvex open set. Then* *the equation* $\bar{\partial}u=f$ *has a solution* $u\in\tilde{\mathscr{E}}^{p,q}(\Omega;E)$ *for every* $f\in\tilde{\mathscr{E}}^{p,q+1}(\Omega;E)$ *such that* $\bar{\partial}f=0$. *Here* p, q *are nonnegative integers.*

Proof. It follows from Theorem 2.2.1 and Corollary to Theorem 2.1.3.

Q. E. D.

2.3. Malgrange's Theorem

The remainder parts of this chapter go well along the line of Junker [15]. So we briefly summarize them for unification. In the following of this chapter E is assumed to be a Fréchet space.

Theorem 2.3.1. *Let* Ω *be an open set in* $\tilde{\boldsymbol{C}}^n$. *Then we have* $H^n(\Omega, {}^E\tilde{\mathcal{O}})=0$.

Proof. See Junker [15], p. 34. Q. E. D.

Corollary. Flabby dim ${}^E\tilde{\mathcal{O}}\leqq n$.

2.4. Serre's duality theorem

Theorem 2.4.1. *Let* Ω *be an open set in* $\tilde{\boldsymbol{C}}^n$ *such that* $\dim H^p(\Omega, \tilde{\mathcal{O}})<\infty$ *holds* $(p\geqq 1)$. *Then we have the isomorphism* $H^p(\Omega, {}^E\tilde{\mathcal{O}})\cong L(H_c^{n-p}(\Omega, \underset{\sim}{\mathcal{O}}); E)$, $0\leqq p\leqq n$.

Proof. By Junker [15], Lemma 3.5, we have the isomorphism $H^p(\Omega, {}^E\tilde{\mathcal{O}})\cong H^p(\Omega, \tilde{\mathcal{O}})\hat{\otimes}_\pi E$. Then, by Theorem 1.4.1, we have the following isomorphisms

$$H^p(\Omega, {}^E\tilde{\mathcal{O}})\cong H^p(\Omega, \tilde{\mathcal{O}})\hat{\otimes}_\pi E\cong [H_c^{n-p}(\Omega, \underset{\sim}{\mathcal{O}})]'\hat{\otimes}_\pi E\cong L(H_c^{n-p}(\Omega, \underset{\sim}{\mathcal{O}}); E)$$

Q. E. D.

2.5. Martineau-Harvey's Theorem

Theorem 2.5.1. *Let* K *be a compact set in* $\tilde{\boldsymbol{C}}^n$ *such that it has an* $\tilde{\mathcal{O}}$*-pseudoconvex open neighborhood* Ω *and satisfies the conditions* $H^p(K, \underset{\sim}{\mathcal{O}})=0$ $(p\geqq 1)$. *Then we have* $H_K^p(\Omega, {}^E\tilde{\mathcal{O}})=0$ *for* $p\neq n$ *and isomorphisms* $H_K^n(\Omega, {}^E\tilde{\mathcal{O}})\cong H^{n-1}(\Omega\backslash K, {}^E\tilde{\mathcal{O}})\cong L(\underset{\sim}{\mathcal{O}}(K); E)$.

Proof. We can assume that Ω is an $\tilde{\mathcal{O}}$-pseudoconvex open neighborhood of K. Then, in the long exact sequence of cohomology groups (cf. Komatsu [21], Theorem II.3.2):

$$\begin{aligned}0 &\longrightarrow H_K^0(\Omega, {}^E\tilde{\mathcal{O}}) \longrightarrow H^0(\Omega, {}^E\tilde{\mathcal{O}}) \longrightarrow H^0(\Omega\backslash K, {}^E\tilde{\mathcal{O}})\\ &\longrightarrow H_K^1(\Omega, {}^E\tilde{\mathcal{O}}) \longrightarrow H^1(\Omega, {}^E\tilde{\mathcal{O}}) \longrightarrow H^1(\Omega\backslash K, {}^E\tilde{\mathcal{O}})\\ &\longrightarrow \cdots\cdots\\ &\longrightarrow H_K^n(\Omega, {}^E\tilde{\mathcal{O}}) \longrightarrow H^n(\Omega, {}^E\tilde{\mathcal{O}}) \longrightarrow H^n(\Omega\backslash K, {}^E\tilde{\mathcal{O}}) \longrightarrow \cdots\cdots,\end{aligned}$$

we have $H^p(\Omega, {}^E\tilde{\mathcal{O}})=0$ for $p\geqq 1$ and $H^0_K(\Omega, {}^E\tilde{\mathcal{O}})=0$ by the unique continuation theorem. Hence we have isomorphisms

$$H^1_K(\Omega, {}^E\tilde{\mathcal{O}})\cong\tilde{\mathcal{O}}(\Omega\backslash K; E)/\tilde{\mathcal{O}}(\Omega; E),$$

$$H^p_K(\Omega, {}^E\tilde{\mathcal{O}})\cong H^{p-1}(\Omega\backslash K, {}^E\tilde{\mathcal{O}}),\ p\geqq 2.$$

But, by Junker [15], Lemma 3.5, we have isomorphisms $H^p(V, {}^E\tilde{\mathcal{O}})\cong H^p(V, \tilde{\mathcal{O}})\hat{\otimes}_\pi E$, $0\leqq p\leqq n$, for any open set V in $\tilde{\boldsymbol{C}}^n$. So that, by Theorem 1.5.1, we have isomorphisms

$$H^p_K(\Omega, {}^E\tilde{\mathcal{O}})\cong H^p_K(\Omega, \tilde{\mathcal{O}})\hat{\otimes}_\pi E=0 \quad \text{for } p\neq n,$$

and

$$H^n_K(\Omega, {}^E\tilde{\mathcal{O}})\cong H^{n-1}(\Omega\backslash K, {}^E\tilde{\mathcal{O}})\cong H^{n-1}(\Omega\backslash K, \tilde{\mathcal{O}})\hat{\otimes}_\pi E\cong H^n_K(\Omega, \tilde{\mathcal{O}})\hat{\otimes}_\pi E$$
$$\cong\underset{\sim}{\mathcal{O}}(K)'\hat{\otimes}_\pi E\cong L(\underset{\sim}{\mathcal{O}}(K); E).$$

Q.E.D.

2.6. Sato's Theorem

In this section we will prove the pure-codimensionality of $\boldsymbol{D}^n$ with respect to ${}^E\tilde{\mathcal{O}}$. Then we will realize E-valued Fourier hyperfunctions as "boundary values" of E-valued slowly increasing holomorphic functions or as (relative) cohomology classes of E-valued slowly increasing holomorphic functions.

Theorem 2.6.1 (Sato's Theorem). *Let Ω be an open set in $\boldsymbol{D}^n$ and V an open set in $\tilde{\boldsymbol{C}}^n$ which contains Ω as its closed subsets. Then we have the following*

(1) *The relative cohomology groups $H^p_\Omega(V, {}^E\tilde{\mathcal{O}})$ are zero for $p\neq n$.*

(2) *The presheaf over $\boldsymbol{D}^n$*

$$\Omega \longrightarrow H^n_\Omega(V, {}^E\tilde{\mathcal{O}})$$

is a flabby sheaf.

(3) *This sheaf (2) is isomorphic to the sheaf ${}^E\mathcal{R}$ of E-valued Fourier hyperfunctions.*

Proof. (1) By the excision theorem, we may assume that V is an $\tilde{\mathcal{O}}$-pseudoconvex open set in $\tilde{\boldsymbol{C}}^n$. Consider the following exact sequence of relative cohomology groups

$$0 \longrightarrow H^0_{\partial\Omega}(V, {}^E\tilde{\mathcal{O}}) \longrightarrow H^0_{\Omega^a}(V, {}^E\tilde{\mathcal{O}}) \longrightarrow H^0_\Omega(V, {}^E\tilde{\mathcal{O}})$$
$$\longrightarrow H^1_{\partial\Omega}(V, {}^E\tilde{\mathcal{O}}) \longrightarrow \quad \cdots\cdots \quad \longrightarrow H^{n-1}_\Omega(V, {}^E\tilde{\mathcal{O}})$$

$$\longrightarrow H^n_{\partial\Omega}(V, {}^E\tilde{\mathcal{O}}) \longrightarrow H^n_{\Omega^a}(V, {}^E\tilde{\mathcal{O}}) \longrightarrow H^n_\Omega(V, {}^E\tilde{\mathcal{O}})$$
$$\longrightarrow H^{n+1}_{\partial\Omega}(V, {}^E\tilde{\mathcal{O}}) \longrightarrow \cdots\cdots.$$

By Theorems 1.1.8 and 2.5.1, we may conclude that $H^p_{\partial\Omega}(V, {}^E\tilde{\mathcal{O}})=H^p_{\Omega^a}(V, {}^E\tilde{\mathcal{O}})=0$ for $p\neq n$. So that, we have $H^p_\Omega(V, {}^E\tilde{\mathcal{O}})=0$ for $p\neq n-1$, n. On the other hand, by Theorems 1.1.8 and 2.5.1, we also have the exact sequence

$$0 \longrightarrow H^{n-1}_\Omega(V, {}^E\tilde{\mathcal{O}}) \longrightarrow L(\mathscr{A}(\partial\Omega); E) \xrightarrow{j} L(\mathscr{A}(\Omega^a); E).$$

Since j is injective, we have $H^{n-1}_\Omega(V, {}^E\tilde{\mathcal{O}})=0$.

(2) By Malgrange's Theorem, we can conclude that flabby dim ${}^E\tilde{\mathcal{O}}\leqq n$. Thus by (1) and by the theorem II.3.24 of Komatsu [21], we have the conclusion.

(3) By the proof of (1), we have the exact sequence

$$0 \longrightarrow H^n_{\partial\Omega}(V, {}^E\tilde{\mathcal{O}}) \longrightarrow H^n_{\Omega^a}(V, {}^E\tilde{\mathcal{O}}) \longrightarrow H^n_\Omega(V, {}^E\tilde{\mathcal{O}}) \longrightarrow 0.$$

Since, by Martineau-Harvey's Theorem, we have isomorphisms

$$H^n_{\partial\Omega}(V, {}^E\tilde{\mathcal{O}})\cong L(\mathscr{A}(\partial\Omega); E),$$
$$H^n_{\Omega^a}(V, {}^E\tilde{\mathcal{O}})\cong L(\mathscr{A}(\Omega^a); E),$$

we obtain the isomorphism

$$H^n_\Omega(V, {}^E\tilde{\mathcal{O}})\cong L(\mathscr{A}(\Omega^a); E)/L(\mathscr{A}(\partial\Omega); E)=\mathscr{R}(\Omega; E).$$

Thus the sheaf $\Omega\to H^n_\Omega(V, {}^E\tilde{\mathcal{O}})$ is isomorphic to the sheaf ${}^E\mathscr{R}$ of E-valued Fourier hyperfunctions over $\boldsymbol{D}^n$. Q.E.D.

In the same notations as in Theorem 1.6.2, we have the following

Theorem 2.6.2. *$H^n_\Omega(V, {}^E\tilde{\mathcal{O}})\cong H^n(\mathfrak{B}, \mathfrak{B}', {}^E\tilde{\mathcal{O}})\cong\tilde{\mathcal{O}}(\cap_j V_j; E)/\sum_{j=1}^n \tilde{\mathcal{O}}(\cap_{i\neq j} V_i; E)$ holds.*

At last we will realize Fourier analytic linear mappings with certain compact carrier as (relative) cohomology classes with coefficients in ${}^E\tilde{\mathcal{O}}$.

Let K be a compact set in $\tilde{\boldsymbol{C}}^n$ of the form $K=K_1\times\cdots\times K_n$ with compact sets K_j in $\tilde{\boldsymbol{C}}$ $(j=1, 2,\ldots, n)$. Assume that K admits a fundamental system of $\tilde{\mathcal{O}}$-pseudoconvex open neighborhoods. Then we have

$$H^p(K, \underset{\sim}{\mathcal{O}})=0 \qquad \text{for} \quad p>0.$$

By virtue of Martineau-Harvey's Theorem, there exists the isomorphism

$$\underset{\sim}{\mathcal{O}}'(K; E)\cong H^n_K(\tilde{\boldsymbol{C}}^n, {}^E\tilde{\mathcal{O}}).$$

Further assume that there exists an $\tilde{\mathcal{O}}$-pseudoconvex open neighborhood Ω of

K such that

$$\Omega_j=\Omega\cap\{z\in\tilde{C}^n;\ z_j\notin K_j\}$$

is also an $\tilde{\mathcal{O}}$-pseudoconvex open set for $j=1, 2,\dots, n$. Then $\mathfrak{V}=\{\Omega_0=\Omega, \Omega_1,\dots, \Omega_n\}$ and $\mathfrak{V}'=\{\Omega_1, \Omega_2,\dots, \Omega_n\}$ form acyclic coverings of Ω and $\Omega\backslash K$. Set

$$\Omega\sharp K=\bigcap_{j=1}^{n}\Omega_j,$$

$$\Omega^j=\bigcap_{i\neq j}\Omega_i.$$

Let $\sum_j \tilde{\mathcal{O}}(\Omega^j; E)$ be the image in $\tilde{\mathcal{O}}(\Omega\sharp K; E)$ of $\prod_{j=1}^{n}\tilde{\mathcal{O}}(\Omega^j; E)$ by the mapping

$$(f_j)_{j=1}^n \longrightarrow \sum_{j=1}^{n}(-1)^{j+1}f'_j,$$

where f'_j denotes the restriction of f_j to $\Omega\sharp K$.

Then, by the same method as that of Theorem 1.6.2, we have the following

Theorem 2.6.3. *We use the notations as above. Then we have the isomorphisms*

$$\underset{\sim}{\mathcal{Q}}'(K; E)\cong H_K^n(\tilde{C}^n, {}^E\tilde{\mathcal{O}})\cong H^n(\mathfrak{V}, \mathfrak{V}', {}^E\tilde{\mathcal{O}})\cong\tilde{\mathcal{O}}(\Omega\sharp K; E)/\sum_j\tilde{\mathcal{O}}(\Omega^j; E).$$

By the above theorem, we can define the canonical mapping

$$b\colon \tilde{\mathcal{O}}(\Omega\sharp K; E)\longrightarrow\tilde{\mathcal{O}}'(K; E)$$

whose kernel is $\sum_j\tilde{\mathcal{O}}(\Omega^j; E)$

Then we have the following

Theorem 2.6.4. *We use the above notations.*

(i) *Let $u\in\underset{\sim}{\mathcal{Q}}'(K; E)$ and put*

$$\tilde{u}(z)=(2i\pi)^{-n}u_\xi((\xi-z)^{-1}\exp(-(\xi-z)^2)).$$

Then $\tilde{u}\in\tilde{\mathcal{O}}(\Omega\sharp K; E)$ and $b(\tilde{u})=u$ holds.

(ii) *Let $f\in\tilde{\mathcal{O}}(\Omega\sharp K; E)$ and $g\in\underset{\sim}{\mathcal{Q}}(K)$. Let $\omega=\omega_1\times\cdots\times\omega_n\subset\Omega$ with open neighborhoods ω_j of K_j in $\tilde{C}$ and $g\in\underset{\sim}{\mathcal{Q}}(\tilde{\omega})$ where $\tilde{\omega}$ is an open neighborhood of ω with $\tilde{\omega}\subset\Omega$. Let Γ_j $(j=1, 2,\dots, n)$ be regular contours in $\omega_j\cap C$ enclosing once $K_j\cap C$ and oriented in the usual way. Then we have*

$$b(f)(g)=(-1)^n\int_{\Gamma_1}\cdots\int_{\Gamma_n}f(z)g(z)dz_1\cdots dz_n.$$

Proof. It goes in the same way as that of Theorem 1.6.3. Q. E. D.

Chapter 3. Cases of sheaves $\tilde{\tilde{\mathcal{O}}}$, $\tilde{\tilde{\mathcal{A}}}$, $\underset{\approx}{\mathcal{O}}$ and $\underset{\approx}{\mathcal{A}}$

3.1. The Oka-Cartan-Kawai Theorem B

In this section we will prove the Oka-Cartan-Kawai Theorem B for the sheaves $\tilde{\tilde{\mathcal{O}}}$ and $\underset{\approx}{\mathcal{O}}$.

We denote by $\boldsymbol{D}^n$ the radial compactification of $\boldsymbol{R}^n$ in the sense of Kawai (see Kawai [19], Definition 1.1.1) and by $\boldsymbol{E}^n$ the radial compactification of $\boldsymbol{C}^n$ considering it as $\boldsymbol{R}^{2n}$.

Definition 3.1.1 (The sheaf $\tilde{\tilde{\mathcal{O}}}$ of germs of slowly increasing holomorphic functions). *We define $\tilde{\tilde{\mathcal{O}}}$ to be the sheafification of the presheaf $\{\tilde{\tilde{\mathcal{O}}}(\Omega);\ \Omega\subset\boldsymbol{E}^n$ open$\}$, where the section module $\tilde{\tilde{\mathcal{O}}}(\Omega)$ on an open set Ω in $\boldsymbol{E}^n$ is the space of all holomorphic functions $f(z)$ on $\Omega\cap\boldsymbol{C}^n$ such that, for any positive number ε and for any compact set K in Ω, the estimate* $\sup\{|f(z)e(-\varepsilon|z|)|;\ z\in K\cap\boldsymbol{C}^n\}<\infty$ *holds.*

Definition 3.1.2 (The sheaf $\underset{\approx}{\mathcal{O}}$ of germs of rapidly decreasing holomorphic functions). *We define $\underset{\approx}{\mathcal{O}}$ to be the sheafification of the presheaf $\{\underset{\approx}{\mathcal{O}}(\Omega);\ \Omega\subset\boldsymbol{E}^n$ open$\}$, where the section module $\underset{\approx}{\mathcal{O}}(\Omega)$ on an open set Ω in $\boldsymbol{E}^n$ is the space of all holomorphic functions $f(z)$ on $\Omega\cap\boldsymbol{C}^n$ such that, for any compact set K in Ω, there exists some positive constant δ so that the estimate* $\sup\{|f(z)e(\delta|z|)|;\ z\in K\cap\boldsymbol{C}^n\}<\infty$ *holds.*

Definition 3.1.3. *An open set V in $\tilde{\boldsymbol{C}}^n$ is said to be an $\tilde{\tilde{\mathcal{O}}}$-pseudoconvex open set if it satisfies the conditions:*

(1) $\sup\{|\operatorname{Im} z|-|\operatorname{Re} z|;\ z\in V\cap\boldsymbol{C}^n\}<\infty$, *where we put* $\operatorname{Re} z=(\operatorname{Re} z_1,\dots,\operatorname{Re} z_n)$ *and* $\operatorname{Im} z=(\operatorname{Im} z_1,\dots,\operatorname{Im} z_n)$.

(2) *There exists a C^∞-plurisubharmonic function $\varphi(z)$ on $V\cap\boldsymbol{C}^n$ having the following two properties:*

(i) *The closure of $V_c=\{z\in V\cap\boldsymbol{C}^n;\ \varphi(z)<c\}$ in $\boldsymbol{E}^n$ is a compact subset of V for any real c.*

(ii) *$\varphi(z)$ is bounded on $L\cap\boldsymbol{C}^n$ for any compact subset L of V.*

Then we can prove the Oka-Cartan-Kawai Theorem B by the same method as in section 1.1.

Theorem 3.1.4 (The Oka-Cartan-Kawai Theorem B). *For any $\tilde{\tilde{\mathcal{O}}}$-pseudoconvex open set V in $\boldsymbol{E}^n$, we have $H^s(V, \tilde{\tilde{\mathcal{O}}}^p)=0$ for $p\geqq 0$ and $s\geqq 1$.*

Proof. Since V is paracompact, $H^s(V, \tilde{\tilde{\mathcal{O}}}^p)$ coincides with the Čech cohomology group. So we have only to prove $\varinjlim_{\mathfrak{U}} H^s(\mathfrak{U}, \tilde{\tilde{\mathcal{O}}}^p)=0$, where $\mathfrak{U}=\{U_j\}_{j\geq 1}$ is a locally finite open covering of V so that $V_j=U_j\cap \boldsymbol{C}^n$ is pseudoconvex. We can choose such a covering of V because V is an $\tilde{\tilde{\mathcal{O}}}$-pseudoconvex open set.

Now we define $C^s(\tilde{\tilde{Z}}{}^{\mathrm{loc}}_{(p,q)}(\{V_j\}))$ to be the set of all cochains $c=\{c_J;\ J=(j_0, j_1,\dots, j_s)\in \boldsymbol{N}^{s+1}\}$ of forms of type (p, q) satisfying the two conditions:

(i) $\bar{\partial}c_J=0$ in $V_J=V_{j_0}\cap V_{j_1}\cap\cdots\cap V_{j_s}$.

(ii) for any positive ε and any finite subset M of $\boldsymbol{N}^{s+1}$, the estimate

$$\sum_{J\in M}\int_{V_J}|c_J|^2 e(-\varepsilon\|z\|)d\lambda<\infty$$

holds, where $d\lambda$ is the Lebesgue measure on $\boldsymbol{C}^n$ and $\|z\|$ denotes the modification of $\sum_{j=1}^{n}|z_j|$ so as to become C^∞ and convex.

Now we will prove the following

Lemma 3.1.5. *If $c\in C^s(\tilde{\tilde{Z}}{}^{\mathrm{loc}}_{(p,q)}(\{V_j\}))$ satisfies the conditions $\delta c=0$, then we can find some $c'\in C^{s-1}(\tilde{\tilde{Z}}{}^{\mathrm{loc}}_{(p,q)}(\{V_j\}))$ such that $\delta c'=c$. Here δ means the coboundary operator.*

If this Lemma is proved, the theorem will follow from this Lemma as the special case where $q=0$ because we can use Cauchy's integral formula to change the L_2-norm to the sup-norm for holomorphic functions.

Proof of Lemma 3.1.5. We denote by $\{\chi_j\}$ the partition of unity subordinate to $\{V_j\}$ and define $b_I=\sum_j \chi_j c_{jI}$ for $I\in \boldsymbol{N}^s$. Since $\delta c=0$, we have $\delta b=c$. So $\delta\bar{\partial}b=0$ because $\bar{\partial}c=0$. Since $\sum\chi_j=1$ and $\chi_j\geq 0$, we have

$$\int_{V_I}|b_I|^2 e(-\varepsilon\|z\|)d\lambda\leq\sum_j\int_{V_I}\chi_j|c_{jI}|^2 e(-\varepsilon\|z\|)d\lambda$$

for any positive number ε by virtue of Cauchy-Schwarz's inequality.

By the assumption of the existence of C^∞ plurisubharmonic function $\varphi(z)$ in Definition 3.1.3, we can find some plurisubharmonic function $\psi(z)$ on $W=V\cap \boldsymbol{C}^n$ which satisfies the following two conditions:

(1) $\sum|\bar{\partial}\chi_j|\leq e(\psi(z))$

(2) $\sup\{\psi(z);\ z\in K\cap \boldsymbol{C}^n\}\leq C_K$ for any $K\Subset W$.

Thus it follows from the condition on c that we have

$$\sum_{I\in N}\int_{V_I}|\bar{\partial}b_I|^2 e(-\varepsilon\|z\|-\psi(z))d\lambda<\infty$$

for any positive number ε and any finite subset N of $\boldsymbol{N}^s$.

Now we consider the case $s=1$. By the fact that $\delta(\bar{\partial}b)=0$, $\bar{\partial}b$ defines a global section f on $W=V\cap \boldsymbol{C}^n$. Then, by Hörmander [4], Theorem 4.4.2, p. 94, we can prove the existence of u such that $\bar{\partial}u=f$ and the estimate

$$\int_{K\cap \boldsymbol{C}^n}|u|^2 e(-\varepsilon\|z\|)(1+|z|^2)^{-2}d\lambda<\infty$$

holds for any positive number ε and any $K\Subset V$.

If we define $c_I'=b_I-u\,|\,V_I$, then $\bar{\partial}c_I'=0$ and $\delta c'=\delta b=c$. Clearly $c'\in C^{s-1}\cdot(\tilde{\tilde{Z}}{}^{\mathrm{loc}}_{(p,q)}(\{V_j\}))$.

Now we go on to the case $s>1$. In this case we use the induction on s. By the induction hypotheses there exists $b'\in C^{s-2}(\tilde{\tilde{Z}}{}^{\mathrm{loc}}_{(p,q+1)}(\{V_j\}))$ such that $\delta b'=\bar{\partial}b$. By virtue of Hörmander [4], Theorem 4.4.2, p. 94, we can also find $b''=\{b_H''\}_{H\in \boldsymbol{N}^{s-1}}$ such that $b_H'=\bar{\partial}b_H''$ and the estimate

$$\sum_{H\in L}\int_{V_H}|b_H''|^2 e(-\varepsilon\|z\|-\psi(z))(1+|z|^2)^{-2}d\lambda<\infty$$

holds for any positive number ε and any finite subset L of $\boldsymbol{N}^{s-1}$. Therefore $c'=b-\delta b''$ satisfies all the required conditions. Q. E. D.

This completes the proof of the theorem. Q. E. D.

Now we will prove the Malgrange theorem for the sheaf $\tilde{\tilde{\mathscr{A}}}$ of germs of slowly increasing real analytic functions. Here we define the sheaf $\tilde{\tilde{\mathscr{A}}}$ to be the restriction of $\tilde{\tilde{\mathcal{O}}}$ to $\boldsymbol{D}^n$: $\tilde{\tilde{\mathscr{A}}}=\tilde{\tilde{\mathcal{O}}}\,|\,\boldsymbol{D}^n$. Then we have the following

Theorem 3.1.6 (Malgrange). *For an arbitrary set Ω in $\boldsymbol{D}^n$, we have $H^s(\Omega,\tilde{\tilde{\mathscr{A}}}{}^p)=0$ for $p\geqq 0$ and $s\geqq 1$.*

Proof. We know, by virtue of Theorem 6.2.1 of Saburi [28], that Ω has a fundamental system $\{\tilde{\Omega}\}$ of $\tilde{\tilde{\mathcal{O}}}$-pseudoconvex open neighborhoods. Then, it follows from the Oka-Cartan-Kawai Theorem B (cf. Theorem 3.1.4) and Theorem B 42 of Schapira [34], p. 38 that, for $p\geqq 0$ and $s>0$, we have

$$H^s(\Omega,\tilde{\tilde{\mathscr{A}}}{}^p)=\varinjlim_{\tilde{\Omega}\cap \boldsymbol{D}^n=\Omega}H^s(\tilde{\Omega},\tilde{\tilde{\mathcal{O}}}{}^p)=0.$$

Q. E. D.

Next we will prove the Oka-Cartan-Kawai Theorem B for the sheaf $\underset{\tilde{}}{\mathcal{O}}$. This can be proved by the same method as Theorem 3.1.4. Thus we have the following

Theorem 3.1.7 (The Oka-Cartan-Kawai Theorem B). *For any $\tilde{\tilde{\mathcal{O}}}$-pseudoconvex open set V in $\boldsymbol{E}^n$, we have $H^s(V,\underset{\tilde{}}{\mathcal{O}}{}^p)=0$ for $p\geqq 0$ and $s\geqq 1$.*

Proof. Since V is paracompact, $H^s(V, \underset{\sim}{\mathcal{O}}^p)$ coincides with the Čech cohomology group. So we have only to prove $\varinjlim_{\mathfrak{U}} H^s(\mathfrak{U}, \underset{\sim}{\mathcal{O}}^p)=0$, where $\mathfrak{U}=\{U_j\}_{j\geqq 1}$ is a locally finite open covering of V so that $V_j=U_j\cap \boldsymbol{C}^n$ is pseudoconvex. We can choose such a covering of V because V is an $\tilde{\mathcal{O}}$-pseudoconvex open set.

Here we use the notations in the proof of Theorem 3.1.4.

For any cocycle $d=\{d_J\}$ representing an element in $H^s(\mathfrak{U}, \underset{\sim}{\mathcal{O}}^p)$, we can define an element $c=\{c_J\}$ in $C^s(\tilde{\tilde{Z}}{}^{\mathrm{loc}}_{(p,0)}(\{V_j\}))$ such as $\delta c=0$ by putting $c_J=d_J\cdot h_\varepsilon(z)$, $h_\varepsilon(z)=\cosh((\varepsilon/2)\sqrt{z^2})$ for some positive ε, where δ denotes the coboundary operator and we put $z^2=z_1^2+\cdots+z_n^2$. Then we can find some $c'\in C^{s-1}(\tilde{\tilde{Z}}{}^{\mathrm{loc}}_{(p,0)}(\{V_j\}))$ such that $\delta c'=c$. If we put $d'_I=c'_I\cdot(h_\varepsilon(z))^{-1}$, then $d'=\{d'_I\}$ is a cochain with values in $\underset{\sim}{\mathcal{O}}$ such that $\delta d'=d$. Thus the element in $H^s(\mathfrak{U}, \underset{\sim}{\mathcal{O}}^p)$ represented by d is zero. Since a class $[d]$ with a representative d is an arbitrary element in $H^s(\mathfrak{U}, \underset{\sim}{\mathcal{O}}^p)$, we have $H^s(\mathfrak{U}, \underset{\sim}{\mathcal{O}}^p)=0$. This completes the proof. Q.E.D.

At last we will prove the Malgrange theorem for the sheaf $\underset{\sim}{\mathscr{A}}$ of germs of rapidly decreasing real analytic functions. Here we define the sheaf $\underset{\sim}{\mathscr{A}}$ to be the restriction of $\underset{\sim}{\mathcal{O}}$ to $\boldsymbol{D}^n$: $\underset{\sim}{\mathscr{A}}=\underset{\sim}{\mathcal{O}}\,|\,\boldsymbol{D}^n$. Then we have the following

Theorem 3.1.8 (Malgrange). *For an arbitrary set Ω in $\boldsymbol{D}^n$, we have $H^s(\Omega, \underset{\sim}{\mathscr{A}}^p)=0$ for $p\geqq 0$ and $s\geqq 1$.*

Proof. We can prove this by the method similar to that of Theorem 3.1.6. Q.E.D.

3.2. The Dolbeault-Grothendieck resolutions of $\tilde{\tilde{\mathcal{O}}}$ and $\underset{\sim}{\mathcal{O}}$

In this section we will construct soft resolutions of $\tilde{\tilde{\mathcal{O}}}$ and $\underset{\sim}{\mathcal{O}}$ and prove some of its consequences.

At first we will recall the definition of the sheaf $\tilde{\tilde{L}}=\tilde{\tilde{L}}_{2,\mathrm{loc}}$ of germs of slowly increasing locally L_2-functions over $\boldsymbol{E}^n$ following Saburi [28].

Definition 3.2.1. *We define the sheaf $\tilde{\tilde{L}}$ to be the sheafification of the presheaf $\{\tilde{\tilde{L}}(\Omega);\ \Omega\subset \boldsymbol{E}^n$ open$\}$, where, for an open set Ω in $\boldsymbol{E}^n$, the section module $\tilde{\tilde{L}}(\Omega)$ is the space of all $f\in L_{2,\mathrm{loc}}(\Omega\cap \boldsymbol{C}^n)$ such as, for any $\varepsilon>0$ and any relatively compact open subset ω of Ω, $e(-\varepsilon\|z\|)f(z)|\omega$ belongs to $L_2(\omega\cap \boldsymbol{C}^n)$. Here $e(-\varepsilon\|z\|)f(z)|\omega$ denotes the restriction of $e(-\varepsilon\|z\|)f(z)$ to ω and $\|z\|$ denotes the modification of $\sum_{j=1}^{n}|z_j|$ so as to become C^∞ and convex.*

Then it is easy to see that $\tilde{\tilde{L}}$ is a soft FS* sheaf. Then we give

Definition 3.2.2. *We define the sheaf $\tilde{\tilde{\mathscr{L}}}{}^{p,q}=\tilde{\tilde{\mathscr{L}}}{}^{p,q}_{2,\mathrm{loc}}$ to be the sheafification of* *the presheaf* $\{\tilde{\tilde{\mathscr{L}}}^{p,q}(\Omega);\ \Omega\subset \boldsymbol{E}^n \text{ open}\}$, *where, for an open set* Ω *in* $\boldsymbol{E}^n$, *the section module* $\tilde{\tilde{\mathscr{L}}}^{p,q}(\Omega)$ *is the space of all* $f\in \tilde{\tilde{L}}^{p,q}(\Omega)=\tilde{\tilde{L}}^{p,q}_{2,\mathrm{loc}}(\Omega)$ *such that* $\bar{\partial}f\in \tilde{\tilde{L}}^{p,q+1}(\Omega)=\tilde{\tilde{L}}^{p,q+1}_{2,\mathrm{loc}}(\Omega)$. *We put* $\tilde{\tilde{\mathscr{L}}}=\tilde{\tilde{\mathscr{L}}}^{0,0}$.

Then $\tilde{\tilde{\mathscr{L}}}^{p,q}$ is a soft FS* sheaf. Then we have the following

Theorem 3.2.3 (The Dolbeault-Grothendieck resolution). *For some* $d>0$, *put* $U=\mathrm{int}\{z\in \boldsymbol{C}^n;\ |\mathrm{Im}\, z|-|\mathrm{Re}\, z|<d\}^a$, *where* $\mathrm{int}\{\ \}^a$ *denotes the interior of the closure in* $\boldsymbol{E}^n$ *of a set* $\{\ \}$. *Then the sequence of sheaves over* U

$$0\longrightarrow \tilde{\tilde{\mathcal{O}}}^p|U\longrightarrow \tilde{\tilde{\mathscr{L}}}^{p,0}|U\xrightarrow{\bar{\partial}}\tilde{\tilde{\mathscr{L}}}^{p,1}|U\xrightarrow{\bar{\partial}}\cdots\cdots\xrightarrow{\bar{\partial}}\tilde{\tilde{\mathscr{L}}}^{p,n}|U\longrightarrow 0$$

is exact.

Proof. The exactness of the sequence

$$0\longrightarrow \tilde{\tilde{\mathcal{O}}}^p|U\longrightarrow \tilde{\tilde{\mathscr{L}}}^{p,0}|U\xrightarrow{\bar{\partial}}\tilde{\tilde{\mathscr{L}}}^{p,1}|U$$

is evident. In fact, let Ω be a relatively compact open set in U. Let $u\in\tilde{\tilde{\mathscr{L}}}^{p,0}(\Omega)$ such that $\bar{\partial}u=0$. Then, if we write u in the form

$$u=\sum_{|I|=p}u_I dz^I,$$

we have

$$\partial u_I/\partial\bar{z}_j=0,\quad j=1,2,\ldots,n,$$

from which we obtain

$$\sum_{j=1}^n\frac{\partial}{\partial z_j}\frac{\partial}{\partial\bar{z}_j}u_I=0.$$

Since the operator (on $\Omega\cap \boldsymbol{R}^{2n}$)

$$\sum_{j=1}^n\frac{\partial}{\partial z_j}\frac{\partial}{\partial\bar{z}_j}$$

is elliptic, it follows from Weyl's Lemma that u_I's are analytic on Ω. So that we can conclude that u_I's are holomorphic. The fact that $u_I\in\tilde{\tilde{\mathcal{O}}}(\Omega)$ follows from the exchangeability of L_2-norm and sup-norm for holomorphic functions. Thus the exactness of the above sequence was proved.

Next we have to prove the exactness of the sequence

$$\tilde{\tilde{\mathscr{L}}}^{p,0}|U\xrightarrow{\bar{\partial}}\tilde{\tilde{\mathscr{L}}}^{p,1}|U\xrightarrow{\bar{\partial}}\cdots\cdots\xrightarrow{\bar{\partial}}\tilde{\tilde{\mathscr{L}}}^{p,n}|U\longrightarrow 0.$$

For this purpose, we have only to prove the exactness of the sequence of stalks

$$\tilde{\tilde{\mathscr{L}}}^{p,0}_z\xrightarrow{\bar{\partial}}\tilde{\tilde{\mathscr{L}}}^{p,1}_z\xrightarrow{\bar{\partial}}\cdots\cdots\xrightarrow{\bar{\partial}}\tilde{\tilde{\mathscr{L}}}^{p,n}_z\longrightarrow 0$$

for every $z \in U$. But this is an easy consequence of Hörmander [4], Theorem 4.4.2 because every $z \in U$ has a fundamental system of $\tilde{\tilde{\mathcal{O}}}$-pseudoconvex open neighborhoods. Q. E. D.

Corollary 1. *Let U be as in Theorem 3.2.3. For an open set Ω in U, we have the following isomorphism*:

$$H^q(\Omega, \tilde{\tilde{\mathcal{O}}}^p) \cong \{f \in \tilde{\tilde{\mathcal{L}}}^{p,q}_{2,\mathrm{loc}}(\Omega);\ \bar{\partial} f = 0\} / \{\bar{\partial} g;\ g \in \tilde{\tilde{\mathcal{L}}}^{p,q-1}_{2,\mathrm{loc}}(\Omega)\}, \quad (p \geqq 0,\ q \geqq 1).$$

Corollary 2. *Let Ω be an $\tilde{\tilde{\mathcal{O}}}$-pseudoconvex open set in E^n. Then the equation $\bar{\partial} u = f$ has a solution $u \in \tilde{\tilde{\mathcal{L}}}^{p,q}_{2,\mathrm{loc}}(\Omega)$ for every $f \in \tilde{\tilde{\mathcal{L}}}^{p,q+1}_{2,\mathrm{loc}}(\Omega)$ such that $\bar{\partial} f = 0$. Here p, q are nonnegative integers.*

Proof. It follows from Theorem 3.1.4 and Corollary 1 to Theorem 3.2.3. Q. E. D.

We will now recall the definition of the sheaf $\underset{\approx}{L} = \underset{\approx}{L}_{2,\mathrm{loc}}$ of germs of rapidly decreasing locally L_2-functions.

Definition 3.2.4. *We define the sheaf $\underset{\approx}{L}$ to be the sheafification of the presheaf $\{\underset{\approx}{L}(\Omega);\ \Omega \subset E^n$ open$\}$, where, for an open set Ω in E^n, the section module $\underset{\approx}{L}(\Omega)$ is the space of all $f \in L_{2,\mathrm{loc}}(\Omega \cap C^n)$ such as, for any relatively compact open subset ω of Ω, there exists some positive δ such that $e(\delta \| z \|) f(z) | \omega \in L_2(\omega \cap C^n)$.*

Then it is easy to see that $\underset{\approx}{L}$ is a soft FS* sheaf.

Definition 3.2.5 (The sheaf $\underset{\approx}{\mathcal{L}}^{p,q}$). *We define the sheaf $\underset{\approx}{\mathcal{L}}^{p,q} = \underset{\approx}{\mathcal{L}}^{p,q}_{2,\mathrm{loc}}$ to be the sheafification of the presheaf $\{\underset{\approx}{\mathcal{L}}^{p,q}(\Omega);\ \Omega \subset E^n$ open$\}$, where, for an open set Ω in E^n, the section module $\underset{\approx}{\mathcal{L}}^{p,q}(\Omega)$ is the space of all $f \in \underset{\approx}{L}^{p,q}(\Omega) = \underset{\approx}{L}^{p,q}_{2,\mathrm{loc}}(\Omega)$ such that $\bar{\partial} f \in \underset{\approx}{L}^{p,q+1}(\Omega) = \underset{\approx}{L}^{p,q+1}_{2,\mathrm{loc}}(\Omega)$. We put $\underset{\approx}{\mathcal{L}} = \underset{\approx}{\mathcal{L}}^{0,0}$.*

Then $\underset{\approx}{\mathcal{L}}^{p,q}$ is a soft FS* sheaf. Then we have the following

Theorem 3.2.6 (The Dolbeault-Grothendieck resolution). *For some $d > 0$, put $U = \mathrm{int}\{z \in C^n;\ |\mathrm{Im}\, z| - |\mathrm{Re}\, z| < d\}^a$. Then the sequence of sheaves over U*:

$$0 \longrightarrow \underset{\approx}{\mathcal{O}}^p | U \longrightarrow \underset{\approx}{\mathcal{L}}^{p,0} | U \xrightarrow{\bar{\partial}} \underset{\approx}{\mathcal{L}}^{p,1} | U \xrightarrow{\bar{\partial}} \cdots\cdots \xrightarrow{\bar{\partial}} \underset{\approx}{\mathcal{L}}^{p,n} | U \longrightarrow 0$$

is exact.

Proof. The exactness of the sequence

$$0 \longrightarrow \underset{\approx}{\mathcal{O}}^p | U \longrightarrow \underset{\approx}{\mathcal{L}}^{p,0} | U \xrightarrow{\bar{\partial}} \underset{\approx}{\mathcal{L}}^{p,1} | U$$

can be proved by the same way as that of Theorem 3.2.3.

Next, in order to prove the exactness of the sequence

$$\underset{\approx}{\mathscr{L}}^{p,0}|U \xrightarrow{\bar{\partial}} \underset{\approx}{\mathscr{L}}^{p,1}|U \xrightarrow{\bar{\partial}} \cdots\cdots \xrightarrow{\bar{\partial}} \underset{\approx}{\mathscr{L}}^{p,n}|U \longrightarrow 0,$$

we will recall the following

Lemma 3.2.7 (Saburi). *For* $\delta>0$ *and* $A>0$, *put*

$$V_{\delta,A}=\{z\in \boldsymbol{C}^n;\ |\mathrm{Im}\, z|^2 \leqq \delta^2|\mathrm{Re}\, z|^2+A^2\}.$$

Then, if $0<\delta<1$, *we have, for* $0<\varepsilon<\sqrt{1-\delta^2}/(\sqrt{2}A)$,

$$|\cosh(\varepsilon\sqrt{z^2})| \geqq Ce\left(\varepsilon\sqrt{\frac{1-\delta^2}{1+\delta^2}}\,|z|\right),\quad z\in V_{\delta,A}.$$

Here C *is a constant independent of* δ, A, ε *and we put* $z^2=z_1^2+z_2^2+\cdots+z_n^2$. *We also have*

$$|\cosh(\varepsilon\sqrt{z^2})| \leqq e^{\varepsilon|z|},\quad z\in \boldsymbol{C}^n.$$

Proof of Lemma 3.2.7. See Saburi [28], Lemma 2.3.8, p. 37. Q.E.D.

Now we will return to prove Theorem 3.2.6. Let $z\in U$. For an open neighborhood Ω of the form $V_{\delta,A}$ in Lemma 3.2.7 for some δ and A such as $0<\delta<1$ and $A>0$, we take an element $f\in \underset{\approx}{\mathscr{L}}^{p,q+1}(\Omega)$ such that $\bar{\partial}f=0$. Then, for some $\varepsilon>0$, we can see that $f\cdot h_\varepsilon(z)\in \underset{\approx}{\tilde{\mathscr{L}}}^{p,q+1}(\Omega)$, where we put $h_\varepsilon(z)=\cosh(\varepsilon\sqrt{z^2}/2)$. Since $\bar{\partial}(f\cdot h_\varepsilon(z))=0$, we can find some $v\in \underset{\approx}{\tilde{\mathscr{L}}}^{p,q}(\Omega')$ for some open neighborhood $\Omega'(\subset\Omega)$ of z such that $\bar{\partial}v=f\cdot h_\varepsilon(z)$. Here we may assume that $h_\varepsilon(z)\neq 0$ on $\Omega'\cap \boldsymbol{C}^n$. Then $u=v/h_\varepsilon(z)$ belongs to $\underset{\approx}{\mathscr{L}}^{p,q}(\Omega')$ and $\bar{\partial}u=f$ holds. This completes the proof.

Q.E.D.

Corollary 1. *Let* U *be as in Theorem* 3.2.6. *For an open set* Ω *in* U, *we have the following isomorphism*:

$$H^q(\Omega, \underset{\approx}{\mathcal{O}}^p)\cong \{f\in \underset{\approx}{\mathscr{L}}^{p,q}_{2,\mathrm{loc}}(\Omega);\ \bar{\partial}f=0\}/\{\bar{\partial}g;\ g\in \underset{\approx}{\mathscr{L}}^{p,q-1}_{2,\mathrm{loc}}(\Omega)\},\qquad (p\geqq 0,\ q\geqq 1).$$

Corollary 2. *Let* Ω *be an* $\tilde{\tilde{\mathcal{O}}}$*-pseudoconvex open set in* $\boldsymbol{E}^n$. *Then the equation* $\bar{\partial}u=f$ *has a solution* $u\in \underset{\approx}{\mathscr{L}}^{p,q}_{2,\mathrm{loc}}(\Omega)$ *for every* $f\in \underset{\approx}{\mathscr{L}}^{p,q+1}_{2,\mathrm{loc}}(\Omega)$ *such that* $\bar{\partial}f=0$. *Here* p *and* q *are nonnegative integers.*

Proof. It follows from Theorem 3.1.7 and Corollary 1 to Theorem 3.2.6.

Q.E.D.

Now, for later applications, we will construct another soft resolutions of $\tilde{\tilde{\mathcal{O}}}$ and $\underset{\approx}{\mathcal{O}}$.

At first, we will give some preliminary facts.

For an integer $s \geqq 0$ and for an open set Ω in $\boldsymbol{C}^n$, we put

$W_{s,\mathrm{loc}}(\Omega)=\{f \in L_{2,\mathrm{loc}}(\Omega)$; for every relatively compact open subset ω of Ω and for every $\alpha \in \bar{\boldsymbol{N}}^{2n}$ such that $|\alpha| \leqq s$, $f^{(\alpha)}(z)|\omega \in L_2(\omega)$ holds$\}$,

and denote by $W^{p,q}_{s,\mathrm{loc}}(\Omega)$ the space of all differential forms of type (p, q) whose coefficients in $W_{s,\mathrm{loc}}(\Omega)$. Further, for an open set Ω in $\boldsymbol{E}^n$, we put

$\tilde{\tilde{W}}_s(\Omega)=\{f \in W_{s,\mathrm{loc}}(\Omega \cap \boldsymbol{C}^n)$; for any positive ε and for every relatively compact open subset ω of Ω and for every $\alpha \in \bar{\boldsymbol{N}}^{2n}$ such that $|\alpha| \leqq s$, $(e(-\varepsilon\|z\|)f^{(\alpha)}(z))\,|\omega \cap \boldsymbol{C}^n \in L_2(\omega \cap \boldsymbol{C}^n)$ holds$\}$,

and denote by $\tilde{\tilde{W}}{}^{p,q}_s(\Omega)$ the space of all differential forms of type (p, q) whose coefficients in $\tilde{\tilde{W}}_s(\Omega)$. Then we have the following

Theorem 3.2.8. *Let Ω be an $\tilde{\tilde{\mathcal{O}}}$-pseudoconvex open set in $\boldsymbol{E}^n$ and s an integer such as $0 \leqq s \leqq \infty$. Then, for every $f \in \tilde{\tilde{W}}{}^{p,q+1}_s(\Omega)$ such as $\bar{\partial}f=0$, we can find a solution $u \in \tilde{\tilde{W}}{}^{p,q}_{s+1}(\Omega)$ of the equation $\bar{\partial}u=f$. Every solution of the equation $\bar{\partial}u=f$ has this property when $q=0$.*

Proof. (a) First assume that $q=0$. We know, from Corollary 2 to Theorem 3.2.3, that the equation $\bar{\partial}u=f$ has a solution $u=\sum' u_I dz^I \in \tilde{\tilde{\mathscr{L}}}{}^{p,0}_{2,\mathrm{loc}}(\Omega)$ because $f \in \tilde{\tilde{\mathscr{L}}}{}^{p,1}_{2,\mathrm{loc}}(\Omega)$ and $\bar{\partial}f=0$. The equation $\bar{\partial}u=f$ means that

$$\partial(u_I|\Omega \cap \boldsymbol{C}^n)/\partial\bar{z}_j=f_{I,j}|\Omega \cap \boldsymbol{C}^n \in W_{s,\mathrm{loc}}(\Omega \cap \boldsymbol{C}^n)$$

for all I and j. Thus, by Hörmander [4], Theorem 4.2.5, we have $u_I \in W_{s+1,\mathrm{loc}}\cdot(\Omega \cap \boldsymbol{C}^n)$. Then, by Nagamachi [25], Lemma 4.3, we can conclude that $u_I \in \tilde{\tilde{W}}_{s+1}(\Omega)$.

(b) Next we assume that $q>0$. Then, by Hörmander [4], Theorem 4.2.5, we can find $u \in W^{p,q}_{s+1,\mathrm{loc}}(\Omega \cap \boldsymbol{C}^n)$ such that $\bar{\partial}u=f$. Then, by Nagamachi [25], Lemma 4.2, we can conclude that $u \in \tilde{\tilde{W}}{}^{p,q}_{s+1}(\Omega)$. Q.E.D.

Now we will define the sheaf $\tilde{\tilde{\mathscr{E}}}$ of germs of slowly increasing C^∞-functions over $\boldsymbol{E}^n$.

Definition 3.2.9. *We define the sheaf $\tilde{\tilde{\mathscr{E}}}$ to be the sheafification of the presheaf $\{\tilde{\tilde{\mathscr{E}}}(\Omega);\ \Omega \subset \boldsymbol{E}^n$ open$\}$, where, for an open set Ω in $\boldsymbol{E}^n$, the section module $\tilde{\tilde{\mathscr{E}}}(\Omega)$ is defined as follows:*

$\tilde{\tilde{\mathscr{E}}}(\Omega)=\{f \in \mathscr{E}(\Omega \cap \boldsymbol{C}^n)$; for any positive ε and any compact set K in Ω and any $\alpha \in \bar{\boldsymbol{N}}^{2n}$, the estimate $\sup\{|f^{(\alpha)}(z)|e(-\varepsilon|z|);\ z \in K \cap \boldsymbol{C}^n\} < \infty$ *holds$\}$.*

Then it is easy to see that $\tilde{\tilde{\mathscr{E}}}$ is a soft nuclear Fréchet sheaf. Then we have the following

Theorem 3.2.10. *Let Ω be an $\tilde{\tilde{\mathcal{O}}}$-pseudoconvex open set in $\boldsymbol{E}^n$. Then the equation $\bar{\partial}u=f$ has a solution $u\in\tilde{\tilde{\mathcal{E}}}^{p,q}(\Omega)$ for every $f\in\tilde{\tilde{\mathcal{E}}}^{p,q+1}(\Omega)$ such that $\bar{\partial}f=0$. Every solution of the equation $\bar{\partial}u=f$ has this property when $q=0$.*

Proof. Since $f\in\tilde{\tilde{W}}{}_s^{p,q+1}(\Omega)$ for every integer $s\geqq 0$, we can find $u\in\tilde{\tilde{W}}{}_{s+1}^{p,q}(\Omega)$ for every s. But, by the well-known Sobolev lemma, we have

$$\tilde{\tilde{W}}{}_{s+2n}^{p,q}(\Omega)\subset\tilde{\tilde{C}}{}_s^{p,q}(\Omega),$$

where we put

$\tilde{\tilde{C}}_s(\Omega)=\{f\in C^s(\Omega\cap\boldsymbol{C}^n)$; for any positive ε and any compact set K in Ω and any $\alpha\in\bar{\boldsymbol{N}}^{2n}$ such that $|\alpha|\leqq s$, the estimate $\sup\{|f^{(\alpha)}(z)|e(-\varepsilon|z|);\ z\in K\cap\boldsymbol{C}^n\}<\infty$ holds$\}$.

Thus we have $u\in\tilde{\tilde{\mathcal{E}}}^{p,q}(\Omega)$. Q.E.D.

Then we have the following

Theorem 3.2.11 (The Dolbeault-Grothendieck resolution). *For some $d>0$, put $U=\operatorname{int}\{z\in\boldsymbol{C}^n;\ |\operatorname{Im}z|-|\operatorname{Re}z|<d\}^a$. Then the sequence of sheaves over U*

$$0\longrightarrow\tilde{\tilde{\mathcal{O}}}{}^p|U\longrightarrow\tilde{\tilde{\mathcal{E}}}{}^{p,0}|U\xrightarrow{\bar{\partial}}\tilde{\tilde{\mathcal{E}}}{}^{p,1}|U\xrightarrow{\bar{\partial}}\cdots\cdots\xrightarrow{\bar{\partial}}\tilde{\tilde{\mathcal{E}}}{}^{p,n}|U\longrightarrow 0$$

is exact.

Proof. It follows immediately from Theorem 3.2.10. Q.E.D.

Corollary. *We use notations in Theorem* 3.2.11. *For an open set Ω in U, we have the following isomorphism*:

$$H^q(\Omega,\tilde{\tilde{\mathcal{O}}}{}^p)\cong\{f\in\tilde{\tilde{\mathcal{E}}}{}^{p,q}(\Omega);\ \bar{\partial}f=0\}/\{\bar{\partial}g;\ g\in\tilde{\tilde{\mathcal{E}}}{}^{p,q-1}(\Omega)\},\quad(p\geqq 0,\ q\geqq 1).$$

Now we will define the sheaf $\underset{\approx}{\mathcal{E}}$ of germs of rapidly decreasing C^∞-functions over $\boldsymbol{E}^n$.

Definition 3.2.12. *We define the sheaf $\underset{\approx}{\mathcal{E}}$ to be the sheafification of the presheaf $\{\underset{\approx}{\mathcal{E}}(\Omega);\ \Omega\subset\boldsymbol{E}^n$ open$\}$, where the section module $\underset{\approx}{\mathcal{E}}(\Omega)$ on an open set Ω in $\boldsymbol{E}^n$ is the space of all C^∞-functions on $\Omega\cap\boldsymbol{C}^n$ such that, for any compact set K in Ω and any $\alpha\in\bar{\boldsymbol{N}}^{2n}$, there exists some positive constant δ so that the estimate* $\sup\cdot$ $\{|f^{(\alpha)}(z)|e(\delta|z|);\ z\in K\cap\boldsymbol{C}^n\}<\infty$ *holds.*

Then $\underset{\approx}{\mathcal{E}}$ becomes a soft nuclear Fréchet sheaf. Then we have the following

Theorem 3.2.13 (The Dolbeault-Grothendieck resolution). *The sequence of sheaves over U*

$$0\longrightarrow\underset{\approx}{\mathcal{O}}{}^p|U\longrightarrow\underset{\approx}{\mathcal{E}}{}^{p,0}|U\xrightarrow{\bar{\partial}}\underset{\approx}{\mathcal{E}}{}^{p,1}|U\xrightarrow{\bar{\partial}}\cdots\cdots\xrightarrow{\bar{\partial}}\underset{\approx}{\mathcal{E}}{}^{p,n}|U\longrightarrow 0$$

is exact, where $U=\operatorname{int}\{z\in \boldsymbol{C}^n;\ |\operatorname{Im} z|-|\operatorname{Re} z|<d\}^a$ *for some* $d>0$.

Proof. Let $z\in U$. For an open neighborhood Ω of the form $V_{\delta,A}$ in Lemma 3.2.7 for some δ and A such as $0<\delta<1$ and $A>0$, we take an element $f\in \underset{\approx}{\mathscr{E}}^{p,q+1}(\Omega)$ such that $\bar{\partial}f=0$. Then, for some $\varepsilon>0$, we can see that $f\cdot h_\varepsilon(z)\in \tilde{\tilde{\mathscr{E}}}^{p,q+1}(\Omega)$, where we put $h_\varepsilon(z)=\cosh(\varepsilon\sqrt{z^2}/2)$. Since $\bar{\partial}(f\cdot h_\varepsilon(z))=0$, we can find some $v\in \tilde{\tilde{\mathscr{E}}}^{p,q}(\Omega')$ for some open neighborhood $\Omega'(\subset\Omega)$ of z such that $\bar{\partial}v=f\cdot h_\varepsilon(z)$ by Theorem 3.2.11. Here we may assume that $h_\varepsilon(z)\neq 0$ on $\Omega'\cap \boldsymbol{C}^n$. Then $u=v/h_\varepsilon(z)$ belongs to $\underset{\approx}{\mathscr{E}}^{p,q}(\Omega')$ and $\bar{\partial}u=f$ holds. This completes the proof. Q. E. D.

Corollary 1. *Let U be as in Theorem 3.2.13. For an open set Ω in U, we have the following isomorphism*:

$$H^q(\Omega,\ \underset{\approx}{\mathscr{O}}^p)\cong\{f\in \underset{\approx}{\mathscr{E}}^{p,q}(\Omega);\ \bar{\partial}f=0\}/\{\bar{\partial}g;\ g\in \underset{\approx}{\mathscr{E}}^{p,q-1}(\Omega)\},\quad (p\geqq 0,\ q\geqq 1).$$

Corollary 2. *Let Ω be an $\tilde{\tilde{\mathscr{O}}}$-pseudoconvex open set in $\boldsymbol{E}^n$. Then the equation $\bar{\partial}u=f$ has a solution $u\in \underset{\approx}{\mathscr{E}}^{p,q}(\Omega)$ for every $f\in \underset{\approx}{\mathscr{E}}^{p,q+1}(\Omega)$ such that $\bar{\partial}f=0$. Here p and q are nonnegative integers.*

Proof. It follows from Theorem 3.1.7 and Corollary 1 to Theorem 3.2.13. Q. E. D.

3.3. Malgrange's Theorem

In the following of this chapter we will go in the same way as Saburi [28].

Theorem 3.3.1 (Malgrange's Theorem). *Let Ω be an open set in $\boldsymbol{E}^n$ such that, for any $z\in\Omega\cap \boldsymbol{C}^n$, $|\operatorname{Im} z|-|\operatorname{Re} z|<d$ holds for some constant $d>0$ independent of $z\in\Omega\cap \boldsymbol{C}^n$. Then we have $H^n(\Omega,\ \tilde{\tilde{\mathscr{O}}})=0$.*

Proof. By virtue of Corollary 1 to Theorem 3.2.3, we have only to prove the exactness of the sequence

$$\tilde{\tilde{\mathscr{L}}}^{0,n-1}(\Omega)\xrightarrow{\ \bar{\partial}\ }\tilde{\tilde{\mathscr{L}}}^{0,n}(\Omega)\longrightarrow 0$$

in the notations of Theorem 3.2.3. But, in order to do so, we have only to prove the injectiveness and the closedness of the range of $\vartheta=(\bar{\partial})'$ in the dual sequence

$$\underset{\approx}{\mathscr{L}}{}_c^{0,1}(\Omega)\xleftarrow{\ \vartheta\ }\underset{\approx}{\mathscr{L}}{}_c^{0,0}(\Omega)\longleftarrow 0$$

in the notations of Theorem 3.2.6 by virtue of the Serre-Komatsu duality theorem for FS*-spaces. Here $\underset{\approx}{\mathscr{L}}{}_c^{p,q}(\Omega)$ denotes the space of sections with compact support of $\underset{\approx}{\mathscr{L}}^{p,q}$ on Ω. This has already been proved by Saburi [28], p. 44. Q. E. D.

Corollary. Flabby dim $\tilde{\tilde{\mathscr{O}}}\leqq n$.

3.4. Serre's duality theorem

In this section we will prove Serre's duality theorem.

Theorem 3.4.1. *Let Ω be such an open set as in Theorem* 3.3.1 *and assume that* $\dim H^p(\Omega, \tilde{\tilde{\mathcal{O}}}) < \infty$ *holds for* $p \geqq 1$. *Then we have the isomorphism* $[H^p(\Omega, \tilde{\tilde{\mathcal{O}}})]' \cong H_c^{n-p}(\Omega, \underset{\approx}{\mathcal{O}})$ $(0 \leqq p \leqq n)$.

Proof. By virtue of Corollary 1 to Theorem 3.2.3 and Corollary 1 to Theorem 3.2.6, cohomology groups $H^p(\Omega, \tilde{\tilde{\mathcal{O}}})$ and $H_c^{n-p}(\Omega, \underset{\approx}{\mathcal{O}})$ are cohomology groups respectively of the complexes

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \tilde{\tilde{\mathcal{L}}}^{0,0}(\Omega) & \xrightarrow{\bar{\partial}} & \tilde{\tilde{\mathcal{L}}}^{0,1}(\Omega) & \xrightarrow{\bar{\partial}} & \cdots\cdots & \xrightarrow{\bar{\partial}} & \tilde{\tilde{\mathcal{L}}}^{0,n}(\Omega) & \longrightarrow & 0 \\
 & & \updownarrow & & \updownarrow & & & & \updownarrow & & \\
0 & \longleftarrow & \underset{\approx}{\mathcal{L}}_c^{0,n}(\Omega) & \xleftarrow{-\bar{\partial}} & \underset{\approx}{\mathcal{L}}_c^{0,n-1}(\Omega) & \xleftarrow{-\bar{\partial}} & \cdots\cdots & \xleftarrow{-\bar{\partial}} & \underset{\approx}{\mathcal{L}}_c^{0,0}(\Omega) & \longleftarrow & 0.
\end{array}$$

Here the upper complex is composed of FS* spaces and the lower complex is composed of DFS* spaces. Since the ranges of operators $\bar{\partial}$ in the upper complex are all closed by virtue of Schwartz's Lemma (cf. Komatsu [20]), the ranges of operators $-\bar{\partial} = (\bar{\partial})'$ in the lower complex are also all closed. Hence we have the isomorphism

$$[H^p(\Omega, \tilde{\tilde{\mathcal{O}}})]' \cong H_c^{n-p}(\Omega, \underset{\approx}{\mathcal{O}})$$

by virtue of Serre's Lemma (cf. Komatsu [20]). Q. E. D.

3.5. Martineau-Harvey's Theorem

In this section we will prove Martineau-Harvey's Theorem.

Theorem 3.5.1. *Let K be a compact set in $\boldsymbol{E}^n$ such that it has an $\tilde{\tilde{\mathcal{O}}}$-pseudoconvex open neighborhood Ω and satisfies the conditions $H^p(K, \underset{\approx}{\mathcal{O}}) = 0$ $(p \geqq 1)$. Then we have $H_K^p(\Omega, \tilde{\tilde{\mathcal{O}}}) = 0$ for $p \neq n$ and isomorphisms $H_K^n(\Omega, \tilde{\tilde{\mathcal{O}}}) \cong H^{n-1}(\Omega \backslash K, \tilde{\tilde{\mathcal{O}}}) \cong \underset{\approx}{\mathcal{O}}(K)'$.*

Remark. If a compact set K in $\boldsymbol{E}^n$ has a fundamental system of $\tilde{\tilde{\mathcal{O}}}$-pseudoconvex open neighborhood, it satisfies the assumptions in Theorem 3.5.1.

Proof. See Saburi [28], p. 63. Q. E. D.

3.6. Sato's Theorem

In this section we will prove the pure-codimensionality of $\boldsymbol{D}^n$ with respect to $\tilde{\tilde{\mathcal{O}}}$. Then we will realize modified Fourier hyperfunctions as "boundary values" of slowly increasing holomorphic functions or as (relative) cohomology classes of slowly increasing holomorphic functions.

Theorem 3.6.1 (Sato's Theorem). *Let Ω be an open set in $\boldsymbol{D}^n$ and V an open set in $\boldsymbol{E}^n$ which contains Ω as its closed subsets. Then we have the following*

(1) *The relative cohomology groups $H^p_\Omega(V, \tilde{\tilde{\mathcal{O}}})$ are zero for $p \neq n$.*

(2) *The presheaf over $\boldsymbol{D}^n$*

$$\Omega \longrightarrow H^n_\Omega(V, \tilde{\tilde{\mathcal{O}}})$$

is a flabby sheaf.

(3) *This sheaf* (2) *is isomorphic to the sheaf $\mathcal{Q}$ of modified Fourier hyperfunctions.*

Proof. (1) It goes in the same way as Saburi [28], p. 66.

(2) By Malgrange's Theorem, we can conclude that flabby dim $\tilde{\tilde{\mathcal{O}}} \leqq n$. Thus, by (1) and by the theorem II. 3.24 of Komatsu [21], we have the conclusion.

(3) Consider the following exact sequence of relative cohomology groups

$$\begin{aligned} 0 &\longrightarrow H^0_{\partial\Omega}(V, \tilde{\tilde{\mathcal{O}}}) \longrightarrow H^0_{\Omega^a}(V, \tilde{\tilde{\mathcal{O}}}) \longrightarrow H^0_\Omega(V, \tilde{\tilde{\mathcal{O}}}) \\ &\longrightarrow H^1_{\partial\Omega}(V, \tilde{\tilde{\mathcal{O}}}) \longrightarrow \cdots\cdots \longrightarrow H^{n-1}_\Omega(V, \tilde{\tilde{\mathcal{O}}}) \\ &\longrightarrow H^n_{\partial\Omega}(V, \tilde{\tilde{\mathcal{O}}}) \longrightarrow H^n_{\Omega^a}(V, \tilde{\tilde{\mathcal{O}}}) \longrightarrow H^n_\Omega(V, \tilde{\tilde{\mathcal{O}}}) \\ &\longrightarrow H^{n+1}_{\partial\Omega}(V, \tilde{\tilde{\mathcal{O}}}) \longrightarrow \cdots\cdots. \end{aligned}$$

Then, by (1) and by Martineau-Harvey's Theorem, we have $H^{n-1}_\Omega(V, \tilde{\tilde{\mathcal{O}}}) = 0$, $H^{n+1}_{\partial\Omega}(V, \tilde{\tilde{\mathcal{O}}}) = 0$. Thus we have the exact sequence

$$0 \longrightarrow H^n_{\partial\Omega}(V, \tilde{\tilde{\mathcal{O}}}) \longrightarrow H^n_{\Omega^a}(V, \tilde{\tilde{\mathcal{O}}}) \longrightarrow H^n_\Omega(V, \tilde{\tilde{\mathcal{O}}}) \longrightarrow 0.$$

Since, by Martineau-Harvey's Theorem, we have isomorphisms

$$H^n_{\partial\Omega}(V, \tilde{\tilde{\mathcal{O}}}) \cong \underset{\approx}{\mathcal{A}}(\partial\Omega)', \quad H^n_{\Omega^a}(V, \tilde{\tilde{\mathcal{O}}}) \cong \underset{\approx}{\mathcal{A}}(\Omega^a)',$$

we obtain the isomorphism

$$H^n_\Omega(V, \tilde{\tilde{\mathcal{O}}}) \cong \underset{\approx}{\mathcal{A}}(\Omega^a)' / \underset{\approx}{\mathcal{A}}(\partial\Omega)' = \mathcal{Q}(\Omega).$$

Thus the sheaf $\Omega \rightarrow H^n_\Omega(V, \tilde{\tilde{\mathcal{O}}})$ is isomorphic to the sheaf $\mathcal{Q}$ of modified Fourier hyperfunctions over $\boldsymbol{D}^n$. Q.E.D

Let Ω be an open set in $\boldsymbol{D}^n$. Then there exists an $\tilde{\tilde{\mathcal{O}}}$-pseudoconvex open neighborhood V of Ω such that $V \cap \boldsymbol{D}^n = \Omega$ (cf. Saburi [28], Theorem 6.2.1). We put $V_0 = V$ and $V_j = V \backslash \{z \in V;\ \operatorname{Im} z_j = 0\}^a$, $j = 1, 2, \ldots, n$. Then $\mathfrak{B} = \{V_0, V_1, \ldots, V_n\}$ and $\mathfrak{B}' = \{V_1, \ldots, V_n\}$ cover V and $V \backslash \Omega$ respectively. Since V_j and their intersections are also $\tilde{\tilde{\mathcal{O}}}$-pseudoconvex open sets, the covering $(\mathfrak{B}, \mathfrak{B}')$ satisfies the conditions of

Leray's Theorem (cf. Komatsu [21]). Thus, by Leray's Theorem, we obtain the isomorphism $H^n_\Omega(V, \tilde{\tilde{\mathcal{O}}}) \cong H^n(\mathfrak{V}, \mathfrak{V}', \tilde{\tilde{\mathcal{O}}})$. Since the covering $\mathfrak{V}$ is composed of only $n+1$ open sets V_j $(j=0, 1,\dots, n)$, we easily obtain the isomorphisms

$$Z^n(\mathfrak{V}, \mathfrak{V}', \tilde{\tilde{\mathcal{O}}}) \cong \tilde{\tilde{\mathcal{O}}}(\cap_j V_j),$$

$$C^{n-1}(\mathfrak{V}, \mathfrak{V}', \tilde{\tilde{\mathcal{O}}}) \cong \bigoplus_{j=1}^{n} \tilde{\tilde{\mathcal{O}}}(\bigcap_{i \neq j} V_i).$$

Hence we have

$$\delta C^{n-1}(\mathfrak{V}, \mathfrak{V}', \tilde{\tilde{\mathcal{O}}}) \cong \sum_{j=1}^{n} \tilde{\tilde{\mathcal{O}}}(\bigcap_{i \neq j} V_i) \,|\, V_1 \cap \cdots \cap V_n.$$

Thus we have the isomorphisms

$$H^n_\Omega(V, \tilde{\tilde{\mathcal{O}}}) \cong H^n(\mathfrak{V}, \mathfrak{V}', \tilde{\tilde{\mathcal{O}}}) \cong Z^n(\mathfrak{V}, \mathfrak{V}', \tilde{\tilde{\mathcal{O}}}) / \delta C^{n-1}(\mathfrak{V}, \mathfrak{V}', \tilde{\tilde{\mathcal{O}}})$$

$$\cong \tilde{\tilde{\mathcal{O}}}(\cap_j V_j) / \sum_{j=1}^{n} \tilde{\tilde{\mathcal{O}}}(\cap_{i \neq j} V_i).$$

Thus we have the following

Theorem 3.6.2. *We use notations as above. Then we have the isomorphisms*

$$H^n_\Omega(V, \tilde{\tilde{\mathcal{O}}}) \cong H^n(\mathfrak{V}, \mathfrak{V}', \tilde{\tilde{\mathcal{O}}}) \cong \tilde{\tilde{\mathcal{O}}}(\cap_j V_j) / \sum_{j=1}^{n} \tilde{\tilde{\mathcal{O}}}(\cap_{i \neq j} V_i).$$

At last we will realize modified Fourier analytic functionals with certain compact carrier as (relative) cohomology classes with coefficients in $\tilde{\tilde{\mathcal{O}}}$.

Let K be a compact set in $\boldsymbol{E}^n$ of the form $K = K_1 \times \cdots \times K_n$ with compact sets K_j in $\boldsymbol{E} = \boldsymbol{E}^1$ $(j=1, 2,\dots, n)$. Assume that K admits a fundamental system of $\tilde{\tilde{\mathcal{O}}}$-pseudoconvex open neighborhoods. Then we have

$$H^p(K, \underset{\sim}{\mathcal{O}}) = 0 \qquad \text{for} \quad p > 0.$$

By virtue of Martineau-Harvey's Theorem, there exists the isomorphism

$$\underset{\sim}{\mathcal{O}}(K)' \cong H^n_K(\Omega, \tilde{\tilde{\mathcal{O}}}).$$

Here Ω denotes an open neighborhood of K. Further assume that there exists an $\tilde{\tilde{\mathcal{O}}}$-pseudoconvex open neighborhood Ω of K such that

$$\Omega_j = \Omega \cap \{z \in \boldsymbol{E}^n;\ z_j \notin K_j\}$$

is also an $\tilde{\tilde{\mathcal{O}}}$-pseudoconvex open set for $j=1, 2,\dots, n$. Then $\mathfrak{U} = \{\Omega_0 = \Omega, \Omega_1,\dots, \Omega_n\}$ and $\mathfrak{U}' = \{\Omega_1, \Omega_2,\dots, \Omega_n\}$ form acyclic coverings of Ω and $\Omega \backslash K$. Set

$$\Omega \# K = \bigcap_{j=1}^{n} \Omega_j, \quad \Omega^j = \bigcap_{i \neq j} \Omega_i.$$

Let $\sum_j \tilde{\tilde{\mathcal{O}}}(\Omega^j)$ be the image in $\tilde{\tilde{\mathcal{O}}}(\Omega\sharp K)$ of $\prod_{j=1}^n \tilde{\tilde{\mathcal{O}}}(\Omega^j)$ by the mapping

$$(f_j)_{j=1}^n \longrightarrow \sum_{j=1}^n (-1)^{j+1} f'_j,$$

where f'_j denotes the restriction of f_j to $\Omega\sharp K$.

Then, by the same method as that of Theorem 3.6.2, we have the following

Theorem 3.6.3. *We use the notations as above. Then we have the isomorphisms*

$$\underset{\sim}{\mathcal{O}}(K)' \cong H_K^n(\Omega, \tilde{\tilde{\mathcal{O}}}) \cong H^n(\mathfrak{V}, \mathfrak{V}', \tilde{\tilde{\mathcal{O}}}) \cong \tilde{\tilde{\mathcal{O}}}(\Omega\sharp K) / \sum_j \tilde{\tilde{\mathcal{O}}}(\Omega^j).$$

Chapter 4. The case of the sheaf ${}^E\tilde{\tilde{\mathcal{O}}}$

4.1. The Dolbeault-Grothendieck resolution of ${}^E\tilde{\tilde{\mathcal{O}}}$

In this section we will construct a soft resolution of ${}^E\tilde{\tilde{\mathcal{O}}}$. In this chapter we always assume that E is a Fréchet space whose topology is defined by a family $\mathcal{T} = \mathcal{T}_E$ of continuous seminorms of E.

At first we will define sheaves ${}^E\tilde{\tilde{\mathcal{O}}}$ and ${}^E\tilde{\tilde{\mathcal{E}}}$.

Definition 4.1.1 (The sheaf ${}^E\tilde{\tilde{\mathcal{O}}}$ of germs of slowly increasing E-valued holomorphic functions over $\boldsymbol{E}^n$). *We define the sheaf ${}^E\tilde{\tilde{\mathcal{O}}}$ to be the sheafification of the presheaf $\{\tilde{\tilde{\mathcal{O}}}(\Omega; E)\}$, where, for an open set Ω in $\boldsymbol{E}^n$, the module $\tilde{\tilde{\mathcal{O}}}(\Omega; E)$ is defined as follows:*

$\tilde{\tilde{\mathcal{O}}}(\Omega; E) = \{f \in \mathcal{O}(\Omega \cap \boldsymbol{C}^n; E)$; *for any positive* ε *and any compact set* K *in* Ω *and any* $q \in \mathcal{T}$, $\sup\{q(f(z))e(-\varepsilon|z|);\ z \in K \cap \boldsymbol{C}^n\} < \infty$ *holds*$\}$.

We call this sheaf ${}^E\tilde{\tilde{\mathcal{O}}}$ the sheaf of germs of slowly increasing E-valued holomorphic functions.

Definition 4.1.2 (The sheaf ${}^E\tilde{\tilde{\mathcal{E}}}$ of germs of slowly increasing E-valued C^∞-functions). *We define ${}^E\tilde{\tilde{\mathcal{E}}}$ to be the sheafification of the presheaf $\{\tilde{\tilde{\mathcal{E}}}(\Omega; E)\}$, where, for an open set Ω in $\boldsymbol{E}^n$, the module $\tilde{\tilde{\mathcal{E}}}(\Omega; E)$ is defined as follows:*

$\tilde{\tilde{\mathcal{E}}}(\Omega; E) = \{f \in \mathcal{E}(\Omega \cap \boldsymbol{C}^n; E)$; *for any positive* ε *and any compact set* K *in* Ω *and any* $\alpha \in \bar{\boldsymbol{N}}^{2n}$ *and any* $q \in \mathcal{T}$, $\sup\{q(f^{(\alpha)}(z))e(-\varepsilon|z|);\ z \in K \cap \boldsymbol{C}^n\} < \infty$ *holds*$\}$.

Then the sheaf ${}^E\tilde{\tilde{\mathcal{E}}}$ is a soft Fréchet sheaf and we have the following

Theorem 4.1.3 (The Dolbeault-Grothendieck resolution of ${}^E\tilde{\tilde{\mathcal{O}}}{}^p$). *The sequence of sheaves*

$$0 \longrightarrow {}^E\tilde{\tilde{\mathcal{O}}}{}^p|U \longrightarrow {}^E\tilde{\tilde{\mathcal{E}}}{}^{p,0}|U \xrightarrow{\bar{\partial}} {}^E\tilde{\tilde{\mathcal{E}}}{}^{p,1}|U \xrightarrow{\bar{\partial}} \cdots\cdots \xrightarrow{\bar{\partial}} {}^E\tilde{\tilde{\mathcal{E}}}{}^{p,n}|U \longrightarrow 0$$

is exact, where $U=\mathrm{int}\{z\in \boldsymbol{C}^n;\ |\mathrm{Im}\, z|-|\mathrm{Re}\, z|<d\}^a$ *for some* $d>0$.

Proof. The exactness of the sequence

$$0 \longrightarrow {}^E\tilde{\tilde{\mathcal{O}}}{}^p|U \longrightarrow {}^E\tilde{\tilde{\mathcal{E}}}{}^{p,1}|U \xrightarrow{\bar{\partial}} {}^E\tilde{\tilde{\mathcal{E}}}{}^{p,1}|U$$

is evident.

Next the exactness of the sequence

$${}^E\tilde{\tilde{\mathcal{E}}}{}^{p,0}|U \xrightarrow{\bar{\partial}} {}^E\tilde{\tilde{\mathcal{E}}}{}^{p,1}|U \xrightarrow{\bar{\partial}} \cdots\cdots \xrightarrow{\bar{\partial}} {}^E\tilde{\tilde{\mathcal{E}}}{}^{p,n}|U \longrightarrow 0$$

follows from the following

Lemma 4.1.4. *Let Ω be an $\tilde{\tilde{\mathcal{O}}}$-pseudoconvex open set in E^n. Then the equation $\bar{\partial}u=f$ has a solution $u\in\tilde{\tilde{\mathcal{E}}}{}^{p,q}(\Omega;\ E)$ for every $f\in\tilde{\tilde{\mathcal{E}}}{}^{p,q+1}(\Omega;\ E)$ such that $\bar{\partial}f=0$. Here $p,\ q\geqq 0$.*

Proof of Lemma 4.1.4. If we put $\tilde{\tilde{Z}}{}^{p,q+1}(\Omega)=\{f\in\tilde{\tilde{\mathcal{E}}}{}^{p,q+1}(\Omega);\ \bar{\partial}f=0\}$ and $\tilde{\tilde{Z}}{}^{p,q+1}(\Omega;\ E)=\{f\in\tilde{\tilde{\mathcal{E}}}{}^{p,q+1}(\Omega;\ E);\ \bar{\partial}f=0\}$, then $\tilde{\tilde{Z}}{}^{p,q+1}(\Omega)$ is a nuclear Fréchet space and

$$\tilde{\tilde{Z}}{}^{p,q+1}(\Omega;\ E)\cong\tilde{\tilde{Z}}{}^{p,q+1}(\Omega)\hat{\otimes}E$$

holds. By virtue of Theorem 3.2.10, we have an exact sequence

$$\tilde{\tilde{\mathcal{E}}}{}^{p,q}(\Omega) \longrightarrow \tilde{\tilde{Z}}{}^{p,q+1}(\Omega) \longrightarrow 0$$

for the $\tilde{\tilde{\mathcal{O}}}$-pseudoconvex open set. Then, since we have also

$$\tilde{\tilde{\mathcal{E}}}{}^{p,q}(\Omega;\ E)\cong\tilde{\tilde{\mathcal{E}}}{}^{p,q}(\Omega)\hat{\otimes}E,$$

we have an exact sequence

$$\tilde{\tilde{\mathcal{E}}}{}^{p,q}(\Omega;\ E) \xrightarrow{\bar{\partial}} \tilde{\tilde{Z}}{}^{p,q+1}(\Omega;\ E) \longrightarrow 0$$

by virtue of Trèves [36], Proposition 4.3.9. Q.E.D.

This completes the proof of Theorem 4.1.3. Q.E.D.

Corollary. *We use notations in Theorem* 4.1.3. *For an open set Ω in U, we have the following isomorphism*:

$$H^q(\Omega,\ \tilde{\tilde{\mathcal{O}}}{}^p)\cong\{f\in\tilde{\tilde{\mathcal{E}}}{}^{p,q}(\Omega;\ E);\ \bar{\partial}f=0\}/\{\bar{\partial}g;\ g\in\tilde{\tilde{\mathcal{E}}}{}^{p,q-1}(\Omega;\ E)\},\quad (p\geqq 0,\ q\geqq 1).$$

Proof. It follows from Theorem 4.1.3 and Komatsu [21], Theorems II.2.9 and II.2.19. Q.E.D.

4.2. The Oka-Cartan-Kawai Theorem B

We will prove the Oka-Cartan-Kawai Theorem B for the sheaf ${}^E\tilde{\tilde{\mathcal{O}}}$.

Theorem 4.2.1 (The Oka-Cartan-Kawai Theorem B). *For any $\tilde{\tilde{\mathcal{O}}}$-pseudoconvex open set Ω in $\boldsymbol{E}^n$, we have $H^q(\Omega, {}^E\tilde{\tilde{\mathcal{O}}}^p)=0$ for $p\geqq 0$ and $q\geqq 1$.*

Proof. Since we have, by the Oka-Cartan-Kawai Theorem B for $\tilde{\tilde{\mathcal{O}}}$,

$$H^s(\Omega, \tilde{\tilde{\mathcal{O}}}^p)=0 \quad (p\geqq 0 \text{ and } s\geqq 1),$$

the complex obtained from Theorem 3.2.11:

$$\tilde{\tilde{\mathscr{E}}}^{p,0}(\Omega) \xrightarrow{\bar{\partial}} \tilde{\tilde{\mathscr{E}}}^{p,1}(\Omega) \xrightarrow{\bar{\partial}} \cdots\cdots \xrightarrow{\bar{\partial}} \tilde{\tilde{\mathscr{E}}}^{p,n}(\Omega) \longrightarrow 0$$

is exact. Since $\tilde{\tilde{\mathscr{E}}}^{p,q}(\Omega)$'s are nuclear Fréchet spaces and E is a Fréchet space, the complex

$$\tilde{\tilde{\mathscr{E}}}^{p,0}(\Omega; E) \xrightarrow{\bar{\partial}} \tilde{\tilde{\mathscr{E}}}^{p,1}(\Omega; E) \xrightarrow{\bar{\partial}} \cdots\cdots \xrightarrow{\bar{\partial}} \tilde{\tilde{\mathscr{E}}}^{p,n}(\Omega; E) \longrightarrow 0$$

is also exact by virtue of the isomorphism

$$\tilde{\tilde{\mathscr{E}}}^{p,q}(\Omega; E)\cong \tilde{\tilde{\mathscr{E}}}^{p,q}(\Omega)\hat{\otimes} E$$

and Ion and Kawai [5], Theorem 1.10. Hence we obtain

$$H^q(\Omega, {}^E\tilde{\tilde{\mathcal{O}}}^p)=0 \quad (p\geqq 0,\ q\geqq 1).$$

This completes the proof. Q.E.D.

Corollary. *Let Ω be an $\tilde{\tilde{\mathcal{O}}}$-pseudoconvex open set in $\boldsymbol{E}^n$. Then the equation $\bar{\partial}u=f$ has a solution $u\in\tilde{\tilde{\mathscr{E}}}^{p,q}(\Omega; E)$ for every $f\in\tilde{\tilde{\mathscr{E}}}^{p,q+1}(\Omega; E)$ such that $\bar{\partial}f=0$. Here p and q are nonnegative integers.*

Proof. It follows from Theorem 4.2.1 and Corollary to Theorem 4.1.3. Q.E.D.

4.3. Malgrange's Theorem

We will prove Malgrange's Theorem for the sheaf ${}^E\tilde{\tilde{\mathcal{O}}}$.

Theorem 4.3.1. *Let Ω be an open set in $\boldsymbol{E}^n$ such that, for any $z\in\Omega\cap\boldsymbol{C}^n$, $|\mathrm{Im}\, z|-|\mathrm{Re}\, z|<d$ holds for some constant $d>0$. Then we have $H^n(\Omega, {}^E\tilde{\tilde{\mathcal{O}}})=0$.*

Proof. By virtue of Theorem 3.2.11 and 3.3.1, we have an exact sequence

$$\tilde{\tilde{\mathscr{E}}}^{0,n-1}(\Omega) \xrightarrow{\bar{\partial}} \tilde{\tilde{\mathscr{E}}}^{0,n}(\Omega) \longrightarrow 0.$$

Thus, by Trèves [36], Proposition 4.3.9, we have the exact sequence

$$\tilde{\tilde{\mathscr{E}}}^{0,n-1}(\Omega)\hat{\otimes} E \xrightarrow{\bar{\partial}} \tilde{\tilde{\mathscr{E}}}^{0,n}(\Omega)\hat{\otimes} E \longrightarrow 0$$

or

$$\tilde{\tilde{\mathscr{E}}}^{0,n-1}(\Omega; E) \xrightarrow{\bar{\partial}} \tilde{\tilde{\mathscr{E}}}^{0,n}(\Omega; E) \longrightarrow 0.$$

Hence we obtain the conclusion. Q. E. D.

Corollary. Flabby dim ${}^{E}\tilde{\tilde{\mathcal{O}}} \leqq n$.

4.4. Serre's duality theorem

Theorem 4.4.1. *Let Ω be an open set in $\boldsymbol{E}^n$ such as in Theorem* 4.3.1 *and such that* $\dim H^p(\Omega, \tilde{\tilde{\mathcal{O}}}) < \infty$ *holds* $(p \geqq 1)$. *Then we have the isomorphism* $H^p(\Omega, {}^{E}\tilde{\tilde{\mathcal{O}}}) \cong L(H_c^{n-p}(\Omega, \underset{\sim}{\mathcal{O}}); E)$, $0 \leqq p \leqq n$.

Proof. By the same method as Junker [15], Lemma 3.5, we can obtain the isomorphism $H^p(\Omega, {}^{E}\tilde{\tilde{\mathcal{O}}}) \cong H^p(\Omega, \tilde{\tilde{\mathcal{O}}}) \hat{\otimes}_\pi E$. Then, by Theorem 3.4.1, we have the following isomorphisms

$$H^p(\Omega, {}^{E}\tilde{\tilde{\mathcal{O}}}) \cong H^p(\Omega, \tilde{\tilde{\mathcal{O}}}) \hat{\otimes}_\pi E \cong [H_c^{n-p}(\Omega, \underset{\sim}{\mathcal{O}})]' \hat{\otimes}_\pi E \cong L(H_c^{n-p}(\Omega, \underset{\sim}{\mathcal{O}}); E).$$

Q. E. D.

4.5. Martineau-Harvey's Theorem

Theorem 4.5.1. *Let K be a compact set in $\boldsymbol{E}^n$ such that it has an $\tilde{\tilde{\mathcal{O}}}$-pseudoconvex open neighborhood Ω and satisfies the conditions $H^p(K, \underset{\sim}{\mathcal{O}}) = 0$ $(p \geqq 1)$. Then we have $H_K^p(\Omega, {}^{E}\tilde{\tilde{\mathcal{O}}}) = 0$ for $p \neq n$ and isomorphisms $H_K^n(\Omega, {}^{E}\tilde{\tilde{\mathcal{O}}}) \cong H^{n-1}(\Omega \backslash K, {}^{E}\tilde{\tilde{\mathcal{O}}}) \cong L(\underset{\sim}{\mathcal{O}}(K); E)$.*

Proof. We can assume that Ω is an $\tilde{\tilde{\mathcal{O}}}$-pseudoconvex open neighborhood of K. Then, in the long exact sequence of cohomology groups (cf. Komatsu [21], Theorem II.3.2):

$$\begin{aligned} 0 &\longrightarrow H_K^0(\Omega, {}^{E}\tilde{\tilde{\mathcal{O}}}) \longrightarrow H^0(\Omega, {}^{E}\tilde{\tilde{\mathcal{O}}}) \longrightarrow H^0(\Omega \backslash K, {}^{E}\tilde{\tilde{\mathcal{O}}}) \\ &\longrightarrow H_K^1(\Omega, {}^{E}\tilde{\tilde{\mathcal{O}}}) \longrightarrow H^1(\Omega, {}^{E}\tilde{\tilde{\mathcal{O}}}) \longrightarrow H^1(\Omega \backslash K, {}^{E}\tilde{\tilde{\mathcal{O}}}) \\ &\longrightarrow \cdots\cdots \\ &\longrightarrow H_K^n(\Omega, {}^{E}\tilde{\tilde{\mathcal{O}}}) \longrightarrow H^n(\Omega, {}^{E}\tilde{\tilde{\mathcal{O}}}) \longrightarrow H^n(\Omega \backslash K, {}^{E}\tilde{\tilde{\mathcal{O}}}) \longrightarrow \cdots\cdots, \end{aligned}$$

we have $H^p(\Omega, {}^{E}\tilde{\tilde{\mathcal{O}}}) = 0$ for $p \geqq 1$ and $H_K^0(\Omega, {}^{E}\tilde{\tilde{\mathcal{O}}}) = 0$ by the unique continuation theorem. Hence we have isomorphisms

$$H_K^1(\Omega, {}^{E}\tilde{\tilde{\mathcal{O}}}) \cong \tilde{\tilde{\mathcal{O}}}(\Omega \backslash K; E) / \tilde{\tilde{\mathcal{O}}}(\Omega; E)$$

$$H_K^p(\Omega, {}^{E}\tilde{\tilde{\mathcal{O}}}) \cong H^{p-1}(\Omega \backslash K, {}^{E}\tilde{\tilde{\mathcal{O}}}), \quad p \geqq 2.$$

But, by the same method as Junker [15], Lemma 3.5, we have isomorphisms $H^p(V, {}^{E}\tilde{\tilde{\mathcal{O}}}) \cong H^p(V, \tilde{\tilde{\mathcal{O}}}) \hat{\otimes}_\pi E$, $0 \leqq p \leqq n$, where V is an open set in $\boldsymbol{E}^n$ such that, for any $z \in V \cap \boldsymbol{C}^n$, $|\mathrm{Im}\, z| - |\mathrm{Re}\, z| < d$ holds for some constant $d > 0$. So that, by Theorem

3.5.1, we have isomorphisms

$$H^p_K(\Omega, {}^E\tilde{\tilde{\mathcal{O}}}) \cong H^p_K(\Omega, \tilde{\tilde{\mathcal{O}}}) \hat{\otimes}_\pi E = 0 \qquad \text{for} \quad p \neq n,$$

and

$$H^n_K(\Omega, {}^E\tilde{\tilde{\mathcal{O}}}) \cong H^{n-1}(\Omega \backslash K, {}^E\tilde{\tilde{\mathcal{O}}}) \cong H^{n-1}(\Omega \backslash K, \tilde{\tilde{\mathcal{O}}}) \hat{\otimes}_\pi E \cong H^n_K(\Omega, \tilde{\tilde{\mathcal{O}}}) \hat{\otimes}_\pi E$$
$$\cong \underset{\approx}{\mathcal{O}}(K)' \hat{\otimes}_\pi E \cong L(\underset{\approx}{\mathcal{O}}(K); E).$$

Q. E. D.

4.6. Sato's Theorem

In this section we will prove the pure-codimensionality of $\boldsymbol{D}^n$ with respect to ${}^E\tilde{\tilde{\mathcal{O}}}$. Then we will realize E-valued modified Fourier hyperfunctions as "boundary values" of E-valued slowly increasing holomorphic functions or as (relative) cohomology classes of E-valued slowly increasing holomorphic functions.

Theorem 4.6.1 (Sato's Theorem). *Let Ω be an open set in $\boldsymbol{D}^n$ and V an open set in $\boldsymbol{E}^n$ which contains Ω as its closed subset. Then we have the following*

(1) *The relative cohomology groups $H^p_\Omega(V, {}^E\tilde{\tilde{\mathcal{O}}})$ are zero for $p \neq n$.*

(2) *The presheaf over $\boldsymbol{D}^n$*

$$\Omega \longrightarrow H^n_\Omega(V, {}^E\tilde{\tilde{\mathcal{O}}})$$

is a flabby sheaf.

(3) *This sheaf* (2) *is isomorphic to the sheaf ${}^E\mathcal{Q}$ of E-valued modified Fourier hyperfunctions.*

Proof. (1) By the excision theorem, we may assume that V is an $\tilde{\tilde{\mathcal{O}}}$-pseudoconvex open set in $\boldsymbol{E}^n$. Consider the following exact sequence of relative cohomology groups

$$\begin{aligned}0 &\longrightarrow H^0_{\partial\Omega}(V, {}^E\tilde{\tilde{\mathcal{O}}}) \longrightarrow H^0_{\Omega^a}(V, {}^E\tilde{\tilde{\mathcal{O}}}) \longrightarrow H^0_\Omega(V, {}^E\tilde{\tilde{\mathcal{O}}}) \\ &\longrightarrow H^1_{\partial\Omega}(V, {}^E\tilde{\tilde{\mathcal{O}}}) \longrightarrow \quad \cdots\cdots \quad \longrightarrow H^{n-1}_\Omega(V, {}^E\tilde{\tilde{\mathcal{O}}}) \\ &\longrightarrow H^n_{\partial\Omega}(V, {}^E\tilde{\tilde{\mathcal{O}}}) \longrightarrow H^n_{\Omega^a}(V, {}^E\tilde{\tilde{\mathcal{O}}}) \longrightarrow H^n_\Omega(V, {}^E\tilde{\tilde{\mathcal{O}}}) \\ &\longrightarrow H^{n+1}_{\partial\Omega}(V, {}^E\tilde{\tilde{\mathcal{O}}}) \longrightarrow \cdots\cdots.\end{aligned}$$

By Theorems 3.1.8 and 4.5.1, we may conclude that $H^p_{\partial\Omega}(V, {}^E\tilde{\tilde{\mathcal{O}}}) = H^p_{\Omega^a}(V, {}^E\tilde{\tilde{\mathcal{O}}}) = 0$ for $p \neq n$. So that, we have $H^p_\Omega(V, {}^E\tilde{\tilde{\mathcal{O}}}) = 0$ for $p \neq n-1, n$. On the other hand, by Theorems 3.1.8 and 4.5.1, we also have the exact sequence

$$0 \longrightarrow H^{n-1}_\Omega(V, {}^E\tilde{\tilde{\mathcal{O}}}) \longrightarrow L(\underset{\approx}{\mathcal{A}}(\partial\Omega); E) \xrightarrow{j} L(\underset{\approx}{\mathcal{A}}(\Omega^a); E).$$

Since j is injective, we have $H_{\Omega}^{n-1}(V, {}^{E}\tilde{\tilde{\mathcal{O}}})=0$.

(2) By Malgrange's Theorem, we can conclude that flabby dim ${}^{E}\tilde{\tilde{\mathcal{O}}}\leqq n$. Thus by (1) and by the theorem II.3.24 of Komatsu [21], we have the conclusion.

(3) By the proof of (1), we have the exact sequence

$$0 \longrightarrow H_{\partial\Omega}^{n}(V, {}^{E}\tilde{\tilde{\mathcal{O}}}) \longrightarrow H_{\Omega^{a}}^{n}(V, {}^{E}\tilde{\tilde{\mathcal{O}}}) \longrightarrow H_{\Omega}^{n}(V, {}^{E}\tilde{\tilde{\mathcal{O}}}) \longrightarrow 0.$$

Since, by Martineau-Harvey's Theorem, we have isomorphisms

$$H_{\partial\Omega}^{n}(V, {}^{E}\tilde{\tilde{\mathcal{O}}})\cong L(\underset{\sim}{\mathscr{A}}(\partial\Omega); E),$$

$$H_{\Omega^{a}}^{n}(V, {}^{E}\tilde{\tilde{\mathcal{O}}})\cong L(\underset{\sim}{\mathscr{A}}(\Omega^{a}); E),$$

we obtain the isomorphism

$$H_{\Omega}^{n}(V, {}^{E}\tilde{\tilde{\mathcal{O}}})\cong L(\underset{\sim}{\mathscr{A}}(\Omega^{a}); E)/L(\underset{\sim}{\mathscr{A}}(\partial\Omega); E)=\mathscr{Q}(\Omega; E).$$

Thus the sheaf $\Omega\rightarrow H_{\Omega}^{n}(V, {}^{E}\tilde{\tilde{\mathcal{O}}})$ is isomorphic to the sheaf ${}^{E}\mathscr{Q}$ of E-valued modified Fourier hyperfunctions over $\boldsymbol{D}^{n}$. Q.E.D.

In the same notations as in Theorem 3.6.2, we have the following

Theorem 4.6.2. *$H_{\Omega}^{n}(V, {}^{E}\tilde{\tilde{\mathcal{O}}})\cong H^{n}(\mathfrak{B}, \mathfrak{B}', {}^{E}\tilde{\tilde{\mathcal{O}}})\cong\tilde{\tilde{\mathcal{O}}}(\cap_{j} V_{j}; E)/\sum_{j=1}^{n}\tilde{\tilde{\mathcal{O}}}(\cap_{i\neq j} V_{i}; E)$ holds.*

At last we will realize modified Fourier analytic linear mapping with certain compact carrier as (relative) cohomology classes with coefficients in ${}^{E}\tilde{\tilde{\mathcal{O}}}$.

Let K be a compact set in $\boldsymbol{E}^{n}$ of the form $K=K_{1}\times\cdots\times K_{n}$ with compact sets K_{j} in $\boldsymbol{E}$ $(j=1, 2,\dots, n)$. Assume that K admits a fundamental system of $\tilde{\tilde{\mathcal{O}}}$-pseudoconvex open neighborhoods. Then we have

$$H^{p}(K, \underset{\sim}{\mathcal{O}})=0 \qquad \text{for} \quad p>0.$$

By virtue of Martineau-Harvey's Theorem, there exists the isomorphism

$$\underset{\sim}{\mathcal{O}}'(K; E)\cong H_{K}^{n}(\Omega, {}^{E}\tilde{\tilde{\mathcal{O}}}).$$

Here Ω denotes an open neighborhood of K. Further assume that there exists an $\tilde{\tilde{\mathcal{O}}}$-pseudoconvex open neighborhood Ω of K such that

$$\Omega_{j}=\Omega\cap\{z\in\boldsymbol{E}^{n};\ z_{j}\notin K_{j}\}$$

is also an $\tilde{\tilde{\mathcal{O}}}$-pseudoconvex open set for $j=1, 2,\dots, n$. Then $\mathfrak{B}=\{\Omega_{0}=\Omega, \Omega_{1},\dots, \Omega_{n}\}$ and $\mathfrak{B}'=\{\Omega_{1}, \Omega_{2},\dots, \Omega_{n}\}$ form acyclic coverings of Ω and $\Omega\backslash K$. Set

$$\Omega\# K=\bigcap_{j=1}^{n}\Omega_{j},$$

$$\Omega^{j}=\bigcap_{i\neq j}\Omega_{i}.$$

Let $\sum_j \tilde{\tilde{\mathcal{O}}}(\Omega^j; E)$ be the image in $\tilde{\tilde{\mathcal{O}}}(\Omega \sharp K; E)$ of $\prod_{j=1}^{n} \tilde{\tilde{\mathcal{O}}}(\Omega^j; E)$ by the mapping

$$(f_j)_{j=1}^n \longrightarrow \sum_{j=1}^{n} (-1)^{j+1} f'_j,$$

where f'_j denotes the restriction of f_j to $\Omega \sharp K$.

Then, by the same method as that of Theorem 4.6.2, we have the following

Theorem 4.6.3. *We use the notations as above. Then we have the isomorphisms*

$$\underset{\approx}{\mathcal{O}}'(K; E) \cong H^n_K(\Omega, {}^E\tilde{\tilde{\mathcal{O}}}) \cong H^n(\mathfrak{B}, \mathfrak{B}', {}^E\tilde{\tilde{\mathcal{O}}}) \cong \tilde{\tilde{\mathcal{O}}}(\Omega \sharp K; E) / \sum_j \tilde{\tilde{\mathcal{O}}}(\Omega^j; E).$$

Department of Mathematics
College of General Education
Tokushima University

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