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The Gauss Map of Kaehler Immersions into Complex Hyperbolic Spaces

By

Toru ISHIHARA

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§1. Introduction

In the previous paper [2], we studied the Gauss maps of Kaehler immersions in compex projective spaces. We will show in this paper that the similar argument is valid for Kaehler immersions in complex hyperbolic spaces.

It will be agreed that our indices have the following ranges throughout this paper; $0 \leqq A, B, C \leqq N$, $1 \leqq a, b, c \leqq N$, $1 \leqq i, j, k \leqq n$, $n+1 \leqq r, s, t \leqq N$, $0 \leqq i^*, j^*, k^* \leqq n$, $n+1 \leqq r^*, s^*, t^* \leqq N$ or $r^*, s^*, t^* = 0$.

Let C^{N+1} be a complex $(N+1)$-space with the indefinite metric $\langle z, w \rangle = -z^0 \bar{z}^0 + \sum z^a \bar{w}^a$. Put

$$C_0^{N+1} = \{z \in C^{N+1}, \langle z, z \rangle < 0\}.$$

Two points $z_1, z_2 \in C_0^{N+1}$ are equivalent, provided that there is a non-zero complex number c such that $z_1 = cz_2$. The quotient of C_0^{N+1} by this equivalent relation is just a complex hyperbolic N-space $H^N(C)$. Let $U(1, N)$ be the group of all linear isometries of C^{N+1}. Obviously $H^N(C)$ is the homogeneous space $U(1, N)/U(1) \times U(N)$. Let π_c: $C_0^{N+1} \to H^N(C)$ be the natural projection. For $z \in C_0^{N+1}$, put $\pi_c(z) = y$. Then $\ker(d\pi_c)_z = \{z\}$, where $\{z\}$ is the complex line in C_0^{N+1} determined by z. Let $\{z\}^\perp$ be the orthonormal complement of $\{z\}$ in C^{N+1}. Then it is an N-subspace with the induced definite Hermitian metric. The restriction of $(d\pi_c)_z$ to $\{z\}^\perp$ gives an isomorphism of $\{z\}^\perp$ onto the holomorphic tangent bundle $T'_y H^N(C)$ of $H^N(C)$ at y. Let λ_z be the inverse of the isomorphism. Then for $v \in T'_y H^N(C)$, $\lambda_{cz}(v) = c \cdot \lambda_z(v)$. Set $\mu_v(z) = \lambda_z(v)$. Thus we have an isomorphism: $v \to \mu_v$ of $T'_y H^N(C)$ onto $\{z\}^* \times \{z\}^\perp$, where $\{z\}^*$ is the dual space of $\{z\}$. Using this isomorphism, we can regard a linear subspace of $T'_y H^N(C)$ as a linear subspace of C^{N+1}.

Let $f: M \to H^N(C)$ be a holomorphic immersion of a Kaehler n-space M to $H^N(C)$. Since $df(T'_x M)$, $x \in M$, is viewed as a complex n-subspace of C^{N+1}, we can define the Gauss map g of M to a flag manifold $F_{n,n+1} = U(1, N)/U(1) \times U(n) \times U(N-n)$ by $g(x) = (\{f(x)\}, df(T'M))$. Let π_1 (resp. π_2) be the projection of $F_{n,n+1}$ to $F_n = U(1, N)/U(1) \times U(1, N-n)$ (resp. $F_{n+1} = U(1, N)/U(1, n) \times U(N-n)$). Put

$g_i=\pi_i g$ $(i=1, 2)$, these will be also called Gauss maps. The map g_2 is the same as the Gauss map in the sense of Nishikawa [3]. He showed that his Gauss map is anti-holomorphic (-holomorphic). We show that if an immersion is not totally geodesic, the corresponding maps g and g_1 are not $\pm$ holomorphic. But they are always harmonic.

§ 2. Kaehler immersions

In the sequel, we denote $H^N(C)$ by H^N simply. Let FH^N be the bundle of unitary frames of H^N. Let F^*H^N be the set of elements $b=(y, e_0, e_1,\dots, e_N)$ such that $e_0 \in C_0^{N+1}$, $\|e_0\|=-1$, $d\pi_c(e_0)=0$ and $(y, e_1,\dots, e_N)\in FH^N$. Then F^*H^N is a principal bundle over H^N with group $U(1)\times U(N)$. Since $T'_y H^N \cong \pi_c^{-1}(y)\times\{\pi_c^{-1}(y)\}^{\perp}$, each e_a is regarded as a linear map e_a: $\pi_c^{-1}(y)\to\{\pi_c^{-1}(y)\}^{\perp}$. By identifying moreover e_a with $e_a(e_0)$, we view F^*H^N as a subspace of $H^N\times U(1, N)$.

Let ϕ_B^A be the Maurer-Cartan forms of the group $U(1, N)$. Then they satisfy the following relations:

$$\phi_0^a-\bar{\phi}_a^0=0, \qquad \phi_b^a+\bar{\phi}_a^b=0$$

and

$$d\phi_B^A=\sum \phi_B^C\wedge\phi_C^A.$$

Let ρ: $H^N\times U(1, N)\to U(1, N)$ be the natural projection. Denote by θ^a (resp. θ_b^a) the restriction of $\rho^*\phi_0^a$ (resp. $\rho^*\phi_b^a-\delta_b^a\rho^*\phi_0^0$) to F^*H^N. Then the Kaehler metric on H^N is given locally by $d\sigma^2=2\sum\theta^a\bar{\theta}^a$ and the structure equations in H^N are written as

$$(1)\qquad \begin{aligned} d\theta^a&=\sum\theta^b\wedge\theta_b^a\\ d\theta_b^a&=\sum\theta_b^c\wedge\theta_c^a-\theta^a\wedge\bar{\theta}^b-\delta_b^a\sum\theta^c\wedge\bar{\theta}^c. \end{aligned}$$

Let M be a Kaehler n-manifold isometrically immersed into H^N by a holomorphic mapping f: $M\to H^N$. Let FM be the bundle of unitary frames on M. Let denote by F^*M the set of elements $b=(x, e_0,\dots, e_N)$ such that $(x, e_1,\dots, e_n)\in FM$ and $(f(x), e_0, df(e_1),\dots, df(e_n), e_{n+1},\dots, e_N)\in F^*H^N$. Then F^*M is a principal bundle over M with group $U(1)\times U(n)\times U(N-n)$. The natural immersion $\tilde{f}$: $F^*M\to F^*H^N$ is defined by $\tilde{f}(b)=(f(x), e_0, df(e_1),\dots, df(e_n), e_{n+1},\dots, e_N)\in F^*H^N$. On putting $\omega^a=\tilde{f}^*\theta^a$ and $\omega_b^a=\tilde{f}^*\theta_b^a$, we have $\omega^r=0$ and we can put $\omega_k^r=\sum A_{kj}^r\omega^k$, where A_{kj}^r are regarded as the components of the second fundamental form of f: $M\to H^N$. The Kaehler metric on M is given locally by $ds^2=2\sum\theta^i\bar{\theta}^i$. From (1), we obtain the following structure equations in M

$$\begin{aligned} d\omega^i&=\sum\omega^k\wedge\omega_k^i\\ d\omega_j^i&=\sum\omega_j^k\wedge\omega_k^i-\sum\delta_{ik}\delta_{jl}+\delta_{ij}\delta_{kl}+\sum A_{jk}^r\bar{A}_{il}^r)\omega^k\wedge\bar{\omega}^l. \end{aligned}$$

Let $-2i \sum R_{l\bar{k}}\omega^l \wedge \bar{\omega}^k$ be the Ricci form of M (in this paper, $\bar{i}$, $\bar{j}$, $\bar{k}$ run from $\bar{1}$ to $\bar{n}$). Then it holds

$$(2) \qquad \sum A^r_{\bar{i}j}\bar{A}^r_{\bar{i}k} = -(n+1)\delta_{jk} - R_{j\bar{k}}.$$

§3. The Gauss map

Let $\tilde{g}$ be the composition of the natural immersion $f\colon F^*M \to F^*H^N \subset H^N \times U(1, N)$ and the natural projection $\rho\colon H^N \times U(1, N) \to U(1, N)$. Then $\tilde{g}$ is viewed as a bundle map

$$\begin{array}{ccc} F^*M & \xrightarrow{\tilde{g}} & U(1, N) \\ \downarrow & & \downarrow \\ M & \longrightarrow & F_{n,n+1}. \end{array}$$

The corresponding base map g is just the Gauss map given in §1. Let g_1 and g_2 be also as in §1.

We introduce a quadratic differential form

$$d\sum{}^2 = -2\sum \phi^i_0\bar{\phi}^i_0 - 2\sum \phi^0_r\bar{\phi}^0_r + 2\sum \phi^i_r\bar{\phi}^i_r$$

and a complex structure J given by $J\phi^j_0 = i\phi^j_0$, $J\phi^{i*}_r = i\phi^{i*}_r$ on the flag manifold $F_{n,n+1}$. Then $(d\sum^2, J)$ gives a Hermitian structure of $F_{n,n+1}$. From the definition of the Gauss map g, we can see

$$(3) \qquad g^*(\phi^i_0) = \omega^i, \quad g^*(\phi^0_r) = 0, \quad g^*(\phi^i_r) = -\sum \bar{A}^r_{\bar{i}j}\bar{\omega}^j.$$

A Kaehler metric $d\sum_1^2$ (resp. $d\sum_2^2$) on F_n (resp. F_{n+1}) is given by $d\sum_1^2 = -2\sum \cdot \phi^i_0\bar{\phi}^i_0 + 2\sum \phi^i_r\bar{\phi}^i_r$ (resp. $d\sum_2^2 = -2\sum \phi^0_r\bar{\phi}^0_r + 2\sum \phi^i_r\bar{\phi}^i_r$) and a complex structure J_1 (resp. J_2) is given by $J_1\phi^i_{r*} = i\phi^i_{r*}$ (resp. $J_2\phi^{i*}_r = i\phi^{i*}_r$) on F_n (resp. F_{n+1}). We have also

$$(4) \qquad g_1^*(\phi^i_0) = \omega^i, \qquad g_1^*(\phi^i_r) = -\sum \bar{A}^r_{\bar{i}j}\bar{\omega}^j,$$

$$(5) \qquad g_2^*(\phi^0_r) = 0, \qquad g_2^*(\phi^i_r) = -\sum \bar{A}^r_{\bar{i}j}\bar{\omega}^j.$$

Using (2), we can show

$$(6) \qquad g^*(d\sum{}^2) = g_1^*(d\sum{}_1^2) = ds^2 + \sum A^r_{\bar{i}j}\bar{A}^r_{\bar{i}k}\omega^j\bar{\omega}^k = -n\,ds^2 - \sum R_{j\bar{k}}\omega^j\bar{\omega}^k,$$

$$(7) \qquad g_2(d\sum{}_2^2) = \sum A^r_{\bar{i}j}\bar{A}^r_{\bar{i}k}\omega^j\bar{\omega}^k = -(n+1)ds^2 - \sum R_{j\bar{k}}\omega^j\bar{\omega}^k.$$

Nishikawa [3] proved that g_2 is anti-holomorphic and that g_2 is a constant map iff the immersion f is totally geodesic. From (6) it follows

Proposition 1. *If the immersion f is totally geodesic, g and g_1 are isometric and holomorphic. Conversely if g is isometric or holomorphic, or if g_1 is isometric* *or holomorphic, h is totally geodesic.*

Proposition 2. *Let $f: M \to H^N$ be a holomorphic and isometric immersion. The following are equivalent*: (1) *M Einstein*; (2) *g a homothety or constant map*; (3) *g_1 a homothety or constant map*; (4) *g_2 a homothety or constant map.*

§ 4. The harmonic Gauss map

Put

$$\phi^{i^*r^*}_{j^*s^*} = \phi^{i^*}_{j^*}\delta^{r^*}_{s^*} - \delta^{i^*}_{j^*}\phi^{s^*}_{r^*}.$$

Then the structure equations in the Kaehler manifold F_n (resp. F_{n+1}) are given by

$$d\phi^i_{r^*} = -\sum \phi^{ir^*}_{js^*} \wedge \phi^j_{s^*} \quad (\text{resp. } d\phi^{i^*}_r = -\sum \phi^{i^*r}_{j^*s} \wedge \phi^j_{s^*}).$$

On the other hand, the structure equations in $F_{n,n+1}$ are given by

$$d\phi^i_0 = -\sum \phi^{i0}_{jr^*} \wedge \phi^j_{r^*},$$

$$d\phi^0_r = -\sum \phi^{0r}_{j^*r} \wedge \phi^{j^*}_r,$$

$$d\phi^i_r = -\sum \phi^{ir^*}_{js} \wedge \phi^j_{s^*} - \sum \phi^{ir}_{0s} \wedge \phi^0_s + \phi^i_0 \wedge \phi^0_r.$$

Thus, by the very same calculation as in §4 of [2], we can show that the Gauss map g is π_1-horizontal and

Theorem 3. *Let $f: M \to H^N$ be a holomorphic and isometric immersion of a Kaehler n-manifold M. Then the corresponding Gauss maps $g: M \to F_{n,n+1}$ and $g_1: M \to F_n$ are harmonic.*

Department of Mathematics
Faculty of Education
Tokushima University

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