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On the G-connections and Motions on a $\{V, G\}$*-manifold*

Dedicated to Prof. Dr. Makoto Matsumoto on his 60th birthday

By

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§1. Preliminaries

The present paper is the continuation of the serial papers concerning a Finsler manifold modeled on a Minkowski space ([9], [10], [12], [13]). We shall, here, treat a $\{V, G\}$-manifold M where V is a Minkowski space and G is a Lie group consisting of all the linear transformations leaving the norm of V invariant. In our case, the manifold M admits a G-structure and thereby admits G-connections relative to the G-structure. But the G-connection can not be determined uniquely unless G is totally disconnected. In the present paper, first, we shall find the relation between the G-connections in our $\{V, G\}$-manifold. Next, we shall treat Killing vector fields with respect to the $\{V, G\}$-Finsler metric, and affine Killing vector fields with respect to the G-connection on M. There, we shall be concerned with a new tensor $Q_{ijk}{}^{h}$, which plays an important role in the following discussion. Our main purpose is to clarify the relations between the Lie group G, the G-connections, the Killing vector fields and the tensor $Q_{ijk}{}^{h}$ in a $\{V, G\}$-manifold.

Now, let us begin with recollecting the definition of a Minkowski space.

Let V be an n-dimensional Minkowski space, that is, an n-dimensional linear space on which a Minkowski norm is defined.

The Minkowski norm on V means a real valued function on V, whose value at $\xi \in V$ we denote by $\|\xi\|$, with properties:

(1) Let $\{e_\alpha\}$ be a fixed base of V, then the norm of any vector $\xi = \xi^\alpha e_\alpha \in V$ can be represented by $\|\xi\| = f(\xi^1, \xi^2, \dots, \xi^n)$ (For brevity we write $f(\xi^1, \xi^2, \dots, \xi^n)$ as $f(\xi^\alpha)$ or $f(\xi)$). Now, the function $f(\xi)$ is 3-times continuously differentiable at $\xi \neq \mathbf{0}$,

(2) $\|\xi\| \geqq 0$,

(3) $\|\xi\| = 0$ if and only if $\xi = \mathbf{0}$,

(4) $\|k\xi\| = k\|\xi\|$ for any $k > 0$,

(5) $\|\xi_1 + \xi_2\| \leqq \|\xi_1\| + \|\xi_2\|$.

Now we put

$$(1.1) \qquad G = \{(g^\alpha_\beta) \mid (g^\alpha_\beta) \in GL(n, R),\, f(g^\alpha_\beta \xi^\beta) = f(\xi^\alpha) \quad \text{for any} \quad \xi \in V\},$$

then G is a Lie group [9]. Let $\mathfrak{g}$ be the Lie algebra of G.

Next, we consider an n-dimensional C^∞-manifold M admitting a G-structure where G is given by (1.1). Let $\{U\}$ be a coordinate neighbourhood system and $\{X_\alpha\}$ be an n-frame on U adapted to the G-structure, and y be any vector in $T_p(M)$ with the expression $y=y^i \frac{\partial}{\partial x^i}=\xi^\alpha X_\alpha$ where $p\in U$. Putting

$$(1.2)\qquad \frac{\partial}{\partial x^i}=\mu_i^\alpha(x)X_\alpha \quad \text{or} \quad X_\alpha=\lambda_\alpha^i(x)\frac{\partial}{\partial x^i},$$

we have proved, in the paper [9], that the function

$$(1.3)\qquad F(x, y)=f(\xi^\alpha)=f(\mu_i^\alpha(x)y^i)$$

gives M globally a Finsler metric. This Finsler metric is called a *$\{V, G\}$-metric*. When M admits a $\{V, G\}$-metric, we say that M is a *$\{V, G\}$-manifold*. In the $\{V, G\}$-manifold, the tangent Minkowski space $T_p(M)$ at any point $p\in M$ is congruent to the given Minkowski space V.

Moreover, we consider a linear connection $\Gamma_{jk}^i(x)$ on M. Adopting $\Gamma_{mj}^i(x)y^m$ as a non-linear connection, we can define the h-covariant derivative ∇_k for any Finsler tensor S. If we assume, for instance, that S is of type (1,2), $\nabla_k S_{ij}^h$ is given by

$$(1.4)\qquad \nabla_k S_{ij}^h=\partial_k S_{ij}^h-\dot{\partial}_m S_{ij}^h \Gamma_{rk}^m y^r+\Gamma_{mk}^h S_{ij}^m-S_{mj}^h\Gamma_{ik}^m-S_{im}^h\Gamma_{jk}^m.$$

Hereafter we call ∇_k the h-covariant derivative with respect to the linear connection $\Gamma_{jk}^i(x)$.

§ 2. G-connections

Let V be a Minkowski space and let G be the Lie group defined by (1.1). As to the $\{V, G\}$-Finsler metric g_{ij} and the G-connection relative to the G-structure, the following theorem has already been proved in the paper [9].

Theorem 2.1. *In a $\{V, G\}$-manifold, if a linear connection $\Gamma_{jk}^i(x)$ is a G-connection relative to the G-structure where G is the Lie group defined by* (1.1), *then the metric tensor g_{ij} of the $\{V, G\}$-Finsler metric is h-covariant constant with respect to $\Gamma_{jk}^i(x)$, i.e., $\nabla_k g_{ij}=0$.*

Conversely, let M be a $\{V, G\}$-manifold and $\Gamma_{jk}^i(x)$ be any linear connection on M, and assume that

$$(2.1)\qquad \nabla_k g_{ij}=0$$

holds where ∇_k denotes the h-covariant derivative with respect to the linear connection $\Gamma_{jk}^i(x)$. In this case, (2.1) leads us, by virtue of $\nabla_k y^i=0$ and $f^2=g_{ij}y^iy^j$, to

(2.2) $$\nabla_k f(\mu_i^\alpha(x) y^i) = 0.$$

Now, let $\{U\}$ be an admissible coordinate neighbourhood system of the $\{V, G\}$-manifold, and $\{X_\alpha\}$ be a frame adapted to the given G-structure on U. Let Φ be the representation of the restricted holonomy group of $\Gamma_{jk}^i(x)$ at any $p \in U$. Let ℓ be a piecewise differentiable curve beginning and ending at p and homotopic to 0, and, for any $v_0 \in T_p(M)$, let $v(t)$ be a vector field on ℓ which is defined by the parallel displacement of v_0 along ℓ with respect to $\Gamma_{jk}^i(x)$. Then we have $\frac{dv^i(t)}{dt} + \Gamma_{jk}^i(x(t)) \cdot v^j(t) \frac{dx^k}{dt} = 0$. On the other hand, from (1.3), we have $\|v(t)\| = \left\| v^i(t) \frac{\partial}{\partial x^i} \right\| = f(\mu_i^\alpha(x(t)) v^i(t))$. Putting $\mu_i^\alpha v^i = \xi^\alpha$, we find, by virtue of (2.2), that

$$\begin{aligned} \frac{d}{dt} \|v(t)\| &= \frac{\partial f(\xi)}{\partial \xi^\alpha} \left(\frac{d\mu_i^\alpha}{dt} v^i + \mu_i^\alpha \frac{dv^i}{dt} \right) \\ &= \frac{\partial f(\xi)}{\partial \xi^\alpha} \left(\partial_k \mu_j^\alpha \frac{dx^k}{dt} v^j - \mu_i^\alpha \Gamma_{jk}^i v^j \frac{dx^k}{dt} \right) \\ &= \frac{\partial f(\xi)}{\partial \xi^\alpha} \nabla_k \mu_j^\alpha v^j \frac{dx^k}{dt} \\ &= 0. \end{aligned}$$

Hence the length of $v(t)$ is constant along ℓ. Thus the element of Φ is a linear isometry on $T_p(M)$. For any $\varphi \in \Phi$, if we put $\varphi(X_\alpha) = g_\alpha^\beta X_\beta$, the matrix (g_α^β) is a representation of φ with respect to the frame $\{X_\alpha\}$. Hence, for any $\xi = \xi^\alpha X_\alpha$, we have $\varphi(\xi) = \xi^\alpha g_\alpha^\beta X_\beta$ and $\|\xi^\alpha X_\alpha\| = \|\xi^\alpha g_\alpha^\beta X_\beta\|$. Thus we get $f(\xi^\alpha) = f(g_\beta^\alpha \xi^\beta)$, that is, $(g_\beta^\alpha) \in G$. Therefore we obtain

(2.3) $$\Phi \subset G.$$

On the other hand, by virtue of the well-known theorem with respect to holonomy groups (see, for example, [22] p. 206), the manifold M admits a Φ-structure and the given linear connection $\Gamma_{jk}^i(x)$ is a G-connection relative to the Φ-structure. Hence, from (2.3), we find that $\Gamma_{jk}^i(x)$ is a G-connection relative to the G-structure under consideration. Thus we obtain

Theorem 2.2. *Let M be a $\{V, G\}$-manifold where G is the Lie group defined by* (1.1). *Let $\Gamma_{jk}^i(x)$ be a linear connection on M and let us denote by ∇_k the h-covariant derivative with respect to the linear connection $\Gamma_{jk}^i(x)$. If $\nabla_k g_{ij} = 0$ holds on M, then the linear connection $\Gamma_{jk}^i(x)$ is a G-connection relative to the G-structure.*

In the sequel, we shall only deal with a $\{V, G\}$-manifold whose structure group G is the Lie group defined by (1.1). And we shall use, in the present paper, the term *G-connection* as a G-connection relative to the G-structure specified on the $\{V, G\}$-

manifold.

§ 3. The tensor $Q_{ijk}{}^h$

In a Finsler space, we can define a tensor field $Q_{ijk}{}^h$ such that

(3.1) $$Q_{ijk}{}^h = 2C_{ijk}y^h + g_{ik}\delta^h_j + g_{jk}\delta^h_i.$$

This tensor is homogeneous of degree 0 with respect to y^i.

Now, we consider the tensor $Q_{ijk}{}^h$ in a $\{V, G\}$-manifold. It is well-known that the G-connection can not be determined uniquely unless the Lie group G is totally disconnected. Now, let $\tilde{\Gamma}^i_{jk}(x)$ and $\Gamma^i_{jk}(x)$ be two G-connections on a $\{V, G\}$-manifold M, and let D^i_{jk} be the difference of these connections, i.e.,

(3.2) $$D^i_{jk} = \tilde{\Gamma}^i_{jk} - \Gamma^i_{jk}.$$

Of course, D^i_{jk} is a (1, 2)-tensor field on M. We denote by ∇ and $\tilde{\nabla}$ the h-covariant derivative with respect to Γ^i_{jk} and $\tilde{\Gamma}^i_{jk}$ respectively. Then we have

$$\begin{aligned}\tilde{\nabla}_k g_{ij} &= \nabla_k g_{ij} - 2C_{ijm}D^m_{rk}y^r - g_{rj}D^r_{ik} - g_{ir}D^r_{jk}\\ &= \nabla_k g_{ij} - Q_{ijm}{}^r D^m_{rk}.\end{aligned}$$

Taking account of $\nabla_k g_{ij} = 0$ and $\tilde{\nabla}_k g_{ij} = 0$, we find

(3.3) $$Q_{ijm}{}^r D^m_{rk} = 0.$$

Conversely, we assume that $\Gamma^i_{jk}(x)$ is a G-connection on a $\{V, G\}$-manifold and $D^i_{jk}(x)$ is a (1, 2)-tensor field on M satisfying the condition (3.3). If we put $\tilde{\Gamma}^i_{jk} = \Gamma^i_{jk} + D^i_{jk}$, then $\tilde{\Gamma}^i_{jk}$ is a linear connection on M and satisfies $\tilde{\nabla}_k g_{ij} = \nabla_k g_{ij} - Q_{ijm}{}^r D^m_{rk}$. Hence our assumptions lead us to $\tilde{\nabla}_k g_{ij} = 0$, that is, $\tilde{\Gamma}^i_{jk}$ is a G-connection. Consequently we obtain

Theorem 3.1. *Let M be a $\{V, G\}$-manifold. The difference D^i_{jk} of two G-connections on M satisfies $Q_{ijm}{}^r D^m_{rk} = 0$. Conversely, let $\Gamma^i_{jk}(x)$ be a G-connection on M and let $D^i_{jk}(x)$ be a (1, 2)-tensor field satisfying $Q_{ijm}{}^r D^m_{rk} = 0$, then $\Gamma^i_{jk} + D^i_{jk}$ is a G-connection on M.*

Next, let Γ be a G-connection and $\{X_\alpha\}$ be a frame adapted to the G-structure. If we denote by $\Gamma^\alpha_{\beta\gamma}$ the components of Γ with respect to the frame $\{X_\alpha\}$ and we put $X_\alpha = \lambda^i_\alpha(x)\dfrac{\partial}{\partial x^i}$ $\left(\text{or } \dfrac{\partial}{\partial x^i} = \mu^\alpha_i(x)X_\alpha\right)$, then, for any vector field $v = v^i\dfrac{\partial}{\partial x^i} = v^\alpha X_\alpha$, we have, by virtue of the definition, $\nabla_v X_\alpha = v^\beta \Gamma^\gamma_{\alpha\beta}X_\gamma$ and $v^\beta\Gamma^\gamma_{\alpha\beta} \in \mathfrak{g}$ where $\mathfrak{g}$ is the Lie algebra of the Lie group G ([9], [22]). On the other hand, it follows, from the relation $v^i = \lambda^i_\beta v^\beta$, that

$$\nabla_v X_\alpha = v^i\left(\partial_i\lambda^j_\alpha \frac{\partial}{\partial x^j} + \lambda^m_\alpha \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^m}\right)$$

$$= v^i(\partial_i\lambda^j_\alpha + \lambda^m_\alpha \Gamma^j_{mi}) \frac{\partial}{\partial x^j}.$$

Hence $v^\beta \Gamma^\gamma_{\alpha\beta}\lambda^j_\gamma = v^i \nabla_i \lambda^j_\alpha$ holds good, that is, $\Gamma^\gamma_{\alpha\beta}\lambda^j_\gamma = \lambda^i_\beta \nabla_i \lambda^j_\alpha$ holds. Thus we obtain

(3.4) $$\Gamma^\gamma_{\alpha\beta} = \lambda^i_\beta \nabla_i \lambda^j_\alpha \mu^\gamma_j,$$

or equivalently

(3.5) $$\nabla_j \lambda^i_\alpha = \lambda^i_\gamma \Gamma^\gamma_{\alpha\beta}\mu^\beta_j.$$

Moreover, for any suffix j, we have

(3.6) $$\Gamma^\alpha_{\beta\gamma}\mu^\gamma_j = \mu^\alpha_m \nabla_j \lambda^m_\beta \in \mathfrak{g}.$$

Let $\tilde{\Gamma}$ be another G-connection, then we have, for any j,

$$\tilde{\Gamma}^\alpha_{\beta\gamma}\mu^\gamma_j = \mu^\alpha_m \tilde{\nabla}_j \lambda^m_\beta \in \mathfrak{g}.$$

Hence $(\tilde{\Gamma}^\alpha_{\beta\gamma} - \Gamma^\alpha_{\beta\gamma})\mu^\gamma_j = \mu^\alpha_m(\tilde{\nabla}_j\lambda^m_\beta - \nabla_j\lambda^m_\beta) \in \mathfrak{g}$ hold for any j. Since $\nabla_j\lambda^m_\beta = \partial_j\lambda^m_\beta + \Gamma^m_{rj}\lambda^r_\beta$, it follows that

$$\tilde{\nabla}_j\lambda^m_\beta - \nabla_j\lambda^m_\beta = \tilde{\Gamma}^m_{rj}\lambda^r_\beta - \Gamma^m_{rj}\lambda^r_\beta = D^m_{rj}\lambda^r_\beta.$$

Hence we obtain

(3.7) $$\mu^\alpha_m D^m_{rj}\lambda^r_\beta \in \mathfrak{g} \quad \text{for any} \quad j.$$

That is to say,

Theorem 3.2. *Let $\tilde{\Gamma}$ and Γ be two distinct G-connections on a $\{V, G\}$-manifold, and let D be their difference tensor such as $D^i_{jk} = \tilde{\Gamma}^i_{jk} - \Gamma^i_{jk}$. Then, for any suffix j, $\mu^\alpha_m D^m_{rj}\lambda^r_\beta$ belongs to the Lie algebra $\mathfrak{g}$ of the Lie group G.*

Moreover, we shall prove

Theorem 3.3. *Let $A^i_j(x)$ be a $(1, 1)$-tensor field on a $\{V, G\}$-manifold M. The following two conditions are mutually equivalent:*

(1) $$Q_{ijm}{}^r A^m_r = 0;$$

(2) $$\mu^\alpha_m A^m_r \lambda^r_\beta \in \mathfrak{g}.$$

PROOF. First we shall show the condition (1) leads us to (2). Let $v_j(x)$ be any covariant vector field on M, and let us put $D^i_{jk} = A^i_j v_k$. Then, (1) implies $Q_{ijm}{}^r D^m_{rk} = 0$. Hence, from Theorem 3.2, we have $\mu^\alpha_m A^m_r v_k \lambda^r_\beta \in \mathfrak{g}$ for any v_k. Thus the condition (2) is satisfied. Conversely, we assume that the condition (2) is satisfied.

If we put $B^{\alpha}_{\beta}=\mu^{\alpha}_{m}A^{m}_{r}\lambda^{r}_{\beta}$ and $B=(B^{\alpha}_{\beta})$, then we see $B\in\mathscr{g}$ and $\exp(tB)\in G$. Thus we have $f(\exp(tB)\cdot\xi)=f(\xi)$. Differentiating this equation with respect to t at $t=0$, we find $\frac{\partial f(\xi)}{\partial\xi^{\gamma}}B^{\gamma}_{\sigma}\xi^{\sigma}=0$. Now we put $\xi^{\alpha}=\mu^{\alpha}_{i}(x)y^{i}$, then we get $g_{ij}=\frac{1}{2}\frac{\partial^{2}f^{2}(\xi)}{\partial\xi^{\alpha}\partial\xi^{\beta}}\mu^{\alpha}_{i}\cdot\mu^{\beta}_{j}$ where g_{ij} is the metric tensor of the $\{V, G\}$-Finsler metric (1.3). This leads us to $g_{im}y^{m}=f\frac{\partial f(\xi)}{\partial\xi^{\gamma}}\mu^{\gamma}_{i}$, that is, $f\frac{\partial f}{\partial\xi^{\gamma}}=\lambda^{r}_{\gamma}g_{rm}y^{m}$. Hence $\lambda^{r}_{\gamma}(x)g_{rm}y^{m}\mu^{\gamma}_{t}(x)A^{t}_{s}(x)y^{s}=0$, which implies $g_{mr}y^{m}A^{r}_{s}(x)y^{s}=0$. Differentiating this equation with respect to y^{j}, we have $g_{rj}A^{r}_{s}y^{s}+g_{rm}y^{m}A^{r}_{j}=0$. Moreover, differentiating this equation with respect to y^{i}, we obtain $2C_{ijm}A^{m}_{s}y^{s}+g_{mj}A^{m}_{i}+g_{im}A^{m}_{j}=0$, that is, $Q_{ijm}{}^{t}A^{m}_{t}=0$. Q. E. D.

It follows obviously, from these results, that

Corollary 3.4. *Let $\Gamma^{i}_{jk}(x)$ be a G-connection on a $\{V, G\}$-manifold M, and let $A^{i}_{j}(x)$ be any (1, 1)-tensor field on M satisfying $\mu^{\alpha}_{m}A^{m}_{r}\lambda^{r}_{\beta}\in\mathscr{g}$. Then for any covariant vector field $s_{i}(x)$ on M, $\tilde{\Gamma}^{i}_{jk}=\Gamma^{i}_{jk}+A^{i}_{j}s_{k}$ becomes a G-connection on M.*

§ 4. Killing vector fields

Let $\Gamma(x)$ be a G-connection on a $\{V, G\}$-manifold M, and $v(x)$ be a contravariant vector field on M. Let us denote by $\underset{v}{\mathscr{L}}$ and ∇_{v} the Lie derivative and the covariant derivative with respect to $\Gamma(x)$ in the direction of $v(x)$ respectively. Then, it is well-known that $\tilde{A}(v)=\nabla_{v}-\underset{v}{\mathscr{L}}$ is a tensor field of type (1, 1) [14]. For any vector X,

$$\tilde{A}(v)X=\nabla_{v}X-\underset{v}{\mathscr{L}}X=v^{i}(\partial_{i}X^{j}+\Gamma^{j}_{mi}X^{m})\frac{\partial}{\partial x^{j}}-(v^{i}\partial_{i}X^{j}-X^{i}\partial_{i}v^{j})\frac{\partial}{\partial x^{j}}$$

$$=(\partial_{m}v^{j}+\Gamma^{j}_{mr}v^{r})X^{m}\frac{\partial}{\partial x^{j}}.$$

Hence, with respect to the canonical coordinate system, $\tilde{A}(v)$ takes the form

$$\tilde{A}(v)^{i}_{j}=\partial_{j}v^{i}+\Gamma^{i}_{jr}v^{r}. \tag{4.1}$$

Let $v(x)$ be a vector field on M and let φ_{t} be a local 1-parameter group of local transformations generated by $v(x)$. If φ_{t} is, for any t and for any point of M, an isometry of $\{V, G\}$-Finsler metric, $v(x)$ is called a Killing vector field on $\{V, G\}$-manifold.

The condition that a vector field $v(x)$ be a killing one is, as is well-known, $\underset{v}{\mathscr{L}}g_{ij}=0$, where $\underset{v}{\mathscr{L}}g_{ij}$ takes the form

$$\underset{v}{\mathscr{L}}g_{ij}=v^{r}\partial_{r}g_{ij}+2y^{r}\partial_{r}v^{m}C_{ijm}+g_{im}\partial_{j}v^{r}+g_{jm}\partial_{i}v^{m}. \tag{4.2}$$

In connection with these, we shall now prove

Theorem 4.1. *Let $\Gamma^i_{jk}(x)$ be a G-connection on a $\{V, G\}$-manifold M. With respect to a vector field $v(x)$ on M, the following conditions are mutually equivalent:*

(1) *v is a Killing vector field;*
(2) $\underset{v}{\mathcal{L}} g_{ij}=0$;
(3) $Q_{ijm}{}^r \tilde{A}(v)^m_r=0$;
(4) $\mu^\alpha_m \tilde{A}(v)^m_r \lambda^r_\beta \in \mathfrak{g}$, *where* $\mathfrak{g}$ *is the Lie algebra of the Lie group G;*
(5) $g_{im}\tilde{A}(v)^m_j y^i y^j=0$.

PROOF. It is obvious that (1) and (2) are equivalent. From Theorem 3.3, the conditions (3) and (4) are also equivalent. To prove the present theorem, it suffices to show that (2) and (3) are mutually equivalent and so are (3) and (5).

To show the implication (2)→(3), we rewrite $\underset{v}{\mathcal{L}} g_{ij}=0$, by using the relation $\nabla_k g_{ij}=0$, as

$$(4.3)\qquad v^r(2C_{ijm}\Gamma^m_{hr}y^h+g_{im}\Gamma^m_{jr}+g_{jm}\Gamma^m_{ir})+2y^r\partial_r v^m C_{ijm}+g_{mj}\partial_i v^m+g_{im}\partial_j v^m=0.$$

This implies $2C_{ijm}y^r(\partial_r v^m+\Gamma^m_{rh}v^h)+g_{im}(\partial_j v^m+\Gamma^m_{jr}v^r)+g_{mj}(\partial_i v^m+\Gamma^m_{ir}v^r)=0$. Thus we obtain (3). Conversely, if (3) is satisfied, (4.3) holds good. Moreover, by virtue of $\nabla_r g_{ij}=0$, (4.3) can be written in the form

$$v^r\partial_r g_{ij}+2y^r\partial_r v^m C_{ijm}+g_{im}\partial_j v^m+g_{mj}\partial_i v^m=0.$$

Thus we obtain (2).

Next, for the implication (3)→(5), we rewrite (3) as

$$(2C_{ijm}y^r+g_{im}\delta^r_j+g_{mj}\delta^r_j)\tilde{A}(v)^m_r=0.$$

Multiplying this equation by $y^i y^j$ and contracting with i and j, we obtain (5). Since $\tilde{A}(v)^i_j$ are functions of x^k only, by differentiating the equation (5) with y^i, we obtain $g_{im}\tilde{A}(v)^m_s y^s+g_{sm}\tilde{A}(v)^m_i y^s=0$. Moreover, differentiating this equation by y^i, we obtain

$$2C_{ijm}\tilde{A}(v)^m_t y^t+g_{im}\tilde{A}(v)^m_j+g_{jm}\tilde{A}(v)^m_i=0.$$

Thus the implication (5)→(3) has been also verified. Q.E.D.

The consequence of the present theorem and Corollary 3.4 lead us to

Theorem 4.2. *Let $v(x)$ be a Killing vector field on a $\{V, G\}$-manifold M and let $\Gamma(x)$ be a G-connection on M. Then, for any covariant vector field $s(x)$ on M, $\tilde{\Gamma}^i_{jk}=\Gamma^i_{jk}(x)+\tilde{A}(v)^i_j s_k(x)$ is a G-connection on M, where $\tilde{A}(v)^i_j$ is the (1,1)-tensor field given by* (4.1).

Next, we shall show

Theorem 4.3. *Let $v(x)$ be a Killing vector field on a $\{V, G\}$-manifold M and denote by $\tilde{A}$ the matrix $(\tilde{A}(v)^i_j)$ where $\tilde{A}(v)^i_j$ is the tensor given by* (4.1). *Then, $\exp t\tilde{A}$ is an element of the isotropy group of the $\{V, G\}$-Finsler metric $F(x, y)=f(\mu^\alpha_i(x)y^i)$.*

PROOF. If we put $\Lambda=(\lambda^i_\alpha)$, we have $\Lambda^{-1}=(\mu^\alpha_i)$. Thus the equation (4) of Theorem 4.1 can be rewritten as $\Lambda^{-1}\tilde{A}\Lambda\in\mathscr{g}$. Hence $\exp t(\Lambda^{-1}\tilde{A}\Lambda)\in G$. On the other hand, $\exp t(\Lambda^{-1}\tilde{A}\Lambda)=\Lambda^{-1}\cdot\exp t\tilde{A}\cdot\Lambda$. Therefore we have $f(\Lambda^{-1}\cdot\exp t\tilde{A}\cdot\Lambda\cdot\Lambda^{-1}y)=f(\Lambda^{-1}y)$, i.e., $f(\Lambda^{-1}\cdot\exp t\tilde{A}\cdot y)=f(\Lambda^{-1}y)$, which implies $F(x, \exp t\tilde{A}\cdot y)=F(x, y)$. Q.E.D.

§ 5. Affine Killing vector fields

Let $\Gamma(x)$ be a G-connection on a $\{V, G\}$-manifold M. A vector field $v(x)$ on M is called an affine Killing vector field with respect to $\Gamma(x)$ if, for each $x\in M$, a local 1-parameter group of local transformations φ_t generated by $v(x)$ preserves the linear connection $\Gamma(x)$, more precisely, if each φ_t is an affine mapping. Now we shall denote by $\Phi_t(\Gamma)$ the induced connection from Γ by φ_t, and put

$$\underset{v}{\mathscr{L}}\Gamma^i_{jk}=\lim_{t\to 0}\frac{(\Phi_t(\Gamma))^i_{jk}-\Gamma^i_{jk}}{t}, \tag{5.1}$$

then we have

$$\underset{v}{\mathscr{L}}\Gamma^i_{jk}=\frac{\partial^2 v}{\partial x^j\partial x^k}+v^r\partial_r\Gamma^i_{jk}+\Gamma^i_{rk}\partial_j v^r+\Gamma^i_{jr}\partial_k v^r-\Gamma^r_{jk}\partial_r v^i. \tag{5.2}$$

Apparently the condition for $v(x)$ to be an affine Killing vector field is

$$\underset{v}{\mathscr{L}}\Gamma^i_{jk}=0. \tag{5.3}$$

It is well-known that the following relation holds good ([14], [20]):

$$\underset{v}{\mathscr{L}}\Gamma^i_{jk}=\nabla_k\tilde{A}(v)^i_j+v^r R_j{}^i{}_{kr}. \tag{5.4}$$

where $R_j{}^i{}_{kr}$ is the curvature tensor of Γ^i_{jk}, i.e.,

$$R_j{}^i{}_{kr}=\partial_r\Gamma^i_{jk}-\partial_k\Gamma^i_{jr}+\Gamma^i_{mr}\Gamma^m_{jk}-\Gamma^i_{mk}\Gamma^m_{jr}. \tag{5.5}$$

In the case where the manifold is Riemannian, any Killing vector field is an affine Killing one with respect to the Riemann-Christoffel's connection $\left\{ {i \atop j\ k} \right\}$.

In a $\{V, G\}$-manifold, however, under the only assumption that $\Gamma(x)$ is a G-connection, a Killing vector field is not necessarily an affine Killing one with respect to $\Gamma(x)$.

Next, we shall show

Theorem 5.1. *Let $\Gamma(x)$ be a G-connection on a $\{V, G\}$-manifold M. Let $v(x)$ be a Killing vector field on M and let φ_t be the local 1-parameter group of local transformations generated by $v(x)$. Let $\Phi_t(\Gamma)$ be the induced connection from $\Gamma(x)$ by φ_t. Then, $\Phi_t(\Gamma)$ is a G-connection on M.*

PROOF. Let $\{U, x^i\}$ and $\{\bar{U}, \bar{x}^a\}$ be coordinate neighbourhoods of M and let us assume $\varphi_t(U)=\bar{U}$. If we put $\Phi_t(\Gamma)=\tilde{\Gamma}$, then, as is well-known, we have

$$\tilde{\Gamma}^i_{jk}=\frac{\partial x^i}{\partial \bar{x}^a}\left\{\frac{\partial^2 \bar{x}^a}{\partial x^j \partial x^k}+\bar{\Gamma}^a_{bc}\frac{\partial \bar{x}^b}{\partial x^j}\frac{\partial \bar{x}^c}{\partial x^k}\right\}. \tag{5.6}$$

By $\tilde{\nabla}_k$ we denote the covariant derivative with respect to $\tilde{\Gamma}$, we see

$$\begin{aligned}\tilde{\nabla}_k g_{ij}=\partial_k g_{ij}-\dot{\partial}_m g_{ij}&\left\{\frac{\partial x^m}{\partial \bar{x}^a}\frac{\partial^2 \bar{x}^a}{\partial x^r \partial x^k}y^r+\frac{\partial x^m}{\partial \bar{x}^a}\bar{\Gamma}^a_{bc}\bar{y}^b\frac{\partial \bar{x}^c}{\partial x^k}\right\}\\ &-g_{mj}\left\{\frac{\partial x^m}{\partial \bar{x}^a}\frac{\partial^2 \bar{x}^a}{\partial x^i \partial x^k}+\frac{\partial x^m}{\partial \bar{x}^\alpha}\bar{\Gamma}^a_{bc}\frac{\partial \bar{x}^b}{\partial x^i}\frac{\partial \bar{x}^c}{\partial x^k}\right\}\\ &-g_{im}\left\{\frac{\partial x^m}{\partial \bar{x}^a}\frac{\partial^2 \bar{x}^a}{\partial x^j \partial x^k}+\frac{\partial x^m}{\partial \bar{x}^a}\bar{\Gamma}^a_{bc}\frac{\partial \bar{x}^b}{\partial x^j}\frac{\partial \bar{x}^c}{\partial x^k}\right\}.\end{aligned} \tag{5.7}$$

Since v is a Killing vector field, we have

$$g_{ij}=\bar{g}_{ab}\frac{\partial \bar{x}^a}{\partial x^i}\frac{\partial \bar{x}^b}{\partial x^j}, \tag{5.8}$$

and also we have $\bar{x}^a=\bar{x}^a(x^i)$ and $\bar{y}^a=\dfrac{\partial \bar{x}^a}{\partial x^m}y^m$. Hence we get

$$\begin{aligned}\partial_k g_{ij}=\partial_c \bar{g}_{ab}\frac{\partial \bar{x}^a}{\partial x^i}\frac{\partial \bar{x}^b}{\partial x^j}\frac{\partial \bar{x}^c}{\partial x^k}&+\dot{\partial}_d \bar{g}_{ab}\frac{\partial \bar{x}^a}{\partial x^i}\frac{\partial \bar{x}^b}{\partial x^j}\frac{\partial^2 \bar{x}^d}{\partial x^m \partial x^k}y^m\\ &+\bar{g}_{ab}\frac{\partial^2 \bar{x}^a}{\partial x^k \partial x^i}\frac{\partial \bar{x}^b}{\partial x^j}+\bar{g}_{ab}\frac{\partial \bar{x}^a}{\partial x^i}\frac{\partial^2 \bar{x}^b}{\partial x^k \partial x^j},\end{aligned}$$

$$\dot{\partial}_m g_{ij}=\dot{\partial}_c \bar{g}_{ab}\frac{\partial \bar{x}^c}{\partial x^m}\frac{\partial \bar{x}^a}{\partial x^i}\frac{\partial \bar{x}^b}{\partial x^j}.$$

Substituting these into the right hand side of (5.7), we obtain

$$\tilde{\nabla}_k g_{ij}=\frac{\partial \bar{x}^a}{\partial x^i}\frac{\partial \bar{x}^b}{\partial x^j}\frac{\partial \bar{x}^c}{\partial x^k}\{\partial_c \bar{g}_{ab}-\dot{\partial}_d \bar{g}_{ab}\bar{\Gamma}^d_{ec}\bar{y}^e-\bar{g}_{db}\bar{\Gamma}^d_{ac}-g_{ad}\bar{\Gamma}^d_{bc}\}=0.$$

Therefore, from Theorem 2.2, we see that the linear connection $\tilde{\Gamma}$ is a G-connection.
Q. E. D.

Here let us assume that $\Gamma(x)$ is a symmetric G-connection. Then, $\Phi_t(\Gamma)$ also becomes a symmetric G-connection. On the other hand, if $\Gamma(x)$ is a symmetric G-

connection, the relation $\nabla_k g_{ij}=0$ and $\Gamma^i_{jk}=\Gamma^i_{kj}$ hold good. These relations show us that Γ^i_{jk} coincides with the Cartan's connection $\overset{*}{\Gamma}{}^i_{jk}$ ([5], [13]). Since Γ^i_{jk} are functions of position only, the given $\{V, G\}$-manifold is nothing but a Berwald space. Moreover the relation $\Phi_t(\overset{*}{\Gamma})=\overset{*}{\Gamma}$ holds good. Consequently φ_t is an affine mapping with respect to Γ^*. Thus

Theorem 5.2. *In a Berwald space, a Killing vector field is, at the same time, an affine Killing vector field with respect to the Cartan's connection.*

Remark. More generally, it has been proved (see, for example, [7], [16]) that *a homothetic Killing vector field in a Finsler space is an affine Killing one with respect to the Cartan's connection.*

§ 6. Killing and affine Killing vector fields

Let $\Gamma(x)$ be a G-connection on a $\{V, G\}$-manifold M. We shall consider a condition for a Killing vector field to be an affine Killing vector field with respect to $\Gamma(x)$.

Let $T(M)$ be the tangent bundle over M. Let $g_{ij}(x, y)$ be the metric tensor of the given $\{V, G\}$-Finsler metric. Let us put

$$\Gamma^i_j=\Gamma^i_{mj}(x)y^m, \tag{6.1}$$

then Γ^i_j is a non-linear connection and gives $T(M)$ a horizontal distribution. If we put

$$\begin{cases} X_i=\dfrac{\partial}{\partial x^i}-\Gamma^m_i\dfrac{\partial}{\partial y^m}, \\ Y_i=\dfrac{\partial}{\partial y^i}, \end{cases} \tag{6.2}$$

then $\{X_i\}$ form a base of a horizontal distribution on $T(M)$, and $\{Y_i\}$ form a base of the vertical distribution on $T(M)$.

As to a vector field $v=v^i(x)\dfrac{\partial}{\partial x^i}$ on M, we denote by v^h and v^v the horizontal lift and vertical lift of v respectively. In the canonical coordinate system $(z^A)=(x^i, y^i)$, v^h and v^v are, as is well-known, given by $v^h=v^iX_i$ and $v^v=v^iY_i$ and take the form

$$v^h=\begin{pmatrix} v^i \\ -\Gamma^i_m v^m \end{pmatrix}, \qquad v^v=\begin{pmatrix} 0 \\ v^i \end{pmatrix}. \tag{6.3}$$

Moreover the complete lift v^c of v takes the form

(6.4)
$$v^c=\begin{pmatrix} v^i \\ y^m\partial_m v^i \end{pmatrix}.$$

Now, let us introduce an inner product of vectors in $T(M)$ by

(6.5)
$$\begin{aligned} &<X_i, X_j>_{(x,y)}=g_{ij}(x, y), \\ &<X_i, Y_j>_{(x,y)}=0, \\ &<Y_i, Y_j>_{(x,y)}=g_{ij}(x, y). \end{aligned}$$

Then, $T(M)$ becomes a Riemannian space. The metric has been called a lifted metric of the Finsler metric $g_{ij}(x, y)$ with respect to the non-linear connection Γ^i_j ([8], [15], [22]). The components of this Riemannian metric with respect to $(Z^A)=(x^i, y^i)$ takes the form

(6.6)
$$(G_{AB})=\begin{pmatrix} G_{ij}, & G_{i\bar{j}} \\ G_{\bar{i}j}, & G_{\bar{i}\bar{j}} \end{pmatrix}=\begin{pmatrix} g_{ij}+g_{rs}\Gamma^r_i\Gamma^s_j, & \Gamma^m_i g_{mj} \\ g_{im}\Gamma^m_j, & g_{ij} \end{pmatrix},$$

where $\bar{i}=n+i$. For a vector field V on $T(M)$, $\underset{V}{\mathscr{L}}G_{AB}$ is given by

$$\underset{V}{\mathscr{L}}G_{AB}=V^C\frac{\partial G_{AB}}{\partial Z^C}+G_{AC}\frac{\partial V^C}{\partial Z^B}+G_{CB}\frac{\partial V^C}{\partial Z^A}.$$

As to a vector field v on M, calculating the above equation in the case where $V=v^C$, we can verify

(6.7)
$$\begin{cases} \underset{v^C}{\mathscr{L}}G_{\bar{i}\bar{j}}=\underset{v}{\mathscr{L}}g_{ij}, \\ \underset{v^C}{\mathscr{L}}G_{\bar{i}j}=\Gamma^r_j\underset{v}{\mathscr{L}}g_{ir}+g_{ir}y^m\underset{v}{\mathscr{L}}\Gamma^r_{mj}, \\ \underset{v^C}{\mathscr{L}}G_{ij}=\underset{v}{\mathscr{L}}g_{ij}+\Gamma^r_i\Gamma^t_j\underset{v}{\mathscr{L}}g_{rt}+y^m g_{rt}\Gamma^r_i\underset{v}{\mathscr{L}}\Gamma^t_{mj}+y^m g_{rt}\Gamma^t_j\underset{v}{\mathscr{L}}\Gamma^r_{mi}. \end{cases}$$

Therefore, if we assume $\underset{v^C}{\mathscr{L}}G_{AB}=0$, $(6.7)_1$ shows $\underset{v}{\mathscr{L}}g_{ij}=0$ and $(6.7)_2$ shows $y^m\underset{v}{\mathscr{L}}\Gamma^i_{mj}=0$. Since $\underset{v}{\mathscr{L}}\Gamma^i_{mj}$ are functions of x^k only, it follows that $\underset{v}{\mathscr{L}}\Gamma^i_{kj}=0$. Conversely, if $\underset{v}{\mathscr{L}}g_{ij}=0$ and $\underset{v}{\mathscr{L}}\Gamma^h_{ij}=0$ hold, (6.7) implies $\underset{v^C}{\mathscr{L}}G_{AB}=0$. Consequently we obtain

Theorem 6.1. *Let M be a $\{V, G\}$-manifold, g_{ij} be the $\{V, G\}$-Finsler metric tensor, Γ^h_{ij} be a G-connection on M, and $T(M)$ be the tangent bundle over M. Let $\tilde{G}$ be the lifted metric of g_{ij} to $T(M)$ with respect to $\Gamma^i_{mj}y^m$. In order that a vector field v on M be a Killing vector with respect to the $\{V, G\}$-Finsler metric and, at the same time, be an affine Killing one with respect to Γ^h_{ij}, it is necessary and sufficient that the complete lift of v is a Killing vector field on $T(M)$ with respect to $\tilde{G}$.*

Moreover, similar calculation gives us

$$(6.8)\qquad \begin{cases} \underset{v^h}{\mathscr{L}} G_{\bar{i}\bar{j}}=0, \\ \underset{v^h}{\mathscr{L}} G_{\bar{i}j}=v^m y^r g_{ih} R_r{}^h{}_{jm}, \\ \underset{v^h}{\mathscr{L}} G_{ij}=\underset{v}{\mathscr{L}} g_{ij}+y^h g_{rt} \Gamma^r_i v^m R_h{}^t{}_{jm}+y^h g_{rt} \Gamma^r_j v^m R_h{}^t{}_{im}, \end{cases}$$

which implies

Theorem 6.2. *Under the same assumption as in Theorem* 6.1, *a necessary and sufficient condition for the horizontal lift of a vector field v on M to be a Killing vector field on $T(M)$ with respect to $\tilde{G}$ is that v is a Killing vector field with respect to the $\{V, G\}$-metric and satisfies $v^m R_j{}^i{}_{km}=0$.*

Furthermore, after the similar calculation, we find

$$(6.9)\qquad \begin{cases} \underset{v^v}{\mathscr{L}} G_{\bar{i}\bar{j}}=2C_{ijm}v^m, \\ \underset{v^v}{\mathscr{L}} G_{\bar{i}j}=2\Gamma^r_j C_{rim} v^m+g_{im} \nabla_j v^m, \\ \underset{v^v}{\mathscr{L}} G_{ij}=2C_{ijm}v^m+2C_{mrt}v^m \Gamma^r_i \Gamma^t_j+g_{rt}\Gamma^r_i \nabla_j v^t+g_{rt}\Gamma^t_j \nabla_i v^r. \end{cases}$$

Thus we obtain

Theorem 6.3. *Under the same assumption as in Theorem* 6.1, *a necessary and sufficient condition for the vertical lift of a vector field v on M to be a Killing vector field on $T(M)$ with respect to $\tilde{G}$ is that v is a parallel vector field with respect to $\Gamma^i_{jk}(x)$ and satisfies $C_{ijm}v^m=0$.*

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