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Galerkin Method for Autonomous Differential Equations with Unknown Parameters

By

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§1. Introduction

We shall consider a real nonlinear autonomous differential system with unknown parameters $B_1,\dots, B_{d-1}$ $(1 \le d \le n)$ of the form

$$\frac{dx}{d\tau} = X(x, B_1,\dots, B_{d-1})\,. \tag{1.1}$$

Here x and $X(x, B_1,\dots, B_{d-1}) \in C^1[D_1 \times R^{d-1}]$ are n-dimensional vectors, where D_1 is a domain in the x-space and R^{d-1} is the $(d-1)$-dimensional Euclidean space. Let $x(\tau)$ be a desired ω-periodic solution of the system (1.1) depending on the unknown parameters $B_1,\dots, B_{d-1}$, where ω is unknown, too. We transform τ to t by $\tau = \frac{\omega}{2\pi} t$, then (1.1) is rewritten in the form

$$\frac{dx}{dt} = \frac{\omega}{2\pi} X(x, B_1,\dots, B_{d-1})\,. \tag{1.2}$$

Then the problem is reduced to the one of finding a 2π-periodic solution of (1.2). Hence, the problem is equivalent to

(1.3) *Looking for a function* $u = u(t) = (x(t), B_1,\dots, B_{d-1}, \omega)$ *satisfying the equation*

$$F(u) = \left[\frac{dx}{dt} - \frac{\omega}{2\pi} X(x, B_1,\dots, B_{d-1}), f(u)\right] = 0,$$

where $f(u) = \begin{pmatrix} x(0) - x(2\pi) \\ L(u) - \beta \end{pmatrix}$, β *is a d-dimensional vector, and L is a linear mapping.*

In § 3, we shall consider the linear mapping L in detail.

In the present paper we discuss the question of existence and numerical approximation of periodic solutions of (1.3). Setting $u_m(t) = (x_m(t), B_1,\dots, B_{d-1}, \omega)$, where $x_m(t) = a_0 + \sqrt{2} \sum_{n=1}^{m} (a_{2n-1} \cos nt + a_{2n} \sin nt)$, $B_1,\dots, B_{d-1}$ are undertermined parameters and ω is an undertermined period, then it may be able to determine these $\{n(2m+1)+d\}$ coefficients by Galerkin method so that $\boldsymbol{u}_m(t)$ satisfies identically the system

$$\frac{d\boldsymbol{x}_m}{dt}=\frac{1}{2\pi}\int_0^{2\pi}\boldsymbol{X}(\boldsymbol{u}_m(s))ds+\frac{1}{\pi}\sum_{n=1}^{m}\left\{\cos nt\int_0^{2\pi}\boldsymbol{X}(\boldsymbol{u}_m(s))\cos ns\ ds\right.$$
$$\left.+\sin nt\int_0^{2\pi}\boldsymbol{X}(\boldsymbol{u}_m(s))\sin ns\ ds\right\},$$

and the equation

$$\boldsymbol{f}(\boldsymbol{u}_m(t))=\boldsymbol{0},$$

or equivalently

$$(1.4)\quad \begin{cases} \boldsymbol{F}_0^{(m)}(\boldsymbol{\alpha})=\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\boldsymbol{X}(\boldsymbol{u}_m(s))ds=\boldsymbol{0}, \\ \boldsymbol{F}_{2n-1}^{(m)}(\boldsymbol{\alpha})=\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_0^{2\pi}\boldsymbol{X}(\boldsymbol{u}_m(s))\cos ns\ ds-n\boldsymbol{\alpha}_{2n}=\boldsymbol{0}, \\ \boldsymbol{F}_{2n}^{(m)}(\boldsymbol{\alpha})=\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_0^{2\pi}\boldsymbol{X}(\boldsymbol{u}_m(s))\sin ns\ ds+n\boldsymbol{\alpha}_{2n-1}=\boldsymbol{0}, \\ \qquad\qquad\qquad\qquad\qquad\qquad (n=1,2,\dots,m), \\ \boldsymbol{G}_f(\boldsymbol{\alpha})=\boldsymbol{f}(\boldsymbol{u}_m(t))=\boldsymbol{0}, \end{cases}$$

where $\boldsymbol{X}(\boldsymbol{u}_m(s))=\frac{\omega}{2\pi}\boldsymbol{X}(\boldsymbol{x}_m(s), B_1,\dots, B_{d-1})$ and $\boldsymbol{\alpha}$ is the $\{n(2m+1)+d\}$-dimensional vector such that $\boldsymbol{\alpha}=\text{col}\,[\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1,\dots, \boldsymbol{\alpha}_{2m-1}, \boldsymbol{\alpha}_{2m}, B_1,\dots, B_{d-1}, \omega]$. It is to be expected that for m sufficiently large $\boldsymbol{u}_m(t)$ determined by (1.4) may be a reasonable approximation to a periodic solution $\hat{\boldsymbol{u}}(t)$ of (1.3).

In the present paper, concerning this point, we also obtain the following two results similar to the ones given by Y. Shinohara [15] in autonomous cases;

1. *The existence of an isolated periodic solution* $\hat{\boldsymbol{u}}(t)$ *of* (1.3) *lying in the interior of the region of definition of* $\boldsymbol{F}(\boldsymbol{u})$ *always implies the existence of Galerkin approximations* $\boldsymbol{u}_m(t)$ *of any order* m *sufficiently high, and these Galerkin approximations* $\boldsymbol{u}_m(t)$ *converge to the exact solution* $\hat{\boldsymbol{u}}(t)$ *uniformly as* $m\to\infty$.

2. *The existence of a "good" approximation* $\bar{\boldsymbol{u}}(t)$ *always implies the existence of an exact solution and an error bound for* $\bar{\boldsymbol{u}}(t)$ *are given at the same time.*

Lastly, in §6, we apply our results to van der Pol equation appeared in the electrical engineering. The results of numerical computations are original and useful.

Now, in order to consider the problem (1.3), let us prepare the following notations.

Let D_1 be a domain in the $\boldsymbol{x}$-space, and consider the product sets $B=D_1\times\boldsymbol{R}^d$ and $\Omega=I\times B$, where $I=[0, 2\pi]$. We put

$$C^1[I]=\{\boldsymbol{x}(t)=\mathrm{col}\,[x_1(t),\dots, x_n(t)];\ x_i(t)\ (i=1,\dots, n)\ \textit{are}\ C^1\textit{-class on}\ I\},$$

$$C[I]=\{\boldsymbol{x}(t)=\mathrm{col}\,[x_1(t),\dots, x_n(t)];\ x_i(t)\ (i=1,\dots, n)\ \textit{are continuous on}\ I\},$$

$$S=\{\boldsymbol{u}(t)=(\boldsymbol{x}(t), B_1,\dots, B_{d-1}, \omega);\ (t, \boldsymbol{u}(t))\in\Omega\ \textit{for all}\ t\in I,\ \boldsymbol{u}(t)\in M\equiv C^1[I]\times\boldsymbol{R}^d\}$$

and

$$S'=\{\boldsymbol{u}(t)=(\boldsymbol{x}(t), B_1,\dots, B_{d-1}, \omega);\ (t, \boldsymbol{u}(t))\in\Omega\ \textit{for all}\ t\in I,\ \boldsymbol{u}(t)\in C[I]\times\boldsymbol{R}^d\}.$$

We shall denote the Euclidean norm in $\boldsymbol{R}^n$ by $\|\cdot\|_n$ and define the norms in the product spaces $C[I]\times\boldsymbol{R}^d$ and $N\equiv C[I]\times\boldsymbol{R}^{n+d}$ by the formulas

$$\|\boldsymbol{u}(t)\|_\infty=\|\boldsymbol{x}(t)\|_c+|B_1|+\cdots+|B_{d-1}|+|\omega| \qquad \text{for}\quad \boldsymbol{u}(t)\in C[I]\times\boldsymbol{R}^d,$$

and

$$|||\boldsymbol{n}|||=\|\boldsymbol{\varphi}(t)\|_c+\|\boldsymbol{v}\|_{n+d} \qquad \text{for}\quad \boldsymbol{n}=(\boldsymbol{\varphi}(t), \boldsymbol{v})\in C[I]\times R^{n+d},$$

respectively, where $\|\boldsymbol{x}(t)\|_c=\sup_{t\in I}\|\boldsymbol{x}(t)\|_n$. Then both the product spaces $C[I]\times\boldsymbol{R}^d$ and N are Banach spaces with respect to the above norms, respectively. Now, the problem (1.3) is summarized as follows:

$$(1.5)\quad \begin{cases} \textit{Look for a function } \boldsymbol{u}(t)\in M=C^1[I]\times\boldsymbol{R}^d \textit{ satisfying the equation}\\ \boldsymbol{F}(\boldsymbol{u})=\left[\dfrac{d\boldsymbol{x}}{dt}-\dfrac{\omega}{2\pi}\boldsymbol{X}(\boldsymbol{x}, B_1,\dots, B_{d-1}), \boldsymbol{f}(\boldsymbol{u})\right]=\boldsymbol{0},\\ \textit{where } L\ (C[I]\times\boldsymbol{R}^d\to\boldsymbol{R}^d) \textit{ is a linear mapping.}\end{cases}$$

In (1.5), we assume that the function $\boldsymbol{F}(\boldsymbol{u})$ with domain $S\subset M$ and range N is continuously weak Fréchet differentiable. The weak Fréchet differential of $\boldsymbol{F}(\boldsymbol{u})$ at $\boldsymbol{u}=\boldsymbol{u}(t)=(\boldsymbol{x}(t), B_1,\dots, B_{d-1}, \omega)$ can be written as follows:

$$\boldsymbol{F}'(\boldsymbol{u})\boldsymbol{h}=\left[\frac{d\boldsymbol{h}_1}{dt}-\boldsymbol{X}_{\boldsymbol{u}}(\boldsymbol{u}(t))\boldsymbol{h}^t, \boldsymbol{f}'(\boldsymbol{u})\boldsymbol{h}\right] \qquad \text{for any}\ \boldsymbol{h}=(\boldsymbol{h}_1(t), h_{n+1},\dots, h_{n+d})\in M,$$

where $$\boldsymbol{X}_{\boldsymbol{u}}(\boldsymbol{u}(t))=\Big(\frac{\omega}{2\pi}\boldsymbol{X}_{\boldsymbol{x}}(\boldsymbol{x}, B_1,\dots, B_{d-1})\quad \frac{\omega}{2\pi}\boldsymbol{X}_{B_1}(\boldsymbol{x}, B_1,\dots, B_{d-1})\cdots$$
$$\cdots\frac{\omega}{2\pi}\boldsymbol{X}_{B_{d-1}}(\boldsymbol{x}, B_1,\dots, B_{d-1})\quad \frac{1}{2\pi}\boldsymbol{X}(\boldsymbol{x}, B_1,\dots, B_{d-1})\Big),$$

$\boldsymbol{f}'(\boldsymbol{u})\boldsymbol{h}=\begin{pmatrix}\boldsymbol{h}_1(0)-\boldsymbol{h}_1(2\pi)\\ L(\boldsymbol{h})\end{pmatrix}$, $\boldsymbol{X}_{\boldsymbol{x}}(\boldsymbol{x}, B_1,\dots, B_{d-1})$ is the Jacobian matrix of $\boldsymbol{X}(\boldsymbol{x}, B_1,\dots, B_{d-1})$ with respect to $\boldsymbol{x}$, and $\boldsymbol{h}^t$ denotes the transpose of $\boldsymbol{h}$.

§ 2. Basic Theorems

We consider a linear operator T mapping M into N of the following form:

$$(2.1)\quad T\boldsymbol{h}=\left[\frac{d\boldsymbol{h}_1}{dt}-A(t)\boldsymbol{h}^t,\ \mathscr{L}\boldsymbol{h}\right]\qquad \text{for any}\quad \boldsymbol{h}=(\boldsymbol{h}_1(t),\ h_{n+1},\dots,\ h_{n+d})\in M,$$

where $A(t)$ is an $n\times(n+d)$-matrix whose elements are continuous on I and $\mathscr{L}$ is a linear mapping $C[I]\times\boldsymbol{R}^d\rightarrow\boldsymbol{R}^{n+d}$.

M. Urabe has obtained the following theorem about the inverse operator of T.

Theorem 1 (M. Urabe [6]).
If the $(n+d)\times(n+d)$ matrix $G=\mathscr{L}[\Psi(t)]$ is non-singular, namely,

$$\det G=\det\mathscr{L}[\Psi(t)]=\det(\mathscr{L}\tilde{\boldsymbol{\varphi}}_1,\ \mathscr{L}\tilde{\boldsymbol{\varphi}}_2,\dots,\ \mathscr{L}\tilde{\boldsymbol{\varphi}}_{n+d})\neq 0,$$

then the operator T has a linear inverse operator T^{-1}. That is, for any $\boldsymbol{n}=(\boldsymbol{\varphi}(t),\boldsymbol{v})\in N$ there exists one and only one solution $\boldsymbol{h}=(\boldsymbol{h}_1(t),\ h_{n+1},\dots,\ h_{n+d})\in M$ satisfying the equation $T\boldsymbol{h}=\boldsymbol{n}$. The solution $\boldsymbol{h}$ can be written as follows:

$$\boldsymbol{h}^t=\mathrm{col}\,[\boldsymbol{h}_1(t),\ h_{n+1},\dots,\ h_{n+d}]=H_1\boldsymbol{\varphi}+H_2\boldsymbol{v},$$

where

$$\begin{cases} H_1\boldsymbol{\varphi}=\Psi(t)\displaystyle\int_0^t\Psi^{-1}(s)\boldsymbol{\phi}(s)ds-\Psi(t)G^{-1}\mathscr{L}\Big[\Psi(t)\displaystyle\int_0^t\Psi^{-1}(s)\boldsymbol{\phi}(s)ds\Big],\\ H_2\boldsymbol{v}=\Psi(t)G^{-1}\boldsymbol{v},\quad \boldsymbol{\phi}(s)=\mathrm{col}\,[\boldsymbol{\varphi}(s),\ \boldsymbol{0}]\qquad (\text{where } \boldsymbol{0} \text{ is a } d\text{-dimensional vector}),\end{cases}$$

and by $\Psi(t)$ we denote the fundamental matrix of the linear homogeneous system

$$\frac{d\boldsymbol{z}}{dt}=\begin{pmatrix}A(t)\\ \boxed{0}\end{pmatrix}\boldsymbol{z}\qquad (\text{where } \boxed{0} \text{ is a } d\times(n+d) \text{ matrix})$$

with the initial condition $\Psi(0)=E((n+d)\times(n+d)$ unit matrix) and by $\mathscr{L}[\Psi(t)]$ we denote the matrix whose column vectors are $\mathscr{L}\tilde{\boldsymbol{\varphi}}_i(t)\,(i=1,2,\dots,n+d)$, where we put $\tilde{\boldsymbol{\varphi}}_i=(\boldsymbol{\varphi}_{i1},\ \psi_{in+1},\dots,\ \psi_{in+d})=\boldsymbol{\varphi}_i^t\in M$ and where $\boldsymbol{\varphi}_i=\mathrm{col}\,[\boldsymbol{\varphi}_{i1},\ \psi_{in+1},\dots,\ \psi_{in+d}]$ are column vectors of the matrix $\Psi(t)$.

Now, we introduce a concept of "*the isolatedness of a solution of* (1.3)".

Definition 1.
Let $\boldsymbol{u}=\hat{\boldsymbol{u}}(t)$ be a solution of (1.3). *Then, the solution $\boldsymbol{u}=\hat{\boldsymbol{u}}(t)$ of* (1.3) *is called an "isolated solution" if*

$$\det\boldsymbol{f}'(\hat{\boldsymbol{u}}(t))\,[\hat{\Psi}(t)]\neq 0,$$

where $\hat{\Psi}(t)$ is the fundamental matrix of the linear homogeneous system

$$\frac{dz}{dt} = \begin{pmatrix} X_u(\hat{u}(t)) \\ \boxed{0} \end{pmatrix} z \qquad (\textit{where } \boxed{0} \textit{ is a } d\times(n+d) \textit{ matrix})$$

satisfying $\hat{\Psi}(0)=E((n+d)\times(n+d)$ *unit matrix*).

Proposition 1.

The following statements are equivalent:

$$(2.2) \qquad \begin{cases} (a) & \det f'(\hat{u})[\hat{\Psi}(t)] \neq 0, \\ (b) & \textit{there exists the inverse operator of } F'(\hat{u}). \end{cases}$$

PROOF. We consider the linear boundary value problem

$$F'(\hat{u})h = \left[\frac{dh_1}{dt} - X_u(\hat{u})h^t, f'(\hat{u})h\right] = (\varphi(t), v) \qquad \text{for any} \quad (\varphi(t), v) \in N.$$

The solution $h=h(t)$ is denoted as follows; $\tilde{h}=h^t=\text{col}\,[h_1(t), h_{n+1},\dots, h_{n+d}]$ and

$$\tilde{h} = \hat{\Psi}(t)c + \hat{\Psi}(t)\int_0^t \hat{\Psi}^{-1}(s)\phi(s)ds, \quad \phi(s)=\text{col}\,[\varphi(s), \mathbf{0}]$$

($\mathbf{0}$ is a d-dimensional zero vector).

Therefore we get

$$f'(\hat{u})h = f'(\hat{u})[\hat{\Psi}(t)]c + f'(\hat{u})[\hat{\Psi}(t)\int_0^t \hat{\Psi}^{-1}(s)\phi(s)ds] = v.$$

This equation shows that if $\det f'(\hat{u})[\hat{\Psi}(t)] \neq 0$, there exists one and only one vector c. That is, there exists the inverse operator of $F'(\hat{u})$. Conversely, if $F'(\hat{u})^{-1}$ exists, we can determine the one and only one $(n+d)$-dimensional vector c for any vector v. This implies

$$\det f'(\hat{u})[\hat{\Psi}(t)] \neq 0. \qquad \text{Q.E.D.}$$

According to (2.2), we may define that $u=\hat{u}(t)$ is "*isolated*" if there exists the inverse of $F'(\hat{u})$.

Now, we devide the linear inverse operator T^{-1} of the linear operator T defined by (2.1) into several linear operators. Setting

$$h = T^{-1}n = T^{-1}(\varphi(t), v) = (T_1^{-1}(\varphi(t), v), T_{n+1}^{-1}(\varphi(t), v), \dots, T_{n+d}^{-1}(\varphi(t), v)),$$

that is,

$$\begin{cases} T_1^{-1}(\varphi(t), v) = h_1(t), \\ T_{n+1}^{-1}(\varphi(t), v) = h_{n+1}, \\ \quad\vdots \qquad\qquad\quad \vdots \\ T_{n+d}^{-1}(\varphi(t), v) = h_{n+d} \end{cases}$$

and

$$H_1\varphi=\begin{pmatrix} H_{11}\varphi \\ H_{1n+1}\varphi \\ \vdots \\ H_{1n+d}\varphi \end{pmatrix}, \quad H_2 \boldsymbol{v}=\begin{pmatrix} H_{21}\boldsymbol{v} \\ H_{2n+1}\boldsymbol{v} \\ \vdots \\ H_{2n+d}\boldsymbol{v} \end{pmatrix},$$

then we have

$$\begin{cases} T_1^{-1}\ (\varphi(t),\ \boldsymbol{v})=H_{11}\varphi+H_{21}\boldsymbol{v}, \\ T_{n+1}^{-1}(\varphi(t),\ \boldsymbol{v})=H_{1n+1}\varphi+H_{2n+1}\boldsymbol{v}, \\ \quad\vdots \\ T_{n+d}^{-1}(\varphi(t),\ \boldsymbol{v})=H_{1n+d}\varphi+H_{2n+d}\boldsymbol{v}. \end{cases}$$

Therefore we obtain

$$\begin{cases} \|T_1^{-1}\|_c \leq \max(\|H_{11}\|_c,\ \|H_{21}\|_c), \\ |T_{n+1}^{-1}|\ \leq \max(|H_{1n+1}|,\ |H_{2n+1}|), \\ \quad\vdots \\ |T_{n+d}^{-1}|\ \leq \max(|H_{1n+d}|,\ |H_{2n+d}|) \end{cases}$$

and

$$\|T^{-1}\|_\infty \leq \|T_1^{-1}\|_c+|T_{n+1}^{-1}|+\cdots+|T_{n+d}^{-1}|.$$

Then we have the following theorem.

Theorem 2.

Assume that the boundary value problem (1.5) *possesses an approximate solution* $\boldsymbol{u}=\bar{\boldsymbol{u}}(t)$ *in* S *such that* $\det G=\det \boldsymbol{f}'(\bar{\boldsymbol{u}})[\Psi(t)]\neq 0$, *where* $\Psi(t)$ *is the fundamental matrix of the linear homogeneous differential system*

$$\frac{d\boldsymbol{z}}{dt}=\begin{pmatrix} \boldsymbol{X}_u(\bar{\boldsymbol{u}}(t)) \\ \boxed{0} \end{pmatrix}\boldsymbol{z} \qquad (\textit{where } \boxed{0} \textit{ is a } d\times(n+d) \textit{ matrix})$$

satisfying the initial condition $\Psi(0)=E((n+d)\times(n+d)$ *unit matrix). Let* μ_1, $\mu_{n+1}, \mu_{n+2},\dots, \mu_{n+d}$ *and* r *be the positive numbers such that*

$$(2.3)\qquad \begin{cases} \mu_1 \geq \max(\|H_{11}\|_c,\ \|H_{21}\|_c), \\ \mu_{n+1} \geq \max(|H_{1n+1}|,\ |H_{2n+1}|), \\ \quad\vdots \\ \mu_{n+d} \geq \max(|H_{1n+d}|,\ |H_{2n+d}|), \end{cases}$$

$$(2.4)\qquad r\geq \||\boldsymbol{F}(\bar{\boldsymbol{u}})\||=\left\|\frac{d\bar{\boldsymbol{x}}}{dt}-\frac{\bar{\omega}}{2\pi}\boldsymbol{X}(\bar{\boldsymbol{x}},\ \bar{B}_1,\dots,\ \bar{B}_{d-1})\right\|_c+\|\boldsymbol{f}(\bar{\boldsymbol{u}})\|_{n+d}.$$

If there exist the positive numbers $\delta_1, \delta_{n+1}, \delta_{n+2},\dots, \delta_{n+d}$ *and a non-negative number*

$\kappa<1$ such that

(2.5) $D'_\delta \equiv \{\boldsymbol{u}(t);\ \|\boldsymbol{x}(t)-\bar{\boldsymbol{x}}(t)\|_c \le \delta_1,\ |B_1-\bar{B}_1|\le\delta_{n+1},\dots,\ |B_{d-1}-\bar{B}_{d-1}|\le\delta_{n+d-1},$

$|\omega-\bar{\omega}|\le\delta_{n+d},\ \boldsymbol{u}(t)\in C[I]\times \boldsymbol{R}^d\}\subset S',$

(2.6) $$\|\boldsymbol{X}_u(\boldsymbol{u})-\boldsymbol{X}_u(\bar{\boldsymbol{u}})\|_c+\|\boldsymbol{f}'(\boldsymbol{u})-\boldsymbol{f}'(\bar{\boldsymbol{u}})\|_{n+d}\le \frac{\kappa}{\mu_1+\mu_{n+1}+\cdots+\mu_{n+d}} \quad \text{on } D'_\delta,$$

(2.7) $$\frac{\mu_1 r}{1-\kappa}\le\delta_1,\ \frac{\mu_{n+1}r}{1-\kappa}\le\delta_{n+1},\dots,\ \frac{\mu_{n+d}r}{1-\kappa}\le\delta_{n+d},$$

then the boundary value problem (1.5) *has one and only one 2π-periodic solution $\boldsymbol{u}=\hat{\boldsymbol{u}}(t)$ in*

$$D_\delta \equiv \{\boldsymbol{u}(t);\ \|\boldsymbol{x}(t)-\bar{\boldsymbol{x}}(t)\|_c \le \delta_1,\ |B_1-\bar{B}_1|\le\delta_{n+1},\dots,\ |B_{d-1}-\bar{B}_{d-1}|\le\delta_{n+d-1},$$
$$|\omega-\bar{\omega}|\le\delta_{n+d},\ \boldsymbol{u}(t)\in M\}$$

and for this solution $\hat{\boldsymbol{u}}(t)$ we have

(2.8) $$\begin{cases} \|\hat{\boldsymbol{x}}(t)-\bar{\boldsymbol{x}}(t)\|_c\le\frac{\mu_1 r}{1-\kappa},\ |\hat{B}_1-\bar{B}_1|\le\frac{\mu_{n+1}r}{1-\kappa},\dots,\ |\hat{B}_{d-1}-\bar{B}_{d-1}|\le\frac{\mu_{n+d-1}r}{1-\kappa}, \\ |\hat{\omega}-\bar{\omega}|\le\frac{\mu_{n+d}r}{1-\kappa}. \end{cases}$$

PROOF. The proof of the theorem is similar to the one of the Theorem 2 in [14]. See [14].

§3. On choice of the boundary and additional condition $f(u)$

According to the previous sections, we may choose $\boldsymbol{f}(\boldsymbol{u})$ such that the $(n+d)\times(n+d)$ matrix $\boldsymbol{f}'(\boldsymbol{u})[\Psi(t)]$ is non-singular. Particularly, when $\boldsymbol{f}(\boldsymbol{u})$ is the form

$$\boldsymbol{f}(\boldsymbol{u})=\begin{pmatrix} \boldsymbol{x}(0)-\boldsymbol{x}(2\pi) \\ L(\boldsymbol{u})-\boldsymbol{\beta}\end{pmatrix},\ \text{where } L(C[I]\times\boldsymbol{R}^d\to\boldsymbol{R}^d) \text{ is a linear mapping,}$$

we consider how to choose the linear mapping L.

Let $\boldsymbol{u}=\hat{\boldsymbol{u}}(t)$ be a solution of (1.5). Firstly, we seek for the fundamental matrix $\hat{\Psi}(t)$ of the linear homogeneous system

$$\frac{d\boldsymbol{z}}{dt}=\begin{pmatrix} \boldsymbol{X}_u(\hat{\boldsymbol{u}}) \\ \boxed{0}\end{pmatrix}\boldsymbol{z} \qquad (\text{where } \boxed{0} \text{ is a } d\times(n+d) \text{ matrix})$$

satisfying the initial condition $\hat{\Psi}(0)=E((n+d)\times(n+d)$ unit matrix). Now, we put

$$\hat{\Psi}(t)=\begin{pmatrix} \hat{\Phi}(t) & \hat{\boldsymbol{p}}(t) \\ \hat{\boldsymbol{q}}^t(t) & \hat{s}(t) \end{pmatrix},$$

where $\hat{\Phi}(t)$, $\hat{\boldsymbol{p}}(t)$, $\hat{\boldsymbol{q}}^t(t)$ and $\hat{s}(t)$ are $(n+d-1)\times(n+d-1)$ matrix, $(n+d-1)$-dimensional column vector, $(n+d-1)$-dimensional row vector and real-valued function (scalar), respectively.

Then $\hat{\Phi}(t)$ is the fundamental matrix of the linear homogeneous system

$$\text{(3.1)} \qquad \frac{d\boldsymbol{y}}{dt}=\begin{pmatrix} \frac{\hat{\omega}}{2\pi}\boldsymbol{X}_{\boldsymbol{x}}(\hat{\boldsymbol{x}}, \hat{B}_1,\dots, \hat{B}_{d-1}) & \frac{\hat{\omega}}{2\pi}\boldsymbol{X}_{B_1}\cdots\frac{\hat{\omega}}{2\pi}\boldsymbol{X}_{B_{d-1}} \\ \multicolumn{2}{c}{0((d-1)\times(n+d-1)\ matrix)} \end{pmatrix}\boldsymbol{y}$$

satisfying the initial condition $\hat{\Phi}(0)=E((n+d-1)\times(n+d-1)$ unit matrix). And $\hat{\boldsymbol{q}}^t\equiv\mathbf{0}$, $\hat{s}(t)\equiv 1$, $\hat{\boldsymbol{p}}(t)=\hat{\Phi}(t)\int_0^t\hat{\Phi}^{-1}(\xi)\begin{pmatrix}\frac{1}{2\pi}\boldsymbol{X}(\hat{\boldsymbol{x}}, \hat{B}_1,\dots, \hat{B}_{d-1}) \\ \mathbf{0}\ ((d-1)\text{-}dimensional\ \ vector)\end{pmatrix}d\xi$. As $\begin{pmatrix}\boldsymbol{X}(\hat{\boldsymbol{x}}, \hat{B}_1,\dots, \hat{B}_{d-1}) \\ \mathbf{0}\end{pmatrix}$ is a 2π-periodic solution of (3.1), we can write

$$\begin{pmatrix}\boldsymbol{X}(\hat{\boldsymbol{x}}, \hat{B}_1,\dots, \hat{B}_{d-1}) \\ \mathbf{0}\end{pmatrix}=\hat{\Phi}(t)\boldsymbol{c}$$

for some constant vector $\boldsymbol{c}\neq\mathbf{0}$. Especially, in this case, the vector $\boldsymbol{c}$ can be expressed in the form $\boldsymbol{c}=\begin{pmatrix}\boldsymbol{c}_1\\ \mathbf{0}\end{pmatrix}$, where $\boldsymbol{c}_1$ is an n-dimensional vector and $\mathbf{0}$ is a $(d-1)$-dimensional vector.

Since $\boldsymbol{X}(\hat{\boldsymbol{x}}(t), \hat{B}_1,\dots, \hat{B}_{d-1})$ is 2π-periodic in t,

$$\boldsymbol{c}_1=\boldsymbol{X}(\hat{\boldsymbol{x}}(0), \hat{B}_1,\dots, \hat{B}_{d-1})=\boldsymbol{X}(\hat{\boldsymbol{x}}(2\pi), \hat{B}_1,\dots, \hat{B}_{d-1}).$$

Moreover, as we can put

$$\hat{\Phi}(t)=(\hat{\Phi}_1(t), \hat{\Phi}_2(t)), \quad \hat{\Phi}_1(t)=\begin{pmatrix}\hat{\Phi}_{11}(t)\\ 0\end{pmatrix}, \quad \hat{\Phi}_2(t)=\begin{pmatrix}\hat{\Phi}_{12}(t)\\ E_{d-1}\end{pmatrix},$$

where $\hat{\Phi}_{11}(t)$, 0, $\hat{\Phi}_{12}(t)$ and E_{d-1} are $n\times n$ matrix, $(d-1)\times n$ matrix, $n\times(d-1)$ matrix and $(d-1)\times(d-1)$ unit matrix, respectively. Ultimately $\hat{\Psi}(t)$ is the form

$$\hat{\Psi}(t)=(\hat{\Psi}_1(t)\ \ \hat{\Psi}_2(t)\ \ \hat{\Psi}_{n+d}(t))=\begin{pmatrix}\hat{\Phi}_{11}(t) & \hat{\Phi}_{12}(t) & \frac{t}{2\pi}\boldsymbol{X}(\hat{\boldsymbol{x}}, \hat{B}_1,\dots, \hat{B}_{d-1}) \\ 0 & E_{d-1} & \begin{pmatrix}0\\ \vdots\\ 0\\ 1\end{pmatrix} \\ 0 & 0 & \end{pmatrix},$$

because $\hat{\boldsymbol{p}}(t)=\hat{\Phi}(t)\int_0^t\frac{1}{2\pi}\boldsymbol{c}\,d\xi=\frac{t}{2\pi}\hat{\Phi}(t)\boldsymbol{c}=\frac{t}{2\pi}\begin{pmatrix}\boldsymbol{X}(\hat{\boldsymbol{x}}, \hat{B}_1,\dots, \hat{B}_{d-1})\\ \mathbf{0}\end{pmatrix}$. Here $\hat{\Psi}_1(t)$

$$=\begin{pmatrix}\hat{\Phi}_{11}(t)\\ 0\\ 0\end{pmatrix},\ \hat{\Psi}_2(t)=\begin{pmatrix}\hat{\Phi}_{12}(t)\\ E_{d-1}\\ 0\end{pmatrix} \text{ and } \hat{\Psi}_{n+d}(t)=\begin{bmatrix}\frac{t}{2\pi}\boldsymbol{X}(\hat{\boldsymbol{x}},\ \hat{B}_1,\dots,\ \hat{B}_{d-1})\\ \begin{bmatrix}0\\ \vdots\\ 0\\ 1\end{bmatrix}\end{bmatrix}.$$ Therefore,

$$\boldsymbol{f}'(\hat{\boldsymbol{u}})[\hat{\Psi}(t)]=\begin{pmatrix}E_n-\hat{\Phi}_{11}(2\pi) & -\hat{\Phi}_{12}(2\pi) & -\boldsymbol{c}_1\\ L(\hat{\Psi}_1) & L(\hat{\Psi}_2) & L(\hat{\Psi}_{n+d})\end{pmatrix}.$$

So, we may choose L such that $\det \boldsymbol{f}'(\hat{\boldsymbol{u}})[\hat{\Psi}(t)]\neq 0$. Specially, when $\boldsymbol{X}_{B_1}=\cdots=\boldsymbol{X}_{B_{d-1}}=\boldsymbol{0}$,

$$\det \boldsymbol{f}'(\hat{\boldsymbol{u}})[\hat{\Psi}(t)]=\det(E_{n-1}-Q_1^t\hat{\Phi}_{11}(2\pi)Q_1) \tag{3.2}$$
$$\times\det\left(L(\hat{\Psi}_2)\ L\begin{bmatrix}\boldsymbol{X}(\hat{\boldsymbol{x}},\ \hat{B}_1,\dots,\ \hat{B}_{d-1})\\ \boldsymbol{0}\\ 0\end{bmatrix}\right).$$

Hence, in this case, we can choose L such that

$$\det\left(L(\hat{\Psi}_2)\ L\begin{bmatrix}\boldsymbol{X}(\hat{\boldsymbol{x}},\ \hat{B}_1,\dots,\ \hat{B}_{d-1})\\ \boldsymbol{0}\\ 0\end{bmatrix}\right)\neq 0.$$

§4. Existence and Uniform Convergence of a Galerkin Approximation

Let $\boldsymbol{x}(t)$ be a continuous periodic vector-function of period 2π, and let its Fourier series be

$$\boldsymbol{x}(t)\sim\boldsymbol{c}_0+\sqrt{2}\sum_{n=1}^{\infty}(\boldsymbol{c}_{2n-1}\cos nt+\boldsymbol{c}_{2n}\sin nt),$$

where $\boldsymbol{c}_0,\ \boldsymbol{c}_1,\ \boldsymbol{c}_2,\dots$ are n-dimensional vectors. Then the trigonometric polynomial

$$\boldsymbol{x}_m(t)=\boldsymbol{c}_0+\sqrt{2}\sum_{n=1}^{m}(\boldsymbol{c}_{2n-1}\cos nt+\boldsymbol{c}_{2n}\sin nt)$$

is a truncated trigonometric polynomial of the given periodic function $\boldsymbol{x}(t)$. In the sequel we shall denote such a truncation of a periodic function by P_m and write a truncated polynomial $\boldsymbol{x}_m(t)$ of a periodic function $\boldsymbol{x}(t)$ as follows:

$$\boldsymbol{x}_m(t)=P_m\boldsymbol{x}(t).$$

In this section, we use $\|\cdot\|_q$ defined as follows:

$$\|x(t)\|_q=\left[\frac{1}{2\pi}\int_0^{2\pi}\|x(t)\|_n^2dt\right]^{\frac{1}{2}}.$$

Now, if we put $\gamma_{x_m}=\mathrm{col}\,[c_0, c_1, c_2,\dots, c_{2m-1}, c_{2m}]$, then

$$\|x_m\|_q^2=\frac{1}{2\pi}\int_0^{2\pi}\|x_m(t)\|_n^2dt=\|c_0\|_n^2+\sum_{i=1}^{m}(\|c_{2i-1}\|_n^2+\|c_{2i}\|_n^2)=\|\gamma_{x_m}\|_{n(2m+1)}^2.$$

We owe to Cesari the following proposition concerning continuously differentiable periodic functions.

Proposition 2.

Let $x(t)$ be a continuously differentiable periodic vector-function of period 2π. Then

$$\text{(4.1)}\qquad \begin{cases} \|x-P_mx\|_c\leq\sigma(m)\|\dot{x}\|_q\leq\sigma(m)\|\dot{x}\|_c, \\ \|x-P_mx\|_q\leq\sigma_1(m)\|\dot{x}\|_q, \end{cases}$$

where $\cdot=\dfrac{d}{dt}$ *and*

$$\sigma(m)=\sqrt{2}\left[\frac{1}{(m+1)^2}+\frac{1}{(m+2)^2}+\cdots\right]^{\frac{1}{2}},$$

$$\sigma_1(m)=\frac{1}{m+1}.$$

Also

$$\frac{\sqrt{2}}{m+1}<\sigma(m)<\frac{\sqrt{2}}{\sqrt{m}}.$$

Here, we introduce the following notations;

For any vector function $u(t)=(x(t), B_1,\dots, B_{d-1}, \omega)\in M$, we define $u_m(t)=P_mu(t)$

$$u_m(t)=P_mu(t)=(P_mx(t), B_1,\dots, B_{d-1}, \omega)=(x_m(t), B_1,\dots, B_{d-1}, \omega),$$

and define the norms $\|\cdot\|_q$ and $\|\cdot\|_\infty$ by

$$\|u(t)\|_q=\|x(t)\|_q+|B_1|+\cdots+|B_{d-1}|+|\omega|,$$

$$\|u(t)\|_\infty=\|x(t)\|_c+|B_1|+\cdots+|B_{d-1}|+|\omega|.$$

For any $n(t)=(\varphi(t), v)\in N\equiv C[I]\times R^{n+d}$,

$$|||n(t)|||_q=\|\varphi(t)\|_q+\|v\|_{n+d},$$

$$|||n(t)|||=\|\varphi(t)\|_c+\|v\|_{n+d}.$$

By Proposition 2, we have the following results;

$$\begin{cases} \|\boldsymbol{u}(t)-P_m\boldsymbol{u}(t)\|_\infty=\|\boldsymbol{x}(t)-P_m\boldsymbol{x}(t)\|_c\le\sigma(m)\|\dot{\boldsymbol{x}}\|_q=\sigma(m)\|\dot{\boldsymbol{u}}\|_q\le\sigma(m)\|\dot{\boldsymbol{x}}\|_c=\sigma(m)\|\dot{\boldsymbol{u}}\|_\infty, \\ \|\boldsymbol{u}(t)-P_m\boldsymbol{u}(t)\|_q=\|\boldsymbol{x}(t)-P_m\boldsymbol{x}(t)\|_q\le\sigma_1(m)\|\dot{\boldsymbol{x}}\|_q=\sigma_1(m)\|\dot{\boldsymbol{u}}\|_q. \end{cases}$$

Moreover, we use the following norm $|||\cdot|||'$;

For $\boldsymbol{u}_m(t)=P_m\boldsymbol{u}(t)=(P_m\boldsymbol{x}(t),\ B_1,\dots,\ B_{d-1},\ \omega)=(\boldsymbol{x}_m(t),\ B_1,\dots,\ B_{d-1},\ \omega)$, where $\boldsymbol{x}_m(t)=\boldsymbol{c}_0+\sqrt{2}\sum_{n=1}^{m}(\boldsymbol{c}_{2n-1}\cos nt+\boldsymbol{c}_{2n}\sin nt)$, we put

$$\boldsymbol{\gamma}_{\boldsymbol{u}_m}=\mathrm{col}\,[\boldsymbol{c}_0,\ \boldsymbol{c}_1,\ \boldsymbol{c}_2,\dots,\ \boldsymbol{c}_{2m-1},\ \boldsymbol{c}_{2m},\ B_1,\dots,\ B_{d-1},\ \omega].$$

Then we define the norm $|||\cdot|||'$ as follows;

$$|||\boldsymbol{\gamma}_{\boldsymbol{u}_m}|||'=\|\boldsymbol{\gamma}_{\boldsymbol{x}_m}\|_{n(2m+1)}+|B_1|+\cdots+|B_{d-1}|+|\omega|.$$

At once, we have the following result.

$$\begin{aligned}\|\boldsymbol{u}_m(t)\|_q&=\|P_m\boldsymbol{u}(t)\|_q=\|\boldsymbol{x}_m(t)\|_q+|B_1|+\cdots+|B_{d-1}|+|\omega|\\ &=\|\boldsymbol{\gamma}_{\boldsymbol{x}_m}\|_{n(2m+1)}+|B_1|+\cdots+|B_{d-1}|+|\omega|=|||\boldsymbol{\gamma}_{\boldsymbol{u}_m}|||'.\end{aligned}$$

By $\|\cdot\|_{n+d,n}$, we denote the norm of a continuous linear operator which maps from $\boldsymbol{R}^{n+d}$ to $\boldsymbol{R}^n$. Similarly, by $\|\cdot\|_{\infty,d}$, we denote the norm of a continuous linear operator mapping from the product space $C[I]\times\boldsymbol{R}^d$ with the norm $\|\cdot\|_\infty$ to $\boldsymbol{R}^d$ with the norm $\|\cdot\|_d$ and by $\|\cdot\|_{q,d}$, we denote the norm of a linear operator mapping from $C[I]\times\boldsymbol{R}^d$ with the norm $\|\cdot\|_q$ to $\boldsymbol{R}^d$ with the norm $\|\cdot\|_d$.

Let D_1 be the closed bounded region of the $\boldsymbol{x}$-space and D_2 be the closed bounded region of $\boldsymbol{R}^d$. We put $E=D_1\times D_2$. We assume that $\boldsymbol{X}(\boldsymbol{u})\equiv\frac{\omega}{2\pi}\boldsymbol{X}(\boldsymbol{x},\ B_1,\dots,\ B_{d-1})$ and $\boldsymbol{X}_{\boldsymbol{u}}(\boldsymbol{u})\equiv\left(\frac{\omega}{2\pi}\boldsymbol{X}_{\boldsymbol{x}}(\boldsymbol{x},\ B_1,\dots,\ B_{d-1})\ \ \frac{\omega}{2\pi}\boldsymbol{X}_{B_1}\cdots\frac{\omega}{2\pi}\boldsymbol{X}_{B_{d-1}}\ \ \frac{1}{2\pi}\boldsymbol{X}(\boldsymbol{x},\ B_1,\dots,\ B_{d-1})\right)$ are defined and continuously differentiable with respect to $\boldsymbol{u}$ on E, and $\boldsymbol{f}(\boldsymbol{u})$ is defined and continuously Fréchet differentiable on D', where

$$D=\{\boldsymbol{u}(t);\ (t,\ \boldsymbol{u}(t))\in I\times E\quad for\quad t\in I=[0,\ 2\pi],\ \boldsymbol{u}(t)\in M=C^1[I]\times\boldsymbol{R}^d\},$$

$$D'=\{\boldsymbol{u}(t);\ (t,\ \boldsymbol{u}(t))\in I\times E\quad for\quad t\in I=[0,\ 2\pi],\ \boldsymbol{u}(t)\in C[I]\times\boldsymbol{R}^d\}.$$

If we apply Proposition 2 to a periodic solution of (1.5), then we easily get the following lemma concerning its truncated trigonometric polynomials.

Lemma 1.

Let K and K_1 be non-negative constants such that

$$K=\max_E\|\boldsymbol{X}(\boldsymbol{u})\|_n,\ K_1=\max_E\|\boldsymbol{X}_{\boldsymbol{u}}(\boldsymbol{u})\|_{n+d,n}.$$

If there exists a 2π-periodic solution $\boldsymbol{u}=\hat{\boldsymbol{u}}(t)$ of (1.5) lying in E, then

$$\begin{cases} \text{(i)} & \|\hat{\boldsymbol{u}}-\hat{\boldsymbol{u}}_m\|_\infty \le K\sigma(m), \\ \text{(ii)} & \|\hat{\boldsymbol{u}}-\hat{\boldsymbol{u}}_m\|_q \le K\sigma_1(m), \\ \text{(iii)} & \|\dot{\hat{\boldsymbol{u}}}-\dot{\hat{\boldsymbol{u}}}_m\|_\infty \le KK_1\sigma(m). \end{cases}$$

PROOF. By Proposition 2, we have

$$\|\hat{\boldsymbol{u}}-\hat{\boldsymbol{u}}_m\|_\infty = \|\hat{\boldsymbol{x}}-\hat{\boldsymbol{x}}_m\|_c \le \sigma(m)\|\dot{\hat{\boldsymbol{x}}}\|_q.$$

Since

$$\|\dot{\hat{\boldsymbol{x}}}\|_q = \left\|\frac{\hat{\omega}}{2\pi}\boldsymbol{X}(\hat{\boldsymbol{x}}, \hat{B}_1,\dots, \hat{B}_{d-1})\right\|_q = \|\boldsymbol{X}(\hat{\boldsymbol{u}})\|_q = \left(\frac{1}{2\pi}\int_0^{2\pi}\|\boldsymbol{X}(\hat{\boldsymbol{u}})\|_n^2 dt\right)^{\frac{1}{2}} \le K,$$

then $\|\hat{\boldsymbol{u}}-\hat{\boldsymbol{u}}_m\|_\infty \le K\sigma(m)$. This proves (i).
Similarly,

$$\|\hat{\boldsymbol{u}}-\hat{\boldsymbol{u}}_m\|_q = \|\hat{\boldsymbol{x}}-\hat{\boldsymbol{x}}_m\|_q \le \sigma_1(m)\|\dot{\boldsymbol{x}}\|_q \le K\sigma_1(m).$$

Now $\dot{\hat{\boldsymbol{u}}}=(\boldsymbol{X}(\hat{\boldsymbol{u}}), 0, 0,\dots, 0)$, then $\dot{\hat{\boldsymbol{u}}}-\dot{\hat{\boldsymbol{u}}}_m=(\boldsymbol{X}(\hat{\boldsymbol{u}})-P_m\boldsymbol{X}(\hat{\boldsymbol{u}}), 0, 0,\dots, 0)$, where $\dot{\hat{\boldsymbol{u}}}_m = P_m\dot{\hat{\boldsymbol{u}}}=(P_m\boldsymbol{X}(\hat{\boldsymbol{u}}), 0, 0,\dots, 0)$.
Therefore

$$\|\dot{\hat{\boldsymbol{u}}}-\dot{\hat{\boldsymbol{u}}}_m\|_\infty = \|\boldsymbol{X}(\hat{\boldsymbol{u}})-P_m\boldsymbol{X}(\hat{\boldsymbol{u}})\|_c \le \sigma(m)\|\dot{\boldsymbol{X}}(\hat{\boldsymbol{u}})\|_q.$$

However

$$\frac{d}{dt}\boldsymbol{X}(\hat{\boldsymbol{u}}) = \boldsymbol{X}_u(\hat{\boldsymbol{u}})\begin{pmatrix} \dfrac{d\hat{\boldsymbol{x}}}{dt} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \text{ then}$$

$$\left\|\frac{d}{dt}\boldsymbol{X}(\hat{\boldsymbol{u}})\right\|_n \le \|\boldsymbol{X}_u(\hat{\boldsymbol{u}})\|_{n+d,n}\left\|\frac{d\hat{\boldsymbol{x}}}{dt}\right\|_n = \|\boldsymbol{X}_u(\hat{\boldsymbol{u}})\|_{n+d,n}\|\boldsymbol{X}(\hat{\boldsymbol{u}})\|_n \le KK_1.$$

This implies

$$\|\dot{\hat{\boldsymbol{u}}}-\dot{\boldsymbol{u}}_m\|_\infty \le KK_1\sigma(m). \qquad \text{Q. E. D.}$$

This lemma yields the following corollary.

Corollary 1.1.

If $\boldsymbol{u}=\hat{\boldsymbol{u}}(t)$ is an isolated periodic solution of (1.5) *lying inside E, then there exists a positive integer m_0 such that, for any $m \ge m_0$,*

(i) $\hat{\boldsymbol{u}}_m(t) \in E$;

(ii) *The linear operator T_m defined by*

$$T_m\boldsymbol{h}=\left[\frac{d\boldsymbol{h}_1}{dt}-\boldsymbol{X}_u(\hat{\boldsymbol{u}}_m)\boldsymbol{h}^t,\ \boldsymbol{f}'(\hat{\boldsymbol{u}}_m)\boldsymbol{h}\right]$$

has a linear inverse operator T_m^{-1} *and there exists a positive constant* M_c *such that*

$$\|T_m^{-1}\|_q\leq M_c,\ \|T_m^{-1}\|_\infty\leq M_c,$$

where $\boldsymbol{h}^t$ *denotes the transpose of* $\boldsymbol{h}$, *that is,*

$$\boldsymbol{h}^t=\mathrm{col}\,[\boldsymbol{h}_1(t),\ h_{n+1},\dots,\ h_{n+d}];$$

(iii) $\dfrac{d}{dt}\boldsymbol{X}_u(\hat{\boldsymbol{u}}_m(t))$ *is equibounded, that is, there exists a non-negative constant* K_2 *such that*

$$\left\|\frac{d}{dt}\boldsymbol{X}_u(\hat{\boldsymbol{u}}_m(t))\right\|_{n+d,n}\leq K_2.$$

PROOF. $\|\hat{\boldsymbol{u}}_m(t)-\hat{\boldsymbol{u}}(t)\|_\infty\to 0$ as $m\to+\infty$ from (i) of Lemma 1. Then, there exists a positive integer m_0 such that, for any $m\geq m_0$, $\hat{\boldsymbol{u}}_m(t)$ lies inside E. Since $\hat{\boldsymbol{u}}(t)$ is an isolated solution of (1.5), then the linear operator T defined by

$$T\boldsymbol{h}\equiv\left[\frac{d\boldsymbol{h}_1}{dt}-\boldsymbol{X}_u(\hat{\boldsymbol{u}})\boldsymbol{h}^t,\ \boldsymbol{f}'(\hat{\boldsymbol{u}})\boldsymbol{h}\right]$$

has a linear inverse operator T^{-1} and there exists a positive constant M' such that $\|T^{-1}\|_\infty\leq M'$. Moreover, if m_0 is sufficiently large, then there exists T_m^{-1} for any $m\geq m_0$. Namely, for any $\boldsymbol{n}=(\varphi(t),\ \boldsymbol{v})\in N=C[I]\times\boldsymbol{R}^{n+d}$, there exists one and only one $\boldsymbol{h}(t)\in M$ such that $T_m\boldsymbol{h}=\boldsymbol{n}=(\varphi(t),\ \boldsymbol{v})$. Then we have

$$T\boldsymbol{h}=T_m\boldsymbol{h}+(T-T_m)\boldsymbol{h}=\boldsymbol{n}+(T-T_m)\boldsymbol{h}.$$

Since $\boldsymbol{h}=T^{-1}\boldsymbol{n}+T^{-1}(T-T_m)\boldsymbol{h}$, then

$$\|\boldsymbol{h}\|_\infty\leq\|T^{-1}\|_\infty|||\boldsymbol{n}|||+\|T^{-1}\|_\infty|||T-T_m|||\,\|\boldsymbol{h}\|_\infty.$$

Here, if m_0 is sufficiently large, for any $m\geq m_0$,

$$\|T^{-1}\|_\infty|||T-T_m|||<1.$$

Therefore

$$(1-\|T^{-1}\|_\infty|||T-T_m|||)\|\boldsymbol{h}\|_\infty\leq\|T^{-1}\|_\infty|||\boldsymbol{n}|||,$$

and this implies

$$\|\boldsymbol{h}\|_\infty\leq\frac{\|T^{-1}\|_\infty|||\boldsymbol{n}|||}{1-\|T^{-1}\|_\infty|||T-T_m|||}.$$

Now, we may take $\frac{\|T^{-1}\|_\infty}{1-\|T^{-1}\|_\infty \|T-T_{m_0}\|}$ as a positive constant M_c. Similarly, we can show that $\|T_m^{-1}\|_q \leq M_c$.
By the definition of $\boldsymbol{X}_u(\hat{\boldsymbol{u}}_m(t))$,

$$\boldsymbol{X}_u(\hat{\boldsymbol{u}}_m(t))=\left(\frac{\hat{\omega}}{2\pi}\boldsymbol{X}_x(\hat{\boldsymbol{x}}_m,\hat{B}_1,\dots,\hat{B}_{d-1})\ \ \frac{\hat{\omega}}{2\pi}\boldsymbol{X}_{B_1}\cdots\frac{\hat{\omega}}{2\pi}\boldsymbol{X}_{B_{d-1}}\ \ \frac{1}{2\pi}\boldsymbol{X}(\hat{\boldsymbol{x}}_m,\hat{B}_1,\dots,\hat{B}_{d-1})\right).$$

Firstly

$$\frac{d}{dt}\left\{\frac{1}{2\pi}\boldsymbol{X}(\hat{\boldsymbol{x}}_m,\hat{B}_1,\dots,\hat{B}_{d-1})\right\}=\frac{1}{2\pi}\boldsymbol{X}_x(\hat{\boldsymbol{x}}_m,\hat{B}_1,\dots,\hat{B}_{d-1})\frac{d\hat{\boldsymbol{x}}_m}{dt}.$$

As $\frac{d\hat{\boldsymbol{x}}_m}{dt}=\frac{d\hat{\boldsymbol{x}}}{dt}+\left(\frac{d\hat{\boldsymbol{x}}_m}{dt}-\frac{d\hat{\boldsymbol{x}}}{dt}\right)$ and

$$\left\|\frac{d\hat{\boldsymbol{x}}_m}{dt}\right\|_n\leq\left\|\frac{d\hat{\boldsymbol{x}}}{dt}\right\|_n+\left\|\frac{d\hat{\boldsymbol{x}}_m}{dt}-\frac{d\hat{\boldsymbol{x}}}{dt}\right\|_n=\|\boldsymbol{X}(\hat{\boldsymbol{u}})\|_n+\left\|\frac{d\hat{\boldsymbol{x}}_m}{dt}-\frac{d\hat{\boldsymbol{x}}}{dt}\right\|_n$$
$$\leq K+KK_1\sigma(m),$$

then

$$\left\|\frac{d}{dt}\left\{\frac{1}{2\pi}\boldsymbol{X}(\hat{\boldsymbol{x}}_m,\hat{B}_1,\dots,\hat{B}_{d-1})\right\}\right\|_n\leq\tilde{M}'(K+KK_1\sigma(m)),$$

where

$$\left\|\frac{1}{2\pi}\boldsymbol{X}_x(\hat{\boldsymbol{x}}_m,\hat{B}_1,\dots,\hat{B}_{d-1})\right\|_{n,n}\leq\tilde{M}'.$$

Besides, since

$$\frac{d}{dt}\left\{\frac{\hat{\omega}}{2\pi}\boldsymbol{X}_{B_i}(\hat{\boldsymbol{x}}_m,\hat{B}_1,\dots,\hat{B}_{d-1})\right\}=\frac{\hat{\omega}}{2\pi}\boldsymbol{X}_{B_ix}(\hat{\boldsymbol{x}}_m,\hat{B}_1,\dots,\hat{B}_{d-1})\frac{d\hat{\boldsymbol{x}}_m}{dt},$$

then

$$\left\|\frac{d}{dt}\left\{\frac{\hat{\omega}}{2\pi}\boldsymbol{X}_{B_i}(\hat{\boldsymbol{x}}_m,\hat{B}_1,\dots,\hat{B}_{d-1})\right\}\right\|_n\leq\tilde{M}''(K+KK_1\sigma(m)),$$

where

$$\left\|\frac{\hat{\omega}}{2\pi}\boldsymbol{X}_{B_ix}(\hat{\boldsymbol{x}}_m,\hat{B}_1,\dots,\hat{B}_{d-1})\right\|_{n,n}\leq\tilde{M}''.$$

Lastly, as

$$\frac{d}{dt}\left\{\frac{\hat{\omega}}{2\pi}\boldsymbol{X}_x(\hat{\boldsymbol{x}}_m,\hat{B}_1,\dots,\hat{B}_{d-1})\right\}=\sum_{k=1}^{n}\frac{\partial\boldsymbol{X}_x}{\partial x_k}(\hat{\boldsymbol{x}}_m,\hat{B}_1,\dots,\hat{B}_{d-1})\frac{d\hat{x}_{mk}}{dt},$$

then

$$\left\| \frac{d}{dt} \left\{ \frac{\hat{\omega}}{2\pi} X_x(\hat{x}_m, \hat{B}_1, \dots, \hat{B}_{d-1}) \right\} \right\|_{n,n} \le \tilde{M}'''(K + KK_1\sigma(m)).$$

These follows that there exists a positive constant $M^{(iv)}$ such that

$$\left\| \frac{d}{dt} \{ X_u(\hat{u}_m(t)) \} \right\|_{n+d,n} \le M^{(iv)}(K + KK_1\sigma(m)). \qquad \text{Q.E.D.}$$

The Jacobian matrix of the determining equation of Galerkin approximations

We put

$$x_m(t) = \alpha_0 + \sqrt{2} \sum_{n=1}^{m} (\alpha_{2n-1} \cos nt + \alpha_{2n} \sin nt),$$

$$u_m(t) = (x_m(t), B_1, B_2, \dots, B_{d-1}, \omega)$$

and

$$\alpha = \mathrm{col}\,[\alpha_0, \alpha_1, \dots, \alpha_{2m-1}, \alpha_{2m}, B_1, B_2, \dots, B_{d-1}, \omega].$$

Now, the determining equation of Galerkin approximations is the system

$$(4.2)\qquad \begin{cases} F_0^{(m)}(\alpha) = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} X(u_m(s))\, ds = 0, \\ F_{2n-1}^{(m)}(\alpha) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_0^{2\pi} X(u_m(s)) \cos ns\, ds - n\alpha_{2n} = 0, \\ F_{2n}^{(m)}(\alpha) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_0^{2\pi} X(u_m(s)) \sin ns\, ds + n\alpha_{2n-1} = 0, \\ \qquad\qquad (n = 1, 2, \dots, m), \\ G_f(\alpha) = f(u_m(t)) = \begin{pmatrix} x_m(0) - x_m(2\pi) \\ L(u_m(t)) - \beta \end{pmatrix} = 0. \end{cases}$$

The system (4.2) is constructed by $\{n(2m+1)+n+d\}$ equations, but the last equation $G_f(\alpha)=0$ are essentially equivalent to d equations $L(u_m(t))-\beta=0$. Then we will solve the following $\{n(2m+1)+d\}$ equations;

$$\begin{cases} F_0^{(m)}(\alpha) = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} X(u_m(s))\, ds = 0, \\ F_{2n-1}^{(m)}(\alpha) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_0^{2\pi} X(u_m(s)) \cos ns\, ds - n\alpha_{2n} = 0, \\ F_{2n}^{(m)}(\alpha) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_0^{2\pi} X(u_m(s)) \sin ns\, ds + n\alpha_{2n-1} = 0 \\ \qquad\qquad (n = 1, 2, \dots, m), \\ F_L^{(m)}(\alpha) = L(u_m(t)) - \beta = 0. \end{cases}$$

Put $\boldsymbol{F}^{(m)}(\boldsymbol{\alpha})=\operatorname{col}[\boldsymbol{F}_0^{(m)}(\boldsymbol{\alpha}), \boldsymbol{F}_1^{(m)}(\boldsymbol{\alpha}),\dots, \boldsymbol{F}_{2m-1}^{(m)}(\boldsymbol{\alpha}), \boldsymbol{F}_{2m}^{(m)}(\boldsymbol{\alpha}), (F_L)_1,\dots, (F_L)_d]$, then the determining equation can be written breifly

$$\boldsymbol{F}^{(m)}(\boldsymbol{\alpha})=\mathbf{0},$$

where $\boldsymbol{F}_L^{(m)}(\boldsymbol{\alpha})=\operatorname{col}[(F_L)_1,\dots, (F_L)_d]$.

Let $J_m(\boldsymbol{\alpha})$ be the Jacobian matrix of $\boldsymbol{F}^{(m)}(\boldsymbol{\alpha})$ with respect to $\boldsymbol{\alpha}$. Then the elements of $J_m(\boldsymbol{\alpha})$ are of the following forms;

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\omega}{2\pi}\boldsymbol{X}_x(\boldsymbol{x}_m(s), B_1,\dots, B_{d-1})ds,$$

$$\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}\frac{\omega}{2\pi}\boldsymbol{X}_x(\boldsymbol{x}_m(s), B_1,\dots, B_{d-1})\cos ns\ ds,$$

(the elements with respect to $\boldsymbol{\alpha}_0$)

$$\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}\frac{\omega}{2\pi}\boldsymbol{X}_x(\boldsymbol{x}_m(s), B_1,\dots, B_{d-1})\sin ns\ ds,$$

$$L(\boldsymbol{u}_{ma_{0i}}(t))\qquad (1\leq i\leq n),$$

where $\boldsymbol{u}_{ma_{0i}}(t)=(\boldsymbol{x}_{ma_{0i}}(t), 0,\dots, 0)$, $\boldsymbol{x}_{ma_{0i}}(t)=\begin{pmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{pmatrix}\subset i.$

$$\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}\frac{\omega}{2\pi}\boldsymbol{X}_x(\boldsymbol{x}_m(s), B_1,\dots, B_{d-1})\cos ks\ ds,$$

$$\frac{1}{\pi}\int_0^{2\pi}\frac{\omega}{2\pi}\boldsymbol{X}_x(\boldsymbol{x}_m(s), B_1,\dots, B_{d-1})\cos ks\cos ns\ ds,$$

(the elements with respect to $\boldsymbol{\alpha}_{2k-1}$)

$$\frac{1}{\pi}\int_0^{2\pi}\frac{\omega}{2\pi}\boldsymbol{X}_x(\boldsymbol{x}_m(s), B_1,\dots, B_{d-1})\cos ks\sin ns\ ds+nE_n,$$

$$L(\boldsymbol{u}_{ma_{2k-1i}}(t))\qquad (1\leq i\leq n),$$

where $\boldsymbol{u}_{ma_{2k-1i}}(t)=(\boldsymbol{x}_{ma_{2k-1i}}(t), 0,\dots, 0)$, $\boldsymbol{x}_{ma_{2k-1i}}(t)=\begin{pmatrix}0\\ \vdots\\ \sqrt{2}\cos kt\\ \vdots\\ 0\end{pmatrix}\subset i$, and E_n is the $n\times n$ unit matrix.

$$\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}\frac{\omega}{2\pi}\boldsymbol{X}_x(\boldsymbol{x}_m(s), B_1,\dots, B_{d-1})\sin ks\ ds,$$

$$\frac{1}{\pi}\int_0^{2\pi}\frac{\omega}{2\pi}\boldsymbol{X}_x(\boldsymbol{x}_m(s), B_1,\dots, B_{d-1})\sin ks\cos ns\ ds-nE_n,$$

$$\frac{1}{\pi}\int_0^{2\pi}\frac{\omega}{2\pi}\boldsymbol{X}_{\boldsymbol{x}}(\boldsymbol{x}_m(s),\ B_1,\dots,\ B_{d-1})\sin ks\sin ns\,ds,$$

(the elements with respect to $\boldsymbol{a}_{2k}$)

$$L(\boldsymbol{u}_{ma_{2ki}}(t)) \qquad (1\le i\le n),$$

where $\boldsymbol{u}_{ma_{2ki}}(t)=(\boldsymbol{x}_{ma_{2ki}}(t),\ 0,\dots,\ 0)$, $\boldsymbol{x}_{ma_{2ki}}(t)=\begin{pmatrix}0\\ \vdots\\ \sqrt{2}\sin kt\\ \vdots\\ 0\end{pmatrix}\subset i.$

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\omega}{2\pi}\boldsymbol{X}_{B_i}(\boldsymbol{x}_m(s),\ B_1,\dots,\ B_{d-1})ds,$$

$$\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}\frac{\omega}{2\pi}\boldsymbol{X}_{B_i}(\boldsymbol{x}_m(s),\ B_1,\dots,\ B_{d-1})\cos ns\ ds,$$

(the elements with respect to B_i $(1\le i\le d-1)$)

$$\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}\frac{\omega}{2\pi}\boldsymbol{X}_{B_i}(\boldsymbol{x}_m(s),\ B_1,\dots,\ B_{d-1})\sin ns\ ds,$$

$$L(\boldsymbol{u}_{mB_i}(t)) \qquad (1\le i\le d-1),$$

where $\boldsymbol{u}_{mB_i}(t)=(\boldsymbol{0},\ 0,\dots,\ 0,\ \underset{\widehat{i}}{1},\ 0,\dots,\ 0)$.

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2\pi}\boldsymbol{X}(\boldsymbol{x}_m(s),\ B_1,\dots,\ B_{d-1})ds,$$

$$\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}\frac{1}{2\pi}\boldsymbol{X}(\boldsymbol{x}_m(s),\ B_1,\dots,\ B_{d-1})\cos ns\ ds,$$

(the elements with respect to ω)

$$\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}\frac{1}{2\pi}\boldsymbol{X}(\boldsymbol{x}_m(s),\ B_1,\dots,\ B_{d-1})\sin ns\ ds,$$

$$L(\boldsymbol{u}_{m\omega}(t)),$$

where $\boldsymbol{u}_{m\omega}(t)=(\boldsymbol{0},\ 0,\dots,\ 0,\ 1)$.

To find the basic properties of $J_m(\boldsymbol{\alpha})$, let us consider the auxiliary linear system

$$J_m(\boldsymbol{\alpha})\boldsymbol{\xi}+\boldsymbol{\gamma}=\boldsymbol{0},$$

where

$$(4.3)\qquad \begin{cases}\boldsymbol{\xi}=\mathrm{col}\,[\boldsymbol{v}_0,\ \boldsymbol{v}_1,\dots,\ \boldsymbol{v}_{2m-1},\ \boldsymbol{v}_{2m},\ V_1,\dots,\ V_{d-1},\ W],\\ \boldsymbol{\gamma}=\mathrm{col}\,[\boldsymbol{c}_0,\ \boldsymbol{c}_1,\dots,\ \boldsymbol{c}_{2m-1},\ \boldsymbol{c}_{2m},\ -C_1,\dots,\ -C_{d-1},\ -\overline{W}].\end{cases}$$

If we put

$$(4.4)\quad \begin{cases} y(t)=v_0+\sqrt{2}\sum_{n=1}^{m}(v_{2n-1}\cos nt+v_{2n}\sin nt)\,, \\ \varphi(t)=c_0+\sqrt{2}\sum_{n=1}^{m}(c_{2n-1}\cos nt+c_{2n}\sin nt)\,, \\ u_y(t)=(y(t),\,V_1,\dots,\,V_{d-1},\,W),\ u_\varphi(t)=(\varphi(t),\,C_1,\dots,\,C_{d-1},\,\overline{W})\,, \\ n_\varphi=\left[\varphi(t),\,\begin{pmatrix}\mathbf{0}\\ w\end{pmatrix}\right],\ \textit{where}\ w=\mathrm{col}\,[C_1,\dots,\,C_{d-1},\,\overline{W}]\,, \end{cases}$$

then the equations $J_m(\alpha)\xi+\gamma=\mathbf{0}$ are equivalent to

$$\left[\frac{dy}{dt}-P_m[X_u(u_m(t))u_y(t)^t],\,f'(u_m(t))u_y(t)\right]=n_\varphi=\left[\varphi(t),\,\begin{pmatrix}\mathbf{0}\\ w\end{pmatrix}\right],$$

or

$$\left[\frac{dy}{dt}-P_m[X_u(u_m(t))u_y(t)^t],\ L(u_y(t))\right]=[\varphi(t),\,w]\,,$$

where $u_y(t)^t$ is the transpose of $u_y(t)$.

First, we shall prove the following lemma.

Lemma 2.

Assume that the conditions of Lemma 1 are satisfied and that (1.5) *has an isolated periodic solution* $u=\hat{u}(t)$ *lying indide E. Taking* m_0 *sufficiently large, we consider the differential system*

$$(4.5)\quad \left[\frac{dy}{dt}-P_m[X_u(\hat{u}_m(t))u_y(t)^t],\,f'(\hat{u}_m(t))u_y(t)\right]=n_\varphi=\left[\varphi(t),\,\begin{pmatrix}\mathbf{0}\\ w\end{pmatrix}\right]$$

for $m\geq m_0$, *where* $\hat{u}_m(t)=P_m\hat{u}(t)$ *and where* $\varphi(t)$ *is an arbitrary continuous periodic function of period* 2π *and* w *is an arbitrary vector of* $\mathbf{R}^d$. *Then, for any periodic solution* $u=u_y(t)$ *of* (4.5) (*if any exists*), *we have*

$$\|u_y\|_q\leq\frac{M_c(\|u_\varphi\|_q+K_1\sigma_1(m)\|\varphi(t)\|_q)}{1-M_c(K_2+K_1^2)\sigma_1(m)}$$

and since

$$|||n_\varphi|||_q\leq\|u_\varphi\|_q\leq\sqrt{d}\,|||n_\varphi|||_q,\ \ \|\varphi(t)\|_q\leq|||n_\varphi|||_q,$$

then we have

$$(4.6)\quad \|u_y\|_q\leq\frac{M_c(\sqrt{d}+K_1\sigma_1(m))}{1-M_c(K_2+K_1^2)\sigma_1(m)}\,|||n_\varphi|||_q\,.$$

PROOF. For brevity let us put

$$\hat{A}_m(t)=X_u(\hat{u}_m(t))\,.$$

Then for any periodic solution $\boldsymbol{u}=\boldsymbol{u}_y(t)$ of (4.5) we have

$$T_m\boldsymbol{u}_y=\left[\frac{dy}{dt}-\hat{A}_m(t)\boldsymbol{u}_y(t)^t,\ \boldsymbol{f}'(\hat{\boldsymbol{u}}_m(t))\boldsymbol{u}_y(t)\right]=\left[\varphi(t)+\boldsymbol{\eta}(t),\ \begin{pmatrix}\mathbf{0}\\ \boldsymbol{w}\end{pmatrix}\right],\ \text{namely,}$$

$$\begin{cases}\dfrac{dy}{dt}=\hat{A}_m(t)\boldsymbol{u}_y(t)^t+\varphi(t)+\boldsymbol{\eta}(t),\\[2ex] \boldsymbol{f}'(\hat{\boldsymbol{u}}_m(t))\boldsymbol{u}_y(t)=\begin{pmatrix}\mathbf{0}\\ \boldsymbol{w}\end{pmatrix},\end{cases}$$

where $\boldsymbol{\eta}(t)=-(I-P_m)\hat{A}_m(t)\boldsymbol{u}_y(t)^t$. Here I is the identity operator. Put

$$\boldsymbol{z}(t)=\hat{A}_m(t)\boldsymbol{u}_y(t)^t.$$

Then

$$\dot{\boldsymbol{z}}(t)=\dot{\hat{A}}_m(t)\boldsymbol{u}_y(t)^t+\hat{A}_m(t)\begin{pmatrix}P_m[\hat{A}_m(t)\boldsymbol{u}_y(t)^t]+\varphi(t)\\ \mathbf{0}\end{pmatrix},$$

from which, it follows that

$$\|\dot{\boldsymbol{z}}\|_q\leq K_2\|\boldsymbol{u}_y\|_q+K_1[\|P_m[\hat{A}_m(t)\boldsymbol{u}_y(t)^t]\|_q+\|\varphi(t)\|_q].$$

But by Bessel's inequality,

$$\|P_m[\hat{A}_m(t)\boldsymbol{u}_y(t)^t]\|_q\leq\|\hat{A}_m(t)\boldsymbol{u}_y(t)^t\|_q\leq K_1\|\boldsymbol{u}_y\|_q.$$

Therefore, we have

$$\|\dot{\boldsymbol{z}}\|_q\leq(K_2+K_1^2)\|\boldsymbol{u}_y\|_q+K_1\|\varphi(t)\|_q.$$

Since $\|\boldsymbol{\eta}(t)\|_q\leq\sigma_1(m)\|\dot{\boldsymbol{z}}\|_q$ by Proposition 2 (4.1), we have then

$$\|\boldsymbol{\eta}(t)\|_q\leq\sigma_1(m)\,[(K_2+K_1^2)\|\boldsymbol{u}_y\|_q+K_1\|\varphi(t)\|_q].$$

On the other hand, $\hat{\boldsymbol{u}}(t)$ is an isolated periodic solution of (1.5), so if m_0 is sufficiently large, there exists T_m^{-1} and $\|T_m^{-1}\|_q\leq M_c$ for $m\geq m_0$. Then

$$\|\boldsymbol{u}_y\|_q\leq M_c\{\|\boldsymbol{u}_\varphi(t)\|_q+\|\boldsymbol{\eta}(t)\|_q\},$$

and

$$\|\boldsymbol{u}_y\|_q\leq M_c[\|\boldsymbol{u}_\varphi(t)\|_q+\sigma_1(m)\,[(K_2+K_1^2)\|\boldsymbol{u}_y\|_q+K_1\|\varphi(t)\|_q]].$$

Since $1-M_c(K_2+K_1^2)\sigma_1(m)>0$ for sufficiently large m, then

$$\|\boldsymbol{u}_y\|_q\leq\frac{M_c(\|\boldsymbol{u}_\varphi(t)\|_q+K_1\sigma_1(m)\|\varphi(t)\|_q)}{1-M_c(K_2+K_1^2)\sigma_1(m)}$$

$$\leq \frac{M_c(\sqrt{d}|||\boldsymbol{n}_\varphi|||_q + K_1\sigma_1(m)\|\boldsymbol{\varphi}(t)\|_q)}{1-M_c(K_2+K_1^2)\sigma_1(m)}$$

$$\leq \frac{M_c(\sqrt{d}+K_1\sigma_1(m))}{1-M_c(K_2+K_1^2)\sigma_1(m)} |||\boldsymbol{n}_\varphi|||_q . \qquad \text{Q.E.D.}$$

Let

$$\hat{\boldsymbol{u}}(t)=(\hat{\boldsymbol{x}}(t), \hat{B}_1,\dots, \hat{B}_{d-1}, \hat{\omega}) \text{ (where } \hat{\boldsymbol{x}}(t)=\hat{\boldsymbol{a}}_0+\sqrt{2}\sum_{n=1}^{\infty}(\hat{\boldsymbol{a}}_{2n-1}\cos nt+\hat{\boldsymbol{a}}_{2n}\sin nt))$$

be an isolated periodic solution of (1.5) lying inside E, and let us consider the Jacobian matrix $J_m(\hat{\boldsymbol{\alpha}})$ where $\hat{\boldsymbol{\alpha}}=\mathrm{col}\,[\hat{\boldsymbol{a}}_0,\dots, \hat{\boldsymbol{a}}_{2m-1}, \hat{\boldsymbol{a}}_{2m}, \hat{B}_1,\dots, \hat{B}_{d-1}, \hat{\omega}]$. Then the lemma above yields the following corollaries.

Corollary 2.1.
There exists a positive integer m_0 such that

$$\det J_m(\hat{\boldsymbol{\alpha}}) \neq 0$$

for any $m \geq m_0$.

PROOF. For $\boldsymbol{u}_y(t)$, $\boldsymbol{n}_\varphi(t)$, $\boldsymbol{\xi}$ and $\boldsymbol{\gamma}$ of the form (4.3) and (4.4), the differential system (4.5) is equivalent to the linear system

$$J_m(\hat{\boldsymbol{\alpha}})\boldsymbol{\xi}+\boldsymbol{\gamma}=\boldsymbol{0} \tag{4.7}$$

as mentioned in the beginning of this section. Now put $\boldsymbol{\gamma}=\boldsymbol{0}$. Then $\boldsymbol{n}_\varphi(t)=\boldsymbol{0}$, and this implies $\boldsymbol{u}_y=\boldsymbol{0}$ by (4.6). Then $\boldsymbol{\xi}=\boldsymbol{0}$ by (4.3), (4.4). Thus, in (4.7), $\boldsymbol{\gamma}=\boldsymbol{0}$ implies $\boldsymbol{\xi}=\boldsymbol{0}$. That is, $\det J_m(\hat{\boldsymbol{\alpha}}) \neq 0$. Q.E.D.

Corollary 2.2.
There is a positive integer m_0 such that, for any $m \geq m_0$, $J_m^{-1}(\hat{\boldsymbol{\alpha}})$ exists and

$$|||J_m^{-1}(\hat{\boldsymbol{\alpha}})|||' \leq \frac{M_c(1+K_1\sigma_1(m))}{1-M_c(K_2+K_1^2)\sigma_1(m)} .$$

PROOF. By Corollary 2.1, $J_m^{-1}(\hat{\boldsymbol{\alpha}})$ certainly exists for $m \geq m_0$. Further for $\boldsymbol{u}_y(t)$, $\boldsymbol{n}_\varphi$, $\boldsymbol{\xi}$ and $\boldsymbol{\gamma}$ of the form (4.3), (4.4), the differential system (4.5) is equivalent to the linear system (4.7). Hence $\boldsymbol{\xi}=-J_m^{-1}(\hat{\boldsymbol{\alpha}})\boldsymbol{\gamma}$. Since $\|\boldsymbol{u}_y(t)\|_q=|||\boldsymbol{\xi}|||'$, $\|\boldsymbol{u}_\varphi(t)\|_q=|||\boldsymbol{\gamma}|||'$ and $\|\boldsymbol{\varphi}(t)\|_q \leq \|\boldsymbol{u}_\varphi\|_q=|||\boldsymbol{\gamma}|||'$, then

$$\|\boldsymbol{u}_y(t)\|_q \leq \frac{M_c(\|\boldsymbol{u}_\varphi\|_q+K_1\sigma_1(m)\|\boldsymbol{\varphi}(t)\|_q)}{1-M_c(K_2+K_1^2)\sigma_1(m)}$$

and so

$$|||\boldsymbol{\xi}|||' \leq \frac{M_c(1+K_1\sigma_1(m))|||\boldsymbol{\gamma}|||'}{1-M_c(K_2+K_1^2)\sigma_1(m)} .$$

Namely,

$$|||J_m^{-1}(\hat{\boldsymbol{\alpha}})|||' \leq \frac{M_c(1+K_1\sigma_1(m))}{1-M_c(K_2+K_1^2)\sigma_1(m)}. \qquad \text{Q.E.D.}$$

Lastly, for the difference $J_m(\boldsymbol{\alpha}')-J_m(\boldsymbol{\alpha}'')$, we shall prove the following lemma.

Lemma 3.

Under the conditions of Lemma 2, let K_4 be a positive constant

$$(4.8) \qquad K_4 \geq \left[\max_E \sum_{k=1}^{n}\sum_{l=1}^{n+d}\left\{\sum_{p=1}^{n}\left(\frac{\partial X_u^{kl}}{\partial x_p}\right)^2+\sum_{i=1}^{d-1}\left(\frac{\partial X_u^{kl}}{\partial B_i}\right)^2+\left(\frac{\partial X_u^{kl}}{\partial \omega}\right)^2\right\}\right]^{\frac{1}{2}},$$

where $X_u^{kl}(\boldsymbol{u})$ $(k=1,\dots,n;\ l=1,\dots,n+d)$ are the elements of the matrix

$$\boldsymbol{X}_u(\boldsymbol{u})=\left(\frac{\omega}{2\pi}\boldsymbol{X}_x(\boldsymbol{x}, B_1,\dots, B_{d-1}) \quad \frac{\omega}{2\pi}\boldsymbol{X}_{B_1}\cdots\frac{\omega}{2\pi}\boldsymbol{X}_{B_{d-1}} \quad \frac{1}{2\pi}\boldsymbol{X}(\boldsymbol{x}, B_1,\dots, B_{d-1})\right)$$

and x_p $(p=1,\dots,n)$ are the components of the vector $\boldsymbol{x}$. Then, if both

$$\boldsymbol{u}'(t)=(\boldsymbol{x}'(t), B_1',\dots, B_{d-1}', \omega') \ (\textit{where } \boldsymbol{x}'(t)=\boldsymbol{\alpha}_0'+\sqrt{2}\sum_{n=1}^{m}(\boldsymbol{\alpha}_{2n-1}'\cos nt+\boldsymbol{\alpha}_{2n}'\sin nt))$$

and

$$\boldsymbol{u}''(t)=(\boldsymbol{x}''(t), B_1'',\dots, B_{d-1}'', \omega'') \ (\textit{where } \boldsymbol{x}''(t)=\boldsymbol{\alpha}_0''+\sqrt{2}\sum_{n=1}^{m}(\boldsymbol{\alpha}_{2n-1}''\cos nt+\boldsymbol{\alpha}_{2n}''\sin nt))$$

belong to E together with $\theta\boldsymbol{u}'(t)+(1-\theta)\boldsymbol{u}''(t)$ $(0\leq\theta\leq 1)$, then

$$(4.9) \qquad |||J_m(\boldsymbol{\alpha}')-J_m(\boldsymbol{\alpha}'')|||' \leq K_4\|\boldsymbol{u}'-\boldsymbol{u}''\|_\infty \leq K_4\sqrt{2m+1}\,|||\boldsymbol{\alpha}'-\boldsymbol{\alpha}''|||',$$

where $\boldsymbol{\alpha}'=\mathrm{col}\,[\boldsymbol{\alpha}_0', \boldsymbol{\alpha}_1',\dots, \boldsymbol{\alpha}_{2m-1}', \boldsymbol{\alpha}_{2m}', B_1',\dots, B_{d-1}', \omega']$ and

$$\boldsymbol{\alpha}''=\mathrm{col}\,[\boldsymbol{\alpha}_0'', \boldsymbol{\alpha}_1'',\dots, \boldsymbol{\alpha}_{2m-1}'', \boldsymbol{\alpha}_{2m}'', B_1'',\dots, B_{d-1}'', \omega''].$$

PROOF. Take an arbitrary $\boldsymbol{\xi}=\mathrm{col}\,[\boldsymbol{v}_0, \boldsymbol{v}_1,\dots, \boldsymbol{v}_{2m-1}, \boldsymbol{v}_{2m}, V_1,\dots, V_{d-1}, W]$, and consider

$$\boldsymbol{u}_y(t)=(\boldsymbol{y}(t), V_1,\dots, V_{d-1}, W), \text{ where } \boldsymbol{y}(t)=\boldsymbol{v}_0+\sqrt{2}\sum_{n=1}^{m}(\boldsymbol{v}_{2n-1}\cos nt+\boldsymbol{v}_{2n}\sin nt).$$

Put

$$(4.10) \qquad \boldsymbol{\gamma}'=-J_m(\boldsymbol{\alpha}')\boldsymbol{\xi},\ \boldsymbol{\gamma}''=-J_m(\boldsymbol{\alpha}'')\boldsymbol{\xi},$$

and let

$$\boldsymbol{\gamma}'=\mathrm{col}\,[\boldsymbol{c}_0', \boldsymbol{c}_1',\dots, \boldsymbol{c}_{2m-1}', \boldsymbol{c}_{2m}', -C_1',\dots, -C_{d-1}', -W'],\ \boldsymbol{w}'=\mathrm{col}\,[C_1',\dots, C_{d-1}', W'],$$

and

$$\boldsymbol{\gamma}''=\mathrm{col}\,[\boldsymbol{c}_0'', \boldsymbol{c}_1'',\dots, \boldsymbol{c}_{2m-1}'', \boldsymbol{c}_{2m}'', -C_1'',\dots, -C_{d-1}'', -W''],\ \boldsymbol{w}''=\mathrm{col}\,[C_1'',\dots, C_{d-1}'', W''].$$

If we put

$$\boldsymbol{u}_{\varphi'}=(\varphi'(t),\ C'_1,\dots,\ C'_{d-1},\ W'),\ \boldsymbol{n}_{\varphi'}=\left[\varphi'(t),\ \begin{pmatrix}\mathbf{0}\\ \boldsymbol{w}'\end{pmatrix}\right],\ \text{and}$$

$$\boldsymbol{u}_{\varphi''}=(\varphi''(t),\ C''_1,\dots,\ C''_{d-1},\ W''),\ \boldsymbol{n}_{\varphi''}=\left[\varphi''(t),\ \begin{pmatrix}\mathbf{0}\\ \boldsymbol{w}''\end{pmatrix}\right],$$

$$\text{(where } \varphi'(t)=\boldsymbol{c}'_0+\sqrt{2}\sum_{n=1}^{m}(\boldsymbol{c}'_{2n-1}\cos nt+\boldsymbol{c}'_{2n}\sin nt)\text{, and}$$

$$\varphi''(t)=\boldsymbol{c}''_0+\sqrt{2}\sum_{n=1}^{m}(\boldsymbol{c}''_{2n-1}\cos nt+\boldsymbol{c}''_{2n}\sin nt)),$$

then by (4.5) and (4.10) we have

$$\left[\frac{dy}{dt}-P_m[\boldsymbol{X}_u(\boldsymbol{u}'(t))\boldsymbol{u}_y(t)^t],\ \boldsymbol{f}'(\boldsymbol{u}'(t))\boldsymbol{u}_y(t)\right]=\boldsymbol{n}_{\varphi'}=\left[\varphi'(t),\ \begin{pmatrix}\mathbf{0}\\ \boldsymbol{w}'\end{pmatrix}\right],\ \text{and}$$

$$\left[\frac{dy}{dt}-P_m[\boldsymbol{X}_u(\boldsymbol{u}''(t))\boldsymbol{u}_y(t)^t],\ \boldsymbol{f}'(\boldsymbol{u}''(t))\boldsymbol{u}_y(t)\right]=\boldsymbol{n}_{\varphi''}=\left[\varphi''(t),\ \begin{pmatrix}\mathbf{0}\\ \boldsymbol{w}''\end{pmatrix}\right].$$

From this it readily follows that

$$\begin{aligned}\boldsymbol{n}_{\varphi'}-\boldsymbol{n}_{\varphi''}&=\left[\varphi'(t)-\varphi''(t),\ \begin{pmatrix}\mathbf{0}\\ \boldsymbol{w}'\end{pmatrix}-\begin{pmatrix}\mathbf{0}\\ \boldsymbol{w}''\end{pmatrix}\right]\\ &=[-P_m[(\boldsymbol{X}_u(\boldsymbol{u}'(t))-\boldsymbol{X}_u(\boldsymbol{u}''(t)))\boldsymbol{u}_y(t)^t],\ \mathbf{0}],\end{aligned}$$

that is,

$$\begin{cases}\varphi'(t)-\varphi''(t)=-P_m[(\boldsymbol{X}_u(\boldsymbol{u}'(t))-\boldsymbol{X}_u(\boldsymbol{u}''(t)))\boldsymbol{u}_y(t)^t],\\ \begin{pmatrix}\mathbf{0}\\ \boldsymbol{w}'\end{pmatrix}-\begin{pmatrix}\mathbf{0}\\ \boldsymbol{w}''\end{pmatrix}=\mathbf{0}\quad \text{(these equations are equivalent to } \boldsymbol{w}'=\boldsymbol{w}''\text{)}.\end{cases}$$

Let us put

$$\varphi(t)=\varphi'(t)-\varphi''(t)\ \text{and}\ \boldsymbol{\gamma}=\boldsymbol{\gamma}'-\boldsymbol{\gamma}''=\mathrm{col}\,[\boldsymbol{c}'_0-\boldsymbol{c}''_0,\dots,\ \boldsymbol{c}'_{2m}-\boldsymbol{c}''_{2m},\ 0,\dots,\ 0].$$

Then we have

$$\varphi(t)=-P_m[(\boldsymbol{X}_u(\boldsymbol{u}'(t))-\boldsymbol{X}_u(\boldsymbol{u}''(t)))\boldsymbol{u}_y(t)^t] \tag{4.11}$$

and

$$\boldsymbol{\gamma}=-[J_m(\boldsymbol{\alpha}')-J_m(\boldsymbol{\alpha}'')]\boldsymbol{\xi}. \tag{4.12}$$

Now $\|\boldsymbol{X}_u(\boldsymbol{u}'(t))-\boldsymbol{X}_u(\boldsymbol{u}''(t))\|^2_{n+d,n}\le\sum_{k=1}^{n}\sum_{l=1}^{n+d}(X_u^{kl}(\boldsymbol{u}'(t))-X_u^{kl}(\boldsymbol{u}''(t)))^2$, where $X_u^{kl}(\boldsymbol{u}'(t))$ and $X_u^{kl}(\boldsymbol{u}''(t))$ are the elements of $\boldsymbol{X}_u(\boldsymbol{u}'(t))$ and $\boldsymbol{X}_u(\boldsymbol{u}''(t))$, respectively. Since $\boldsymbol{u}''(t)+\theta(\boldsymbol{u}'(t)-\boldsymbol{u}''(t))\in E(0\le\theta\le 1)$ by the assumption, the quantity in the right member of the above inequality is estimated successively by means of Schwarz's inequality as follows:

$$[X_u^{kl}(\boldsymbol{u}'(t))-X_u^{kl}(\boldsymbol{u}''(t))]^2$$

$$=\left[\int_0^1\left\{\sum_{p=1}^{n}\frac{\partial X_u^{kl}(\boldsymbol{u}''+\theta(\boldsymbol{u}'-\boldsymbol{u}''))}{\partial x_p}(x'_p-x''_p)+\sum_{i=1}^{d-1}\frac{\partial X_u^{kl}(\boldsymbol{u}''+\theta(\boldsymbol{u}'-\boldsymbol{u}''))}{\partial B_i}(B'_i-B''_i)\right.\right.$$

$$\left.\left.+\frac{\partial X_u^{kl}(\boldsymbol{u}''+\theta(\boldsymbol{u}'-\boldsymbol{u}''))}{\partial\omega}(\omega'-\omega'')\right\}d\theta\right]^2$$

$$=\left[\sum_{p=1}^{n}\int_0^1\frac{\partial X_u^{kl}}{\partial x_p}d\theta(x'_p-x''_p)+\sum_{i=1}^{d-1}\int_0^1\frac{\partial X_u^{kl}}{\partial B_i}d\theta(B'_i-B''_i)+\int_0^1\frac{\partial X_u^{kl}}{\partial\omega}d\theta(\omega'-\omega'')\right]^2$$

$$\leq\left[\sum_{p=1}^{n}\left(\int_0^1\frac{\partial X_u^{kl}}{\partial x_p}d\theta\right)^2+\sum_{i=1}^{d-1}\left(\int_0^1\frac{\partial X_u^{kl}}{\partial B_i}d\theta\right)^2+\left(\int_0^1\frac{\partial X_u^{kl}}{\partial\omega}d\theta\right)^2\right]$$

$$\times\left[\sum_{p=1}^{n}(x'_p-x''_p)^2+\sum_{i=1}^{d-1}(B'_i-B''_i)^2+(\omega'-\omega'')^2\right]$$

$$\leq\left[\sum_{p=1}^{n}\int_0^1\left(\frac{\partial X_u^{kl}}{\partial x_p}\right)^2d\theta+\sum_{i=1}^{d-1}\int_0^1\left(\frac{\partial X_u^{kl}}{\partial B_i}\right)^2d\theta+\int_0^1\left(\frac{\partial X_u^{kl}}{\partial\omega}\right)^2d\theta\right]$$

$$\times\left[\|\boldsymbol{x}'(t)-\boldsymbol{x}''(t)\|_n^2+\sum_{i=1}^{d-1}(B'_i-B''_i)^2+(\omega'-\omega'')^2\right]$$

$$\leq\left[\sum_{p=1}^{n}\int_0^1\left(\frac{\partial X_u^{kl}}{\partial x_p}\right)^2d\theta+\sum_{i=1}^{d-1}\int_0^1\left(\frac{\partial X_u^{kl}}{\partial B_i}\right)^2d\theta+\int_0^1\left(\frac{\partial X_u^{kl}}{\partial\omega}\right)^2d\theta\right]$$

$$\times\left[\|\boldsymbol{x}'-\boldsymbol{x}''\|_c^2+\sum_{i=1}^{d-1}(B'_i-B''_i)^2+(\omega'-\omega'')^2\right]$$

$$\leq\left[\sum_{p=1}^{n}\int_0^1\left(\frac{\partial X_u^{kl}}{\partial x_p}\right)^2d\theta+\sum_{i=1}^{d-1}\int_0^1\left(\frac{\partial X_u^{kl}}{\partial B_i}\right)^2d\theta+\int_0^1\left(\frac{\partial X_u^{kl}}{\partial\omega}\right)^2d\theta\right]\times\|\boldsymbol{u}'-\boldsymbol{u}''\|_\infty^2,$$

Then

$$\sum_{k,l}[X_u^{kl}(\boldsymbol{u}'(t))-X_u^{kl}(\boldsymbol{u}''(t))]^2$$

$$\leq\int_0^1\left[\sum_{k,l}\left\{\sum_{p=1}^{n}\left(\frac{\partial X_u^{kl}}{\partial x_p}\right)^2+\sum_{i=1}^{d-1}\left(\frac{\partial X_u^{kl}}{\partial B_i}\right)^2+\left(\frac{\partial X_u^{kl}}{\partial\omega}\right)^2\right\}\right]d\theta\times\|\boldsymbol{u}'-\boldsymbol{u}''\|_\infty^2$$

$$\leq K_4^2\|\boldsymbol{u}'-\boldsymbol{u}''\|_\infty^2.$$

Hence

$$\|(\boldsymbol{X}_u(\boldsymbol{u}'(t))-\boldsymbol{X}_u(\boldsymbol{u}''(t)))\boldsymbol{u}_y(t)^t\|_n$$

$$\leq\|\boldsymbol{X}_u(\boldsymbol{u}'(t))-\boldsymbol{X}_u(\boldsymbol{u}''(t))\|_{n+d,n}\|\boldsymbol{u}_y(t)^t\|_{n+d}$$

$$\leq K_4\|\boldsymbol{u}'-\boldsymbol{u}''\|_\infty\|\boldsymbol{u}_y(t)^t\|_{n+d}.$$

Then by Bessel's inequality it follows from (4.11) that

$$\|\boldsymbol{\varphi}(t)\|_q \leq K_4 \|\boldsymbol{u}' - \boldsymbol{u}''\|_\infty \|\boldsymbol{u}_y\|_q.$$

Since $|||\boldsymbol{\gamma}|||' = \|\boldsymbol{\varphi}\|_q$ and $|||\boldsymbol{\xi}|||' = \|\boldsymbol{u}_y\|_q$, from (4.12) we have

$$|||(J_m(\boldsymbol{\alpha}') - J_m(\boldsymbol{\alpha}''))\boldsymbol{\xi}|||' = |||\boldsymbol{\gamma}|||' = \|\boldsymbol{\varphi}\|_q \leq K_4\|\boldsymbol{u}' - \boldsymbol{u}''\|_\infty \|\boldsymbol{u}_y\|_q = K_4 \|\boldsymbol{u}' - \boldsymbol{u}''\|_\infty |||\boldsymbol{\xi}|||',$$

which implies

$$|||J_m(\boldsymbol{\alpha}') - J_m(\boldsymbol{\alpha}'')|||' \leq K_4 \|\boldsymbol{u}' - \boldsymbol{u}''\|_\infty .$$

Put $\boldsymbol{\alpha} = \boldsymbol{\alpha}' - \boldsymbol{\alpha}''$, and suppose $\boldsymbol{\alpha} = \mathrm{col}\,[\boldsymbol{a}_0, \boldsymbol{a}_1, \dots, \boldsymbol{a}_{2m-1}, \boldsymbol{a}_{2m}, B_1, \dots, B_{d-1}, \omega]$. Then

$$\begin{aligned} \boldsymbol{u}(t) &= (\boldsymbol{x}_m(t), B_1, \dots, B_{d-1}, \omega) = \boldsymbol{u}'(t) - \boldsymbol{u}''(t) \\ &= (\boldsymbol{x}'_m(t) - \boldsymbol{x}''_m(t), B'_1 - B''_1, \dots, B'_{d-1} - B''_{d-1}, \omega' - \omega''), \end{aligned}$$

from this implies

$$\begin{cases} \boldsymbol{x}_m(t) = \boldsymbol{x}'_m(t) - \boldsymbol{x}''_m(t) = \boldsymbol{a}_0 + \sqrt{2}\sum_{n=1}^{m} (\boldsymbol{a}_{2n-1}\cos nt + \boldsymbol{a}_{2n}\sin nt), \\ B_i = B'_i - B''_i \qquad (i = 1, 2, \dots, d-1), \\ \omega = \omega' - \omega'', \end{cases}$$

and therefore

$$\begin{aligned} \|\boldsymbol{u}(t)\|_\infty &= \|\boldsymbol{u}' - \boldsymbol{u}''\|_\infty = \|\boldsymbol{x}' - \boldsymbol{x}''\|_c + \sum_{i=1}^{d-1} |B'_i - B''_i| + |\omega' - \omega''| \\ &\leq \{\sum_k [|a_{0k}| + \sqrt{2}\sum_{n=1}^{m} \sqrt{a_{2n-1k}^2 + a_{2nk}^2}]^2\}^{\frac{1}{2}} + \sum_{i=1}^{d-1} |B'_i - B''_i| + |\omega' - \omega''| \\ &\leq \{\sum_k (1+2m)[a_{0k}^2 + \sum_{n=1}^{m} (a_{2n-1k}^2 + a_{2nk}^2)]\}^{\frac{1}{2}} + \sum_{i=1}^{d-1} |B'_i - B''_i| + |\omega' - \omega''| \\ &= (2m+1)^{\frac{1}{2}}[\|\boldsymbol{a}_0\|_n^2 + \sum_{i=1}^{m} (\|\boldsymbol{a}_{2i-1}\|_n^2 + \|\boldsymbol{a}_{2i}\|_n^2)]^{\frac{1}{2}} + \sum_{i=1}^{d-1} |B'_i - B''_i| + |\omega' - \omega''| \\ &\leq \sqrt{2m+1}\,|||\boldsymbol{\alpha}' - \boldsymbol{\alpha}''|||'. \end{aligned}$$

Q.E.D.

The existence of a Galerkin approximation

The existence of a Galerkin approximation to an isolated periodic solution is proved by the following theorem.

Theorem 3.

Let

$$\left[\frac{d\boldsymbol{x}}{dt} - \frac{\omega}{2\pi}\boldsymbol{X}(\boldsymbol{x}, B_1, \dots, B_{d-1}), \boldsymbol{f}(\boldsymbol{u})\right] = \boldsymbol{0} \tag{4.13}$$

be a given boundary value problem, where $\boldsymbol{u}=(\boldsymbol{x}(t), B_1,\dots, B_{d-1}, \omega)$, $\boldsymbol{f}(\boldsymbol{u})=\begin{pmatrix}\boldsymbol{x}(0)-\boldsymbol{x}(2\pi)\\ L(\boldsymbol{u})-\boldsymbol{\beta}\end{pmatrix}$ *and where* $\boldsymbol{x}$ *and* $\boldsymbol{X}(\boldsymbol{x}, B_1,\dots, B_{d-1})$ *are real n-dimensional vectors, and* $\boldsymbol{\beta}$ *is a d-dimensional constant vector. We assume that* $\frac{\omega}{2\pi}\boldsymbol{X}(\boldsymbol{x}, B_1,\dots, B_{d-1})$ *and its first partial derivatives with respect to* $\boldsymbol{u}$ *are continuously differentiable with respect to* $\boldsymbol{u}$ *in the region* $E=D_1\times D_2$, *where* D_1 *is a closed bounded region of the* $\boldsymbol{x}$*-space and* D_2 *is a closed bounded region of* $\boldsymbol{R}^d$. *Moreover we assume that a linear operator* $L(C[I]\times\boldsymbol{R}^d\to\boldsymbol{R}^d)$ *is continuous, namely, there exists a positive constant* K_5 *such that* $\|L\|_{\infty,d}\le K_5$, *where* $\|\cdot\|_{\infty,d}$ *denotes the norm of a continuous linear operator mapping from the product space* $C[I]\times\boldsymbol{R}^d$ *with the norm* $\|\cdot\|_\infty$ *to* $\boldsymbol{R}^d$ *with the norm* $\|\cdot\|_d$. *If there is an isolated periodic solution* $\boldsymbol{u}=\hat{\boldsymbol{u}}(t)$ *of* (4.13) *lying inside E, then there exists a Galerkin approximation* $\boldsymbol{u}=\bar{\boldsymbol{u}}_m(t)=(\bar{\boldsymbol{x}}_m(t), \bar{B}_1, \dots, \bar{B}_{d-1}, \bar{\omega})$ *of any order* $m\ge m_0$ *lying in E provided* m_0 *is sufficiently large.*

PROOF. Setting

$$P_m\hat{\boldsymbol{u}}(t)=(P_m\hat{\boldsymbol{x}}(t), \hat{B}_1,\dots, \hat{B}_{d-1}, \hat{\omega})=(\hat{\boldsymbol{x}}_m(t), \hat{B}_1,\dots, \hat{B}_{d-1}, \hat{\omega})=\hat{\boldsymbol{u}}_m(t),$$

we have

$$\frac{d\hat{\boldsymbol{x}}_m}{dt}=P_m\frac{d\hat{\boldsymbol{x}}}{dt}=P_m\boldsymbol{X}(\hat{\boldsymbol{u}})\quad\left(\text{where } \boldsymbol{X}(\hat{\boldsymbol{u}})=\frac{\hat{\omega}}{2\pi}\boldsymbol{X}(\hat{\boldsymbol{x}}(t), \hat{B}_1,\dots, \hat{B}_{d-1})\right). \tag{4.14}$$

Now let us take a small positive number δ_0 so that

$$U=\left\{\boldsymbol{u}=(\boldsymbol{x}, B_1,\dots, B_{d-1}, \omega)\,;\ \left\|\begin{pmatrix}\boldsymbol{x}-\hat{\boldsymbol{x}}(t)\\ B_1-\hat{B}_1\\ \vdots\\ B_{d-1}-\hat{B}_{d-1}\\ \omega-\hat{\omega}\end{pmatrix}\right\|_{n+d}\le\delta_0\quad \textit{for some } t\in\boldsymbol{R}\right\}\subset E.$$

This is possible because $\boldsymbol{u}=\hat{\boldsymbol{u}}(t)$ lies inside E by the assumption. Then by Lemma 1 – (i), $\hat{\boldsymbol{u}}_m(t)\in U\subset E$ for all $t\in\boldsymbol{R}$ and for any $m\ge m_0$ provided m_0 is sufficiently large. For such m equation (4.14) can be rewritten as follows:

$$\frac{d\hat{\boldsymbol{x}}_m}{dt}=P_m\boldsymbol{X}(\hat{\boldsymbol{u}}_m(t))+\boldsymbol{R}_m(t), \tag{4.15}$$

where

$$\boldsymbol{R}_m(t)=P_m[\boldsymbol{X}(\hat{\boldsymbol{u}}(t))-\boldsymbol{X}(\hat{\boldsymbol{u}}_m(t))].$$

Now

$$\boldsymbol{X}(\hat{\boldsymbol{u}}(t))-\boldsymbol{X}(\hat{\boldsymbol{u}}_m(t))=\int_0^1\boldsymbol{X}_{\boldsymbol{u}}(\hat{\boldsymbol{u}}_m(t)+\theta(\hat{\boldsymbol{u}}(t)-\hat{\boldsymbol{u}}_m(t)))\begin{pmatrix}\hat{\boldsymbol{x}}-\hat{\boldsymbol{x}}_m\\ \boldsymbol{0}\end{pmatrix}d\theta,$$

and $\|X_u(\hat{u}_m(t)+\theta(\hat{u}(t)-\hat{u}_m(t)))\|_{n+d,n}\leq K_1$,
hence

$$\|X(\hat{u}(t))-X(\hat{u}_m(t))\|_n\leq K_1\left\|\begin{pmatrix}\hat{x}-\hat{x}_m\\ \mathbf{0}\end{pmatrix}\right\|_{n+d}=K_1\|\hat{x}-\hat{x}_m\|_n.$$

Then by the proof of Lemma 1-(ii) we have

$$\|X(\hat{u}(t))-X(\hat{u}_m(t))\|_q\leq K_1\|\hat{x}-\hat{x}_m\|_q\leq KK_1\sigma_1(m).$$

Hence, from Bessel's inequality, we see that

$$\|R_m\|_q\leq KK_1\sigma_1(m).$$

Let us put

$$\hat{u}(t)=(\hat{x}(t),\ \hat{B}_1,\dots,\ \hat{B}_{d-1},\ \hat{\omega}),\ \hat{x}(t)=\hat{a}_0+\sqrt{2}\sum_{n=1}^{\infty}(\hat{a}_{2n-1}\cos nt+\hat{a}_{2n}\sin nt)$$

and

$$R_m(t)=r_0^{(m)}+\sqrt{2}\sum_{n=1}^{m}(r_{2n-1}^{(m)}\cos nt+r_{2n}^{(m)}\sin nt).$$

Now setting

$$f(\hat{u}_m(t))=\begin{pmatrix}\hat{x}_m(0)-\hat{x}_m(2\pi)\\ L(\hat{u}_m(t))-\beta\end{pmatrix}=\begin{pmatrix}\mathbf{0}\\ v\end{pmatrix}, \tag{4.16}$$

since

$$f(\hat{u}(t))=\begin{pmatrix}\hat{x}(0)-\hat{x}(2\pi)\\ L(\hat{u}(t))-\beta\end{pmatrix}=\mathbf{0},\ \text{then we have}$$

$$v=L(\hat{u}_m(t))-\beta-(L(\hat{u}(t))-\beta)=L[\hat{u}_m(t)-\hat{u}(t)].$$

From this equation and the assumption of the theorem, we see that

$$\|v\|_d\leq\|L\|_{\infty,d}\|\hat{u}_m(t)-\hat{u}(t)\|_\infty\leq K_5\|\hat{u}_m(t)-\hat{u}(t)\|_\infty.$$

From (4.15) and (4.16), we have

$$\left[\frac{d\hat{x}_m}{dt}-P_mX(\hat{u}_m(t)),\ f(\hat{u}_m(t))\right]=\left[R_m(t),\ \begin{pmatrix}\mathbf{0}\\ v\end{pmatrix}\right]. \tag{4.17}$$

Then (4.17) is equivalent to the following system:

$$\left\{\ F_0^{(m)}(\hat{a})=\frac{1}{2\pi}\int_0^{2\pi}X(\hat{u}_m(t))\,dt=-r_0^{(m)},\right.$$

$$(4.18)\quad \begin{cases} \boldsymbol{F}_{2n-1}^{(m)}(\hat{\boldsymbol{\alpha}}) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_0^{2\pi} \boldsymbol{X}(\hat{\boldsymbol{u}}_m(t)) \cos nt \, dt - n\hat{\boldsymbol{\alpha}}_{2n} = -\boldsymbol{r}_{2n-1}^{(m)}, \\ \boldsymbol{F}_{2n}^{(m)}(\hat{\boldsymbol{\alpha}}) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_0^{2\pi} \boldsymbol{X}(\hat{\boldsymbol{u}}_m(t)) \sin nt \, dt + n\hat{\boldsymbol{\alpha}}_{2n-1} = -\boldsymbol{r}_{2n}^{(m)}, \\ \qquad\qquad (n=1, 2, \ldots, m), \\ \boldsymbol{F}_L^{(m)}(\hat{\boldsymbol{\alpha}}) = L(\hat{\boldsymbol{u}}_m(t)) - \boldsymbol{\beta} = L[\hat{\boldsymbol{u}}_m(t) - \hat{\boldsymbol{u}}(t)] = \boldsymbol{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix}, \end{cases}$$

where

$$\begin{cases} \hat{\boldsymbol{\alpha}} = \mathrm{col}\,[\hat{\boldsymbol{\alpha}}_0, \hat{\alpha}_1, \ldots, \hat{\boldsymbol{\alpha}}_{2m-1}, \hat{\boldsymbol{\alpha}}_{2m}, \hat{B}_1, \ldots, \hat{B}_{d-1}, \hat{\omega}], \\ \boldsymbol{\rho}^{(m)} = \mathrm{col}\,[\boldsymbol{r}_0^{(m)}, \boldsymbol{r}_1^{(m)}, \ldots, \boldsymbol{r}_{2m-1}^{(m)}, \boldsymbol{r}_{2m}^{(m)}, -v_1, \ldots, -v_d], \\ \hat{\boldsymbol{u}}_m(t) = (\hat{\boldsymbol{x}}_m(t), \hat{B}_1, \ldots, \hat{B}_{d-1}, \hat{\omega}), \text{ and} \\ \hat{\boldsymbol{x}}_m(t) = \hat{\boldsymbol{\alpha}}_0 + \sqrt{2} \displaystyle\sum_{n=1}^{m} (\hat{\boldsymbol{\alpha}}_{2n-1} \cos nt + \hat{\boldsymbol{\alpha}}_{2n} \sin nt). \end{cases}$$

Put $\boldsymbol{F}^{(m)}(\hat{\boldsymbol{\alpha}}) = \mathrm{col}\,[\boldsymbol{F}_0^{(m)}(\hat{\boldsymbol{\alpha}}), \boldsymbol{F}_1^{(m)}(\hat{\boldsymbol{\alpha}}), \ldots, \boldsymbol{F}_{2m-1}^{(m)}(\hat{\boldsymbol{\alpha}}), \boldsymbol{F}_{2m}^{(m)}(\hat{\boldsymbol{\alpha}}), (F_L)_1, \ldots, (F_L)_d]$, then (4.18) can be rewritten as follows:

$$\boldsymbol{F}^{(m)}(\hat{\boldsymbol{\alpha}}) = -\boldsymbol{\rho}^{(m)},$$

where, $\boldsymbol{F}_L^{(m)}(\hat{\boldsymbol{\alpha}}) = \mathrm{col}\,[(F_L)_1, \ldots, (F_L)_d]$.
Then

$$|||\boldsymbol{\rho}^{(m)}|||' = |||\boldsymbol{F}^{(m)}(\hat{\boldsymbol{\alpha}})|||' = \|\boldsymbol{R}_m(t)\|_q + \sum_{i=1}^{d} |v_i| \le \|\boldsymbol{R}_m(t)\|_q + \sqrt{d}\,\|\boldsymbol{v}\|_d$$
$$\le KK_1\sigma_1(m) + \sqrt{d}K_5 \|\hat{\boldsymbol{u}}_m(t) - \hat{\boldsymbol{u}}(t)\|_\infty.$$

Now, for $m \ge m_0$ and m_0 sufficiently large, let us consider the region

$$V_m = \left\{ \boldsymbol{u} = (\boldsymbol{x}, B_1, \ldots, B_{d-1}, \omega);\ \left\| \begin{pmatrix} \boldsymbol{x} - \hat{\boldsymbol{x}}_m(t) \\ B_1 - \hat{B}_1 \\ \vdots \\ B_{d-1} - \hat{B}_{d-1} \\ \omega - \hat{\omega} \end{pmatrix} \right\|_{n+d} \le \delta_0 - K\sigma(m) \text{ for some } t \in \boldsymbol{R} \right\}.$$

For any $\boldsymbol{u} = (\boldsymbol{x}, B_1, \ldots, B_{d-1}, \omega) \in V_m$, then

$$\left\| \begin{pmatrix} \boldsymbol{x} - \hat{\boldsymbol{x}}(t) \\ B_1 - \hat{B}_1 \\ \vdots \\ B_{d-1} - \hat{B}_{d-1} \\ \omega - \hat{\omega} \end{pmatrix} \right\|_{n+d} \le \left\| \begin{pmatrix} \boldsymbol{x} - \hat{\boldsymbol{x}}_m(t) \\ B_1 - \hat{B}_1 \\ \vdots \\ B_{d-1} - \hat{B}_{d-1} \\ \omega - \hat{\omega} \end{pmatrix} \right\|_{n+d} + \left\| \begin{pmatrix} \hat{\boldsymbol{x}}_m(t) - \hat{\boldsymbol{x}}(t) \\ \hat{B}_1 - \hat{B}_1 \\ \vdots \\ \hat{B}_{d-1} - \hat{B}_{d-1} \\ \hat{\omega} - \hat{\omega} \end{pmatrix} \right\|_{n+d}$$

$$= \left\| \begin{pmatrix} \boldsymbol{x} - \hat{\boldsymbol{x}}_m(t) \\ B_1 - \hat{B}_1 \\ \vdots \\ B_{d-1} - \hat{B}_{d-1} \\ \omega - \hat{\omega} \end{pmatrix} \right\|_{n+d} + \| \hat{\boldsymbol{x}}_m(t) - \hat{\boldsymbol{x}}(t) \|_n \leq \delta_0 - K\sigma(m) + K\sigma(m) = \delta_0 .$$

This implies $\boldsymbol{u} = (\boldsymbol{x}, B_1, \dots, B_{d-1}, \omega) \in U \subset E$. That is,

$$V_m \subset U \subset E \qquad \text{for any} \quad m \geq m_0.$$

Consider

$$\Omega_m = \left\{ \boldsymbol{\alpha};\ \||\boldsymbol{\alpha} - \hat{\boldsymbol{\alpha}}\||' \leq \frac{\delta_0 - K\sigma(m)}{\sqrt{2m+1}} \right\},$$

where $\boldsymbol{\alpha} = \mathrm{col}\,[\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_{2m-1}, \boldsymbol{\alpha}_{2m}, B_1, \dots, B_{d-1}, \omega]$.
Then, as is shown in the proof of Lemma 3, for

$$\boldsymbol{u}(t) = (\boldsymbol{x}(t), B_1, \dots, B_{d-1}, \omega) \quad (\text{where } \boldsymbol{x}(t) = \boldsymbol{\alpha}_0 + \sqrt{2} \sum_{n=1}^{m} (\boldsymbol{\alpha}_{2n-1} \cos nt + \boldsymbol{\alpha}_{2n} \sin nt))$$

with $\boldsymbol{\alpha} = \mathrm{col}\,[\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_{2m-1}, \boldsymbol{\alpha}_{2m}, B_1, \dots, B_{d-1}, \omega] \in \Omega_m$, we have

$$\left\| \begin{pmatrix} \boldsymbol{x}(t) - \dot{\boldsymbol{x}}_m(t) \\ B_1 - \hat{B}_1 \\ \vdots \\ B_{d-1} - \hat{B}_{d-1} \\ \omega - \hat{\omega} \end{pmatrix} \right\|_{n+d} \leq \| \boldsymbol{u}(t) - \hat{\boldsymbol{u}}_m(t) \|_\infty \leq \sqrt{2m+1}\, \||\boldsymbol{\alpha} - \hat{\boldsymbol{\alpha}}\||' \leq \delta_0 - K\sigma(m)$$

for any $m \geq m_0$,
and hence $\boldsymbol{u}(t) = (\boldsymbol{x}(t), B_1, \dots, B_{d-1}, \omega) \in V_m \subset E$. Thus, it is proved that $\boldsymbol{F}^{(m)}(\boldsymbol{\alpha})$ is well defined for any $\boldsymbol{\alpha} \in \Omega_m$.

From (4.18) we note that a Galerkin approximation is a trigonometric polynomial whose Fourier coefficients satisfy the equation

$$\boldsymbol{F}^{(m)}(\boldsymbol{\alpha}) = \boldsymbol{0}. \tag{4.19}$$

Since $\boldsymbol{\alpha} = \hat{\boldsymbol{\alpha}}$ is an approximate solution of the above equation, we shall apply Proposition 3 (M. Urabe [5]) to the above equation in order to prove the existence of an exact solution, namely, the existence of a Galerkin approximation.

Let us take m_0 sufficiently large. Then by Corollary 2.2 of Lemma 2 for any $m \geq m_0$, $J_m^{-1}(\hat{\boldsymbol{\alpha}})$ exists and

$$\||J_m^{-1}(\hat{\boldsymbol{\alpha}})\||' \leq \frac{M_c(1 + K_1\sigma_1(m))}{1 - M_c(K_2 + K_1^2)\sigma_1(m)} .$$

This implies that

(C.1) $$|||J_m^{-1}(\hat{\boldsymbol{\alpha}})|||' \leq M' \qquad \text{for any } m \geq m_0,$$

where

(4.20) $$M' = \frac{M_c(1+K_1\sigma_1(m_0))}{1-M_c(K_2+K_1^2)\sigma_1(m_0)}.$$

Further by Lemma 3

$$|||J_m(\boldsymbol{\alpha})-J_m(\hat{\boldsymbol{\alpha}})|||' \leq K_4\sqrt{2m+1}\,|||\boldsymbol{\alpha}-\hat{\boldsymbol{\alpha}}|||'$$

for any $\boldsymbol{\alpha} \in \Omega_m$ provided $m \geq m_0$.

Take an arbitrary number κ such that $0<\kappa<1$, and put

$$\delta_1 = \min\left(\frac{\kappa}{K_4 M'},\ \delta_0 - K\sigma(m_0)\right).$$

Let us take $m_1 \geq m_0$ so that, for any $m \geq m_1$,

(4.21) $$\frac{M'\{KK_1\sigma_1(m)+\sqrt{d}K_5\|\hat{\boldsymbol{u}}_m(t)-\hat{\boldsymbol{u}}(t)\|_\infty\}}{1-\kappa} < \frac{\delta_1}{\sqrt{2m+1}}.$$

This is possible because

$$\sqrt{2m+1}\,\sigma_1(m) = \frac{\sqrt{2m+1}}{m+1} \longrightarrow 0 \quad \text{as} \quad m \longrightarrow +\infty,$$

and

$$\sqrt{2m+1}\,\|\hat{\boldsymbol{u}}_m(t)-\hat{\boldsymbol{u}}(t)\|_\infty \longrightarrow 0 \quad \text{as} \quad m \longrightarrow +\infty.$$

By (4.21) we can take a positive number δ_m such that

$$\frac{M'\{KK_1\sigma_1(m)+\sqrt{d}K_5\|\hat{\boldsymbol{u}}_m(t)-\boldsymbol{u}(t)\|_\infty\}}{1-\kappa} \leq \delta_m \leq \frac{\delta_1}{\sqrt{2m+1}}.$$

Let us consider the set

$$\Omega_{\delta_m} = \{\boldsymbol{\alpha};\ |||\boldsymbol{\alpha}-\hat{\boldsymbol{\alpha}}|||' \leq \delta_m\}.$$

For any $\boldsymbol{\alpha} \in \Omega_{\delta_m}$ we have

$$|||\boldsymbol{\alpha}-\hat{\boldsymbol{\alpha}}|||' \leq \frac{\delta_1}{\sqrt{2m+1}} \leq \frac{\delta_0-K\sigma(m_0)}{\sqrt{2m+1}} \leq \frac{\delta_0-K\sigma(m)}{\sqrt{2m+1}} \qquad (m \geq m_1 \geq m_0),$$

and consequently,

$$\Omega_{\delta_m} \subset \Omega_m.$$

Then, for any $\boldsymbol{\alpha} \in \Omega_{\delta_m}$, we have

$$\text{(C.2)} \qquad |||J_m(\boldsymbol{\alpha}) - J_m(\hat{\boldsymbol{\alpha}})|||' \leq K_4\sqrt{2m+1}\delta_m \leq K_4\delta_1 \leq K_4 \frac{\kappa}{K_4 M'} = \frac{\kappa}{M'}.$$

Further,

$$\text{(C.3)} \qquad \frac{M' |||\boldsymbol{\rho}^{(m)}|||'}{1-\kappa} \leq \frac{M'\{KK_1\sigma_1(m) + \sqrt{d}K_5\|\hat{\boldsymbol{u}}_m(t) - \hat{\boldsymbol{u}}(t)\|_\infty\}}{1-\kappa} \leq \delta_m.$$

The expressions (C.1)–(C.3) show that the conditions of Proposition 3 (M. Urabe [5]) are all fulfilled. Thus, by that proposition we see that the equation $\boldsymbol{F}^{(m)}(\boldsymbol{\alpha}) = \boldsymbol{0}$ has one and only one solution $\boldsymbol{\alpha} = \bar{\boldsymbol{\alpha}}$ lying in Ω_{δ_m}. This proves the theorem. Q.E.D.

Uniformly convergence of a Galerkin approximation of a periodic solution of (4.13) (or (1.5))

Theorem 4.

Assume that the conditions of Theorem 3 are satisfied. Let $\boldsymbol{u} = \hat{\boldsymbol{u}}(t)$ be an isolated periodic solution of (4.13) lying inside E and $\boldsymbol{u} = \bar{\boldsymbol{u}}_m(t)$ be its Galerkin approximation as stated in Theorem 3.
If m_0 is sufficiently large, then for any positive integer $m \geq m_0$,

$$\text{(4.22)} \qquad \|\bar{\boldsymbol{u}}_m - \hat{\boldsymbol{u}}\|_\infty \leq \frac{\sqrt{2m+1}M'\{KK_1\sigma_1(m) + \sqrt{d}K_5\|\hat{\boldsymbol{u}}_m - \hat{\boldsymbol{u}}\|_\infty\}}{1-\kappa} + K\sigma(m),$$

$$\text{(4.23)} \qquad \|\dot{\bar{\boldsymbol{u}}}_m - \dot{\hat{\boldsymbol{u}}}\|_\infty \leq 2KK_1\sigma(m) + \frac{\sqrt{2m+1}M'K_1\{KK_1\sigma_1(m) + \sqrt{d}K_5\|\hat{\boldsymbol{u}}_m - \hat{\boldsymbol{u}}\|_\infty\}}{1-\kappa},$$

where κ is an arbitrary fixed number such that $0 < \kappa < 1$, K and K_1 are the numbers defined in Lemma 1, M' is the number defined in (4.20), and K_5 is the number such that $\|L\|_{\infty,d} \leq K_5$.

PROOF. Put
$\bar{\boldsymbol{u}}_m(t) = (\bar{\boldsymbol{x}}_m(t), \bar{B}_1, \dots, \bar{B}_{d-1}, \bar{\omega})$ (where $\bar{\boldsymbol{x}}_m(t) = \bar{\boldsymbol{\alpha}}_0 + \sqrt{2}\sum_{n=1}^{m} (\bar{\boldsymbol{\alpha}}_{2n-1}\cos nt + \bar{\boldsymbol{\alpha}}_{2n}\sin nt)$).
As shown in the proof of Theorem 3, $\bar{\boldsymbol{\alpha}} = \text{col}\,[\bar{\boldsymbol{\alpha}}_0, \bar{\boldsymbol{\alpha}}_1, \dots, \bar{\boldsymbol{\alpha}}_{2m-1}, \bar{\boldsymbol{\alpha}}_{2m}, \bar{B}_1, \dots, \bar{B}_{d-1}, \bar{\omega}]$ is a solution of $\boldsymbol{F}^{(m)}(\boldsymbol{\alpha}) = \boldsymbol{0}$ lying in Ω_{δ_m}, and by Proposition 3 (M. Urabe [5]) we have

$$\text{(4.24)} \qquad |||\bar{\boldsymbol{\alpha}} - \hat{\boldsymbol{\alpha}}|||' \leq \frac{M'\{KK_1\sigma_1(m) + \sqrt{d}K_5\|\hat{\boldsymbol{u}}_m(t) - \hat{\boldsymbol{u}}(t)\|_\infty\}}{1-\kappa},$$

where $\hat{\boldsymbol{\alpha}} = \text{col}\,[\hat{\boldsymbol{\alpha}}_0, \hat{\boldsymbol{\alpha}}_1, \dots, \hat{\boldsymbol{\alpha}}_{2m-1}, \hat{\boldsymbol{\alpha}}_{2m}, \hat{B}_1, \dots, \hat{B}_{d-1}, \hat{\omega}]$ is such that

$$\hat{\boldsymbol{u}}_m(t) = P_m\hat{\boldsymbol{u}}(t) = (\hat{\boldsymbol{x}}_m(t), \hat{B}_1, \dots, \hat{B}_{d-1}, \hat{\omega}), \text{ and}$$

$$\hat{\boldsymbol{x}}_m(t) = P_m\hat{\boldsymbol{x}}(t) = \hat{\boldsymbol{\alpha}}_0 + \sqrt{2}\sum_{n=1}^{m} (\hat{\boldsymbol{\alpha}}_{2n-1}\cos nt + \hat{\boldsymbol{\alpha}}_{2n}\sin nt).$$

From (4.9), we have

$$\|\bar{\boldsymbol{u}}_m(t)-\hat{\boldsymbol{u}}_m(t)\|_\infty \le \sqrt{2m+1}\,\|\bar{\boldsymbol{\alpha}}-\hat{\boldsymbol{\alpha}}\|'$$

$$\le \sqrt{2m+1}\,\frac{M'\{KK_1\sigma_1(m)+\sqrt{d}K_5\|\hat{\boldsymbol{u}}_m(t)-\hat{\boldsymbol{u}}(t)\|_\infty\}}{1-\kappa}\,.$$

On the other hand, by Lemma 1 $-$(i)

$$\|\hat{\boldsymbol{u}}_m(t)-\hat{\boldsymbol{u}}(t)\|_\infty \le K\sigma(m)\,.$$

Thus,

$$\|\bar{\boldsymbol{u}}_m(t)-\hat{\boldsymbol{u}}(t)\|_\infty \le \|\bar{\boldsymbol{u}}_m(t)-\hat{\boldsymbol{u}}_m(t)\|_\infty + \|\hat{\boldsymbol{u}}_m(t)-\hat{\boldsymbol{u}}(t)\|_\infty$$

$$\le \sqrt{2m+1}\,\frac{M'\{KK_1\sigma_1(m)+\sqrt{d}K_5\|\boldsymbol{u}_m(t)-\hat{\boldsymbol{u}}(t)\|_\infty\}}{1-\kappa}+K\sigma(m)\,.$$

This proves (4.22).
Since $\bar{\boldsymbol{\alpha}}$ is a solution of $\boldsymbol{F}^{(m)}(\boldsymbol{\alpha})=\boldsymbol{0}$, for $\bar{\boldsymbol{u}}_m(t)$ we have

$$\left[\frac{d\bar{\boldsymbol{x}}_m}{dt}-P_m\boldsymbol{X}(\bar{\boldsymbol{u}}_m(t)),\,\boldsymbol{f}(\bar{\boldsymbol{u}}_m(t))\right]=\boldsymbol{0}$$

$$\left(\text{where } \boldsymbol{X}(\bar{\boldsymbol{u}}_m(t))=\frac{\bar{\omega}}{2\pi}\boldsymbol{X}(\bar{\boldsymbol{x}}_m(t),\,\bar{B}_1,\dots,\,\bar{B}_{d-1})\right).$$

This can be rewritten as follows:

$$\left[\frac{d\bar{\boldsymbol{x}}_m}{dt}-\boldsymbol{X}(\bar{\boldsymbol{u}}_m(t)),\,\boldsymbol{f}(\bar{\boldsymbol{u}}_m(t))\right]=[\boldsymbol{\eta}_m(t),\,\boldsymbol{0}]\,, \tag{4.25}$$

where $\boldsymbol{\eta}_m(t)=-(I-P_m)\boldsymbol{X}(\bar{\boldsymbol{u}}_m(t))$, and I is the identity operator.
Since

$$\frac{d}{dt}\boldsymbol{X}(\bar{\boldsymbol{u}}_m(t))=\boldsymbol{X}_{\boldsymbol{x}}(\bar{\boldsymbol{u}}_m(t))\,\frac{d\bar{\boldsymbol{x}}_m}{dt}$$

$$\left(\text{where } \boldsymbol{X}_{\boldsymbol{x}}(\bar{\boldsymbol{u}}_m(t))=\frac{\bar{\omega}}{2\pi}\boldsymbol{X}_{\boldsymbol{x}}(\bar{\boldsymbol{x}}_m(t),\,\bar{B}_1,\dots,\,\bar{B}_{d-1})\right)$$

$$=\boldsymbol{X}_{\boldsymbol{x}}(\bar{\boldsymbol{u}}_m(t))\,P_m\boldsymbol{X}(\bar{\boldsymbol{u}}_m(t))\,,$$

by Bessel's inequality we have

$$\left\|\frac{d}{dt}\boldsymbol{X}(\bar{\boldsymbol{u}}_m(t))\right\|_q \le K_1\|P_m\boldsymbol{X}(\bar{\boldsymbol{u}}_m(t))\|_q \le K_1\|\boldsymbol{X}(\bar{\boldsymbol{u}}_m(t))\|_q \le K_1K.$$

Then, by Proposition 2 we have

$$\|\boldsymbol{\eta}_m(t)\|_c=\|(I-P_m)\boldsymbol{X}(\bar{\boldsymbol{u}}_m(t))\|_c \le \sigma(m)\left\|\frac{d}{dt}\boldsymbol{X}(\bar{\boldsymbol{u}}_m(t))\right\|_q \le KK_1\sigma(m)\,.$$

On the other hand, $\hat{\boldsymbol{u}}(t)=(\hat{\boldsymbol{x}}(t), \hat{B}_1,\dots, \hat{B}_{d-1}, \hat{\omega})$ satisfies

$$\left[\frac{d\hat{\boldsymbol{x}}}{dt}-\boldsymbol{X}(\hat{\boldsymbol{u}}), \boldsymbol{f}(\hat{\boldsymbol{u}})\right]=\boldsymbol{0} \quad \left(\text{where } \boldsymbol{X}(\hat{\boldsymbol{u}})=\frac{\hat{\omega}}{2\pi}\boldsymbol{X}(\hat{\boldsymbol{x}}, \hat{B}_1,\dots, \hat{B}_{d-1})\right). \tag{4.26}$$

Now, we have

$$\frac{d\bar{\boldsymbol{u}}_m}{dt}-\frac{d\hat{\boldsymbol{u}}}{dt}=\left(\frac{d\bar{\boldsymbol{x}}_m}{dt}-\frac{d\hat{\boldsymbol{x}}}{dt}, 0, 0,\dots, 0\right).$$

By (4.25) and (4.26), we have

$$\begin{aligned}\frac{d\bar{\boldsymbol{x}}_m}{dt}-\frac{d\hat{\boldsymbol{x}}}{dt}&=\{\boldsymbol{X}(\bar{\boldsymbol{u}}_m(t))-\boldsymbol{X}(\hat{\boldsymbol{u}}(t))\}+\boldsymbol{\eta}_m(t)\\ &=\int_0^1 \boldsymbol{X}_u(\hat{\boldsymbol{u}}(t)+\theta(\bar{\boldsymbol{u}}_m(t)-\hat{\boldsymbol{u}}(t)))\begin{pmatrix}\bar{\boldsymbol{x}}_m(t)-\hat{\boldsymbol{x}}(t)\\ \bar{B}_1-\hat{B}_1\\ \vdots\\ \bar{B}_{d-1}-\hat{B}_{d-1}\\ \bar{\omega}-\hat{\omega}\end{pmatrix}d\theta+\boldsymbol{\eta}_m(t),\end{aligned}$$

and consequently

$$\left\|\frac{d\bar{\boldsymbol{x}}_m}{dt}-\frac{d\hat{\boldsymbol{x}}}{dt}\right\|_n\le\|\boldsymbol{X}_u\|_{n+d,n}\left\|\begin{pmatrix}\bar{\boldsymbol{x}}_m(t)-\hat{\boldsymbol{x}}(t)\\ \bar{B}_1-\hat{B}_1\\ \vdots\\ \bar{B}_{d-1}-\hat{B}_{d-1}\\ \bar{\omega}-\hat{\omega}\end{pmatrix}\right\|_{n+d}+\|\boldsymbol{\eta}_m(t)\|_n.$$

Since

$$\left\|\begin{pmatrix}\bar{\boldsymbol{x}}_m(t)-\hat{\boldsymbol{x}}(t)\\ \bar{B}_1-\hat{B}_1\\ \vdots\\ \bar{B}_{d-1}-\hat{B}_{d-1}\\ \bar{\omega}-\hat{\omega}\end{pmatrix}\right\|_{n+d}\le\|\bar{\boldsymbol{x}}_m(t)-\hat{\boldsymbol{x}}(t)\|_n+|\bar{B}_1-\hat{B}_1|+\cdots+|\bar{B}_{d-1}-\hat{B}_{d-1}|+|\bar{\omega}-\hat{\omega}|,$$

then we have

$$\begin{aligned}\left\|\frac{d\bar{\boldsymbol{x}}_m}{dt}-\frac{d\hat{\boldsymbol{x}}}{dt}\right\|_c&\le K_1\{\|\bar{\boldsymbol{x}}_m(t)-\hat{\boldsymbol{x}}(t)\|_c+|\bar{B}_1-\hat{B}_1|+\cdots+|\bar{B}_{d-1}-\hat{B}_{d-1}|+|\bar{\omega}-\hat{\omega}|\}\\ &\quad+\|\boldsymbol{\eta}_m(t)\|_c\le K_1\|\bar{\boldsymbol{u}}_m(t)-\hat{\boldsymbol{u}}(t)\|_\infty+KK_1\sigma(m).\end{aligned}$$

Thus, by (4.22), then we have

$$\left\|\frac{d\bar{\boldsymbol{u}}_m}{dt}-\frac{d\hat{\boldsymbol{u}}}{dt}\right\|_\infty=\left\|\frac{d\bar{\boldsymbol{x}}_m}{dt}-\frac{d\hat{\boldsymbol{x}}}{dt}\right\|_c\le K_1\|\bar{\boldsymbol{u}}_m-\hat{\boldsymbol{u}}\|_\infty+KK_1\sigma(m)$$

$$\leq \frac{\sqrt{2m+1}M'K_1\{KK_1\sigma_1(m)+\sqrt{d}K_5\|\hat{\boldsymbol{u}}_m(t)-\hat{\boldsymbol{u}}(t)\|_\infty\}}{1-\kappa}+KK_1\sigma(m)+KK_1\sigma(m)$$

$$=2KK_1\sigma(m)+\frac{\sqrt{2m+1}M'K_1\{KK_1\sigma_1(m)+\sqrt{d}K_5\|\hat{\boldsymbol{u}}_m(t)-\hat{\boldsymbol{u}}(t)\|_\infty\}}{1-\kappa}.$$

This proves (4.23). Q.E.D.

§5. Appendix

We show that the solution $\boldsymbol{u}=\hat{\boldsymbol{u}}(t)$ guaranteed by Theorem 2 in §2 is an isolated solution. Namely, we show that

$$\det \boldsymbol{f}'(\hat{\boldsymbol{u}}(t))[\hat{\Psi}(t)]\neq 0,$$

where $\hat{\Psi}(t)$ is the fundamental matrix of the linear homogeneous system

$$\frac{d\boldsymbol{z}}{dt}=\begin{pmatrix} \boldsymbol{X}_u(\hat{\boldsymbol{u}}(t)) \\ \boxed{0} \end{pmatrix}\boldsymbol{z} \qquad (\text{where } \boxed{0} \text{ is a } d\times(n+d) \text{ matrix})$$

satisfying the initial condition $\hat{\Psi}(0)=E((n+d)\times(n+d)$ unit matrix).

Proof. Suppose that $\boldsymbol{u}=\hat{\boldsymbol{u}}(t)$ is not an isolated solution of (1.5). Namely,

$$\det \boldsymbol{f}'(\hat{\boldsymbol{u}})[\hat{\Psi}(t)]=0.$$

Then, there exists a nonzero vector $\hat{\boldsymbol{e}}\in \boldsymbol{R}^{n+d}$ such that

$$\boldsymbol{f}'(\boldsymbol{u})[\hat{\Psi}(t)]\hat{\boldsymbol{e}}=\boldsymbol{0}.$$

Put $\hat{\boldsymbol{h}}=\hat{\Psi}(t)\hat{\boldsymbol{e}}$. Then by the definition of $\hat{\Psi}(t)$ we have

$$\frac{d\hat{\boldsymbol{h}}}{dt}=\frac{d}{dt}(\hat{\Psi}(t)\hat{\boldsymbol{e}})=\frac{d\hat{\Psi}}{dt}\hat{\boldsymbol{e}}=\begin{pmatrix} \boldsymbol{X}_u(\hat{\boldsymbol{u}}) \\ \boxed{0} \end{pmatrix}\hat{\Psi}(t)\hat{\boldsymbol{e}}=\begin{pmatrix} \boldsymbol{X}_u(\hat{\boldsymbol{u}}) \\ \boxed{0} \end{pmatrix}\hat{\boldsymbol{h}},$$

and

$$\boldsymbol{f}'(\hat{\boldsymbol{u}})[\tilde{\boldsymbol{h}}]=\boldsymbol{f}'(\hat{\boldsymbol{u}})[\hat{\Psi}(t)]\hat{\boldsymbol{e}}=\boldsymbol{0},$$

where $\hat{\boldsymbol{h}}=\mathrm{col}\,[\hat{\boldsymbol{h}}_1(t), \hat{h}_{n+1},\dots, \hat{h}_{n+d}]$, and $\tilde{\boldsymbol{h}}=(\hat{\boldsymbol{h}}_1(t), \hat{h}_{n+1},\dots, \hat{h}_{n+d})\in M$. Therefore

$$\boldsymbol{F}'(\hat{\boldsymbol{u}})\tilde{\boldsymbol{h}}=\left[\frac{d\hat{\boldsymbol{h}}_1}{dt}-\boldsymbol{X}_u(\hat{\boldsymbol{u}})\hat{\boldsymbol{h}}, \boldsymbol{f}'(\hat{\boldsymbol{u}})[\tilde{\boldsymbol{h}}]\right]=\boldsymbol{0}.$$

Then

(5.1) $\qquad \tilde{\boldsymbol{h}}=\tilde{\boldsymbol{h}}-T^{-1}\boldsymbol{F}'(\hat{\boldsymbol{u}})\tilde{\boldsymbol{h}}=(I-T^{-1}\boldsymbol{F}'(\hat{\boldsymbol{u}}))\tilde{\boldsymbol{h}} \qquad$ (I is the identity operator).

Since

$$\|I-T^{-1}\boldsymbol{F}'(\hat{\boldsymbol{u}})\|_\infty \leq \|T^{-1}\|_\infty \|| T-\boldsymbol{F}'(\hat{\boldsymbol{u}}) \||$$

$$\leq (\mu_1+\mu_{n+1}+\cdots+\mu_{n+d})\frac{\kappa}{\mu_1+\mu_{n+1}+\cdots+\mu_{n+d}}=\kappa,$$

(5.1) implies

$$\|\tilde{\boldsymbol{h}}\|_\infty \leq \kappa \|\tilde{\boldsymbol{h}}\|_\infty.$$

As $0\leq\kappa<1$, $(1-\kappa)\|\tilde{\boldsymbol{h}}\|_\infty\leq 0$. Hence $\|\tilde{\boldsymbol{h}}\|_\infty=0$. Therefore $\tilde{\boldsymbol{h}}=\boldsymbol{0}$, namely, $\hat{\boldsymbol{h}}=\boldsymbol{0}$, it follows that $\hat{\boldsymbol{e}}=\boldsymbol{0}$, which is a contradiction and the proof is complete.

Proof of (3.2) in §3.

In §3, we have obtained

$$f'(\hat{\boldsymbol{u}})[\hat{\Psi}(t)]=\begin{pmatrix} E_n-\hat{\Phi}_{11}(2\pi) & -\hat{\Phi}_{12}(2\pi) & -\boldsymbol{c}_1 \\ L(\hat{\Psi}_1) & L(\hat{\Psi}_2) & L(\hat{\Psi}_{n+d}(t)) \end{pmatrix}, \tag{5.2}$$

where E_n is an $n\times n$ unit matrix.

Let set $\hat{\boldsymbol{c}}=\frac{\boldsymbol{c}_1}{\|\boldsymbol{c}_1\|_n}$ and an $n\times n$ matrix Q be an orthogonal matrix whose first column vector is $\hat{\boldsymbol{c}}$. And we put

$$K=\begin{pmatrix} Q & 0 \\ 0 & E_d \end{pmatrix} \qquad ((n+d)\times(n+d) \text{ matrix})$$

then, K is also an orthogonal matrix. Write Q as

$$Q=[\hat{\boldsymbol{c}}, Q_1],$$

where Q_1 is an $n\times(n-1)$ matrix whose column vectors are unit vectors and moreover they are mutually orthogonal. By (5.2), we then have

$$K^{-1}f'(\hat{\boldsymbol{u}})[\hat{\Psi}(t)]K=K^t f'(\hat{\boldsymbol{u}})[\hat{\Psi}(t)]K$$

$$=\begin{pmatrix} Q^t(E_n-\hat{\Phi}_{11}(2\pi))Q & -Q^t(\hat{\Phi}_{12}(2\pi), \boldsymbol{c}_1) \\ L(\hat{\Psi}_1)Q & (L(\hat{\Psi}_2), L(\hat{\Psi}_{n+d}(t))) \end{pmatrix},$$

where K^t and Q^t denote the transposed matrices of K and Q, respectively. Now, we have

$$Q^t\hat{\Phi}_{11}(2\pi)Q=\begin{pmatrix} \hat{\boldsymbol{c}}^t \\ Q_1^t \end{pmatrix}\hat{\Phi}_{11}(2\pi)(\hat{\boldsymbol{c}}, Q_1)=\begin{pmatrix} \hat{\boldsymbol{c}}^t \\ Q_1^t \end{pmatrix}(\hat{\boldsymbol{c}}, \hat{\Phi}_{11}(2\pi)Q_1)$$

$$=\begin{pmatrix} 1 & \hat{\boldsymbol{c}}^t\hat{\Phi}_{11}(2\pi)Q_1 \\ \boldsymbol{0} & Q_1^t\hat{\Phi}_{11}(2\pi)Q_1 \end{pmatrix}, \quad \text{(where } \hat{\boldsymbol{c}}^t \text{ is the transposed vector of } \hat{\boldsymbol{c}})$$

therefore,

$$Q^t(E_n-\hat{\Phi}_{11}(2\pi))Q=\begin{pmatrix} 0 & -\hat{\boldsymbol{c}}^t\hat{\Phi}_{11}(2\pi)Q_1 \\ \boldsymbol{0} & E_{n-1}-Q_1^t\hat{\Phi}_{11}(2\pi)Q_1 \end{pmatrix}.$$

Moreover, we have

$$Q^t(\hat{\Phi}_{12}(2\pi),\ \boldsymbol{c}_1)=\begin{pmatrix} \hat{\boldsymbol{c}}^t \\ Q_1^t \end{pmatrix}(\hat{\Phi}_{12}(2\pi),\ \boldsymbol{c}_1)=\left(\begin{pmatrix} \hat{\boldsymbol{c}}^t \\ Q_1^t \end{pmatrix}\hat{\Phi}_{12}(2\pi),\ \begin{matrix} \|\boldsymbol{c}_1\|_n \\ \boldsymbol{0} \end{matrix}\right)$$

and

$$\begin{aligned} L(\hat{\Psi}_1)Q&=L(\hat{\Psi}_1)(\hat{\boldsymbol{c}},\ Q_1)=(L(\hat{\Psi}_1\hat{\boldsymbol{c}}),\ L(\hat{\Psi}_1)Q_1) \\ &=\left(\frac{1}{\|\boldsymbol{c}_1\|_n}L(\boldsymbol{X}_0(\hat{\boldsymbol{x}},\ \hat{B}_1,\dots,\ \hat{B}_{d-1})),\ L(\hat{\Psi}_1)Q_1\right), \end{aligned}$$

where $\boldsymbol{X}_0(\hat{\boldsymbol{x}},\ \hat{B}_1,\dots,\ \hat{B}_{d-1})=\begin{pmatrix} \boldsymbol{X}(\hat{\boldsymbol{x}},\ \hat{B}_1,\dots,\ \hat{B}_{d-1}) \\ \boldsymbol{0} \\ 0 \end{pmatrix}$.

Hence we have

$K^t\boldsymbol{f}'(\hat{\boldsymbol{u}})[\hat{\Psi}(t)]K$

$$=\begin{pmatrix} 0 & -\hat{\boldsymbol{c}}^t\hat{\Phi}_{11}(2\pi)Q_1 & -\begin{pmatrix} \hat{\boldsymbol{c}}^t \\ Q_1^t \end{pmatrix}\hat{\Phi}_{12}(2\pi) & -\|\boldsymbol{c}_1\|_n \\ \boldsymbol{0} & E_{n-1}-Q_1^t\hat{\Phi}_{11}(2\pi)Q_1 & & \boldsymbol{0} \\ \frac{1}{\|\boldsymbol{c}_1\|_n}L(\boldsymbol{X}_0(\hat{\boldsymbol{x}},\ \hat{B}_1,\dots,\ \hat{B}_{d-1})) & L(\hat{\Psi}_1)Q_1 & L(\hat{\Psi}_2) & L(\hat{\Psi}_{n+d}(t)) \end{pmatrix}.$$

Thus, we get

$\det G=\det\{K^t\boldsymbol{f}'(\hat{\boldsymbol{u}})[\hat{\Psi}(t)]K\}$

$$=\det\begin{pmatrix} -\|\boldsymbol{c}_1\|_n & -\hat{\boldsymbol{c}}^t\hat{\Phi}_{11}(2\pi)Q_1 & -\begin{pmatrix} \hat{\boldsymbol{c}}^t \\ Q_1^t \end{pmatrix}\hat{\Phi}_{12}(2\pi) & 0 \\ \boldsymbol{0} & E_{n-1}-Q_1^t\hat{\Phi}_{11}(2\pi)Q_1 & & \boldsymbol{0} \\ \frac{1}{\|\boldsymbol{c}_1\|_n}L(\boldsymbol{X}_0(\hat{\boldsymbol{x}},\ \hat{B}_1,\dots,\ \hat{B}_{d-1})) + L(\hat{\Psi}_{n+d}(t)) & L(\hat{\Psi}_1)Q_1 & L(\hat{\Psi}_2) & \frac{-1}{\|\boldsymbol{c}_1\|_n}L(\boldsymbol{X}_0(\hat{\boldsymbol{x}},\ \hat{B}_1,\dots,\ \hat{B}_{d-1})) \end{pmatrix}.$$

Particularly, when $\boldsymbol{X}_{B_1}=\cdots=\boldsymbol{X}_{B_{d-1}}=\boldsymbol{0}$, in this case, as $\hat{\Psi}_{12}(2\pi)=0$, then

$$\begin{aligned} \det G&=-\|\boldsymbol{c}_1\|_n\cdot\det(E_{n-1}-Q_1^t\hat{\Phi}_{11}(2\pi)Q_1)\times\det\left\{L(\hat{\Psi}_2),\ \frac{-1}{\|\boldsymbol{c}_1\|_n}L(\boldsymbol{X}_0(\hat{\boldsymbol{x}},\ \hat{B}_1,\dots,\ \hat{B}_{d-1}))\right\} \\ &=\det(E_{n-1}-Q_1^t\hat{\Phi}_{11}(2\pi)Q_1)\times\det\{L(\hat{\Psi}_2),\ L(\boldsymbol{X}_0(\hat{\boldsymbol{x}},\ \hat{B}_1,\dots,\ \hat{B}_{d-1}))\}. \end{aligned}$$

However, $\det(E_{n-1}-Q_1^t\hat{\Phi}_{11}(2\pi)Q_1)\neq 0$.

This tells us that det $G \neq 0$ is equivalent to

$$\det\{L(\hat{\Psi}_2),\ L(\boldsymbol{X}_0(\hat{\boldsymbol{x}},\ \hat{B}_1,\dots,\ \hat{B}_{d-1}))\} \neq 0.$$

This completes the proof of (3.2) in §3.

Lastly, we write the Proposition 3 (M. Urabe [5]) quoted in §4.

Proposition 3 (M. Urabe [5]).
Let

$$\boldsymbol{F}(\boldsymbol{\alpha})=\mathbf{0} \tag{5.3}$$

be a given real system of equations where $\boldsymbol{\alpha}$ *and* $\boldsymbol{F}(\boldsymbol{\alpha})$ *are vectors of n-dimensional and* $\boldsymbol{F}(\boldsymbol{\alpha})$ *is a continuously differentiable function of defined in some region* Ω *of* $\boldsymbol{\alpha}$. *Assume that* (5.3) *has an approximate solution* $\boldsymbol{\alpha}=\hat{\boldsymbol{\alpha}}$ *for which the determinant of the Jacobian matrix* $J(\boldsymbol{\alpha})$ *of* $\boldsymbol{F}(\boldsymbol{\alpha})$ *with respect to* $\boldsymbol{\alpha}$ *does not vanish at* $\boldsymbol{\alpha}=\hat{\boldsymbol{\alpha}}$ *and there is a positive constant* δ *and a non-negative constant* $\kappa<1$ *such that*

(i) $\Omega_\delta=\{\boldsymbol{\alpha};\ \|\boldsymbol{\alpha}-\hat{\boldsymbol{\alpha}}\|_n\leq\delta\}\subset\Omega$,

(ii) $\|J(\boldsymbol{\alpha})-J(\hat{\boldsymbol{\alpha}})\|_n\leq\dfrac{\kappa}{M'}$ for any $\boldsymbol{\alpha}\in\Omega_\delta$,

(iii) $\dfrac{M'r}{1-\kappa}\leq\delta$,

where r *and* $M'(>0)$ *are numbers such that*

$$\|\boldsymbol{F}(\hat{\boldsymbol{\alpha}})\|_n\leq r \quad \text{and} \quad \|J^{-1}(\hat{\boldsymbol{\alpha}})\|_n\leq M'.$$

Then the system (5.3) *has one and only one solution* $\boldsymbol{\alpha}=\bar{\boldsymbol{\alpha}}$ *in* Ω_δ *and*

$$\|\bar{\boldsymbol{\alpha}}-\hat{\boldsymbol{\alpha}}\|_n\leq\frac{M'r}{1-\kappa}.$$

§6. Numerical Example

We seek for the largest amplitude of periodic solutions of van der Pol equation

$$\frac{d^2x}{d\tau^2}-\lambda(1-x^2)\frac{dx}{d\tau}+x=0. \tag{6.1}$$

We transfrom τ to t by $\tau=\dfrac{\omega}{2\pi}t$, then

$$\frac{d^2x}{dt^2}-\frac{\omega}{2\pi}\lambda(1-x^2)\frac{dx}{dt}+\left(\frac{\omega}{2\pi}\right)^2x=0. \tag{6.2}$$

By the result of M. Urabe, H. Yanagiwara and Y. Shinohara [3], there exists λ between 3 and 3.5 for which the periodic solution of van der Pol equation has the largest amplitude. Later, H. Yanagiwara [17] has computed the largest amplitude to be 2.0235 in van der Pol equation (6.1) with $\lambda=3.2651$. But, he did not give any error estimates of his numerical results. Hence it is not clear how many significant figures his numerical results contain. In the present paper we make clear this point. Let a periodic solution and its period of (6.2) be $x=x(t, \lambda)$ and $\omega=\omega(\lambda)$, respectively. Then we set $x_1=x(t, \lambda)$ and $x_3=x_\lambda(t, \lambda)$, where $x_\lambda(t, \lambda)$ is the derivative of $x(t, \lambda)$ with respect to λ. Then we get the following system

$$(6.3)\quad \begin{cases} \dfrac{dx_1}{dt}=x_2, \\ \dfrac{dx_2}{dt}=-\left(\dfrac{\omega}{2\pi}\right)^2 x_1+\dfrac{\omega}{2\pi}\lambda(1-x_1^2)x_2, \\ \dfrac{dx_3}{dt}=x_4, \\ \dfrac{dx_4}{dt}=-\left\{\left(\dfrac{\omega}{2\pi}\right)^2+\dfrac{\omega}{\pi}\lambda x_1 x_2\right\}x_3+\dfrac{\omega}{2\pi}\lambda(1-x_1^2)x_4+\dfrac{\omega_\lambda}{2\pi}\lambda(1-x_1^2)x_2 \\ \qquad +\dfrac{\omega}{2\pi}(1-x_1^2)x_2-\dfrac{\omega\cdot\omega_\lambda}{2\pi^2}x_1, \end{cases}$$

where ω_λ is the derivative of $\omega=\omega(\lambda)$ with respect to λ. In this case, we consider $x_0(=x(0, \lambda))$, λ and ω_λ as the unknown parameters $B_1, B_2, \ldots, B_{d-1}(d=4)$ in Theorem 2 in §2. Since the case satisfying the condition $x_3(0)=x_\lambda(0, \lambda)=0$ is the one of realizing the largest amplitude, we adopt the following conditions

$$(6.4)\qquad L(\boldsymbol{u})-\boldsymbol{\beta}=\begin{pmatrix} x_1(0)-x_0 \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{pmatrix}=\mathbf{0} \qquad (\boldsymbol{u}=(\boldsymbol{x}, x_0, \lambda, \omega_\lambda, \omega))$$

as the additional conditions. Then, in this case,

$$(6.5)\qquad \boldsymbol{f}(\boldsymbol{u})=\begin{pmatrix} \boldsymbol{x}(0)-\boldsymbol{x}(2\pi) \\ L(\boldsymbol{u})-\boldsymbol{\beta} \end{pmatrix}=\begin{pmatrix} \boldsymbol{x}(0)-\boldsymbol{x}(2\pi) \\ x_1(0)-x_0 \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{pmatrix}=\mathbf{0}.$$

We apply our Galerkin method with unknown parameters to the problem (6.3) and (6.5), and we caluculate the largest amplitude x_0. The results of numerical computations are as follows;

$$\begin{cases} \bar{x}_0=2.02342\ 22556\ 06113,\ \bar{\lambda}=3.29401\ 26635\ 69712,\ \bar{\omega}_\lambda=1.32864\ 47643\ 62845, \\ \bar{\omega}=9.24555\ 82365\ 78715,\ \det G=-0.214\times 10^{-1},\ r=0.13\times 10^{-13}, \\ \mu_1=83500.0,\ \mu_2=2.0,\ \mu_3=600.0,\ \mu_4=25.0,\ \mu_5=22.0,\ \kappa=0.4\times 10^{-1}, \\ \delta_1=0.114\times 10^{-8},\ \delta_2=0.271\times 10^{-13},\ \delta_3=0.813\times 10^{-11},\ \delta_4=0.339\times 10^{-12}, \\ \delta_5=0.300\times 10^{-12}. \end{cases}$$

Applying Theorem 2 in §2 to the above numerical results, then we have the following error estimates:

$$\begin{cases} \|\hat{\boldsymbol{x}}(t)-\bar{\boldsymbol{x}}(t)\|_c \le \dfrac{\mu_1 r}{1-\kappa} \le \delta_1=0.114\times 10^{-8}, \\ |\hat{x}_0-\bar{x}_0| \le \dfrac{\mu_2 r}{1-\kappa} \le \delta_2=0.271\times 10^{-13}, \\ |\hat{\lambda}-\bar{\lambda}| \le \dfrac{\mu_3 r}{1-\kappa} \le \delta_3=0.813\times 10^{-11}, \\ |\hat{\omega}_\lambda-\bar{\omega}_\lambda| \le \dfrac{\mu_4 r}{1-\kappa} \le \delta_4=0.339\times 10^{-12}, \\ |\hat{\omega}-\bar{\omega}| \le \dfrac{\mu_5 r}{1-\kappa} \le \delta_5=0.300\times 10^{-12}. \end{cases}$$

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