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An Error Analysis of Galerkin Approximations of Periodic Solution and its Period to Autonomous Differential System

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§1. Introduction

This work is concerned with a numerical analysis of periodic solution and its period to the n-dimensional autonomous differential system

$$\frac{dx}{d\tau}=X(x)\,, \tag{1.1}$$

where $X(x)\in C_x^1[D]$, D being a domain in the x-space. For the numerical computation of ω-periodic solution $x(\tau)$ to (1.1), we transform τ to t by

$$\tau=\frac{\omega t}{2\pi}\,, \tag{1.2}$$

then the equation (1.1) is rewritten in the form

$$\frac{dx}{dt}=\frac{\omega}{2\pi}X(x)\,. \tag{1.3}$$

The problem then is reduced to the one of finding a 2π-periodic solution to (1.3), but in our case ω is also unknown. Hence, we consider the differential system

$$\begin{cases}\dfrac{dx}{dt}=\dfrac{\omega}{2\pi}X(x)\,,\\[2ex] \dfrac{d\omega}{dt}=0,\end{cases} \tag{1.4}$$

where x and ω are unknown functions. The periodic boundary condition for (1.4) is then as follows:

$$x(0)=x(2\pi)\,. \tag{1.5}$$

As is well known, when $x(t)$ is a solution to the autonomous system (1.3), $x(t+\alpha)$ is also a solution for an arbitrary constant α. The fact tells us that no 2π-periodic solution to (1.3) is uniquely determined by the only boundary condition (1.5).

Hence, we will add one more condition, say,

$$l(\boldsymbol{u})=\beta, \tag{1.6}$$

where $\boldsymbol{u}(t)=(\boldsymbol{x}(t), \omega(t))$ and l is a linear functional satisfying the isolatedness condition $l\left[\left(\frac{\omega}{2\pi}\boldsymbol{X}(\boldsymbol{x}), 0\right)\right] \neq 0$ (see [10]) and β is a constant number.

We shall write the set of boundary conditions (1.5) and (1.6) in the following form:

$$\boldsymbol{f}(\boldsymbol{u})=\boldsymbol{0}, \tag{1.7}$$

where

$$\boldsymbol{f}(\boldsymbol{u})=\mathrm{col}\,[\boldsymbol{x}(0)-\boldsymbol{x}(2\pi),\ l(\boldsymbol{u})-\beta]. \tag{1.8}$$

Then the boundary value problem (1.4)–(1.6) can be rewritten as follows:

$$\begin{cases} \dfrac{d\boldsymbol{u}}{dt}=\boldsymbol{V}(\boldsymbol{u}), \\ \boldsymbol{f}(\boldsymbol{u})=\boldsymbol{0}, \end{cases} \tag{1.9}$$

where $\boldsymbol{V}(\boldsymbol{u})=\left(\frac{\omega}{2\pi}\boldsymbol{X}(\boldsymbol{x}), 0\right)$.

In the next section, we will establish an existence theorem of the boundary value problem (1.9) and propose a novel method of calculating the error bounds on the approximate solutions of periodic solution and its period, respectively.

In the last section, we have first tested our method on numerical analysis of periodic solution and its period to a nonlinear differential equation whose exact solution is known. The results are shown in Table 1. Secondary, we have given the Galerkin approximations and their error bounds of periodic solutions and their periods to van der Pol equations with damping coefficients $\lambda=3, 4, 5$. Numerical results are shown in Tables 2∼4. The present error bound of the Galerkin approximation of period to van der Pol equation with $\lambda=3$ is better than the one given in the previous paper [12]. Note that we also make use of the least square technique to compute the Galerkin approximations of periodic solutions and their periods to van der Pol equations with $\lambda=4, 5$.

Finally, we summarize the various results in Table 5. The table shows the usefulness of our results.

§2. Basic Theorems

Let D be a domain in the $\boldsymbol{x}$-space. Consider the product spaces $B=D\times\boldsymbol{R}$ and $\Omega=I\times B$, where $I=[0, 2\pi]$ and $\boldsymbol{R}$ is the real space.

Put

$$C^1[I]=\{x(t)=\mathrm{col}\,[x_1(t),\dots,x_n(t)];\ x_i(t)\ (i=1,\dots,n)\text{ are }C^1\text{-class on }I\},$$

$$C[I]=\{x(t)=\mathrm{col}\,[x_1(t),\dots,x_n(t)];\ x_i(t)\ (i=1,\dots,n)\text{ are }C^0\text{-class on }I\},$$

$$S=\{u(t)=(x(t),\omega);\ (t,u(t))\in\Omega\ \text{ for all }\ t\in I,\ u(t)\in C^1[I]\times\boldsymbol{R}\}$$

and

$$S'=\{u(t)=(x(t),\omega);\ (t,u(t))\in\Omega\ \text{ for all }\ t\in I,\ u(t)\in C[I]\times\boldsymbol{R}\}.$$

We shall denote the Euclidean norm in $\boldsymbol{R}^n$ by $\|\cdot\|_n$ and define the norms in the product spaces $C[I]\times\boldsymbol{R}$ and $N\equiv C[I]\times\boldsymbol{R}^{n+1}$ by the formulas

$$\|u(t)\|_\infty=\|x(t)\|_c+|\omega| \tag{2.1}$$

and

$$|||n|||=\|x(t)\|_c+\|v\|_{n+1}, \tag{2.2}$$

respectively. Then the product spaces $C[I]\times\boldsymbol{R}$ and N are evidently Banach spaces with respect to the above norms, respectively.

Now put $M\equiv C^1[I]\times\boldsymbol{R}$, then the boundary value problem (1.9) is reduced to the one of finding $u\in M$ satisfying the equation

$$F(u)=\left[\frac{dx}{dt}-\frac{\omega}{2\pi}X(x),f(u)\right]=0. \tag{2.3}$$

In (2.3), we assume that the function $F(u)$ with domain $S\subset M$ and range N is continuously weak Fréchet differentiable.

Now we consider also a linear operator T mapping M into N of the following form

$$Th=\left[\frac{dh_1}{dt}-A(t)\begin{pmatrix}h_1(t)\\ h_{n+1}\end{pmatrix},\ \tilde{L}h\right]\qquad\text{for every } h(t)=(h_1(t),h_{n+1})\in M, \tag{2.4}$$

where $A(t)$ is an $n\times(n+1)$ matrix continuous on I and $\tilde{L}$ is a linear operator mapping $C^1[I]\times\boldsymbol{R}$ into $\boldsymbol{R}^{n+1}$.

By $\Psi(t)$ let us denote the fundamental matrix of the linear homogeneous system

$$\frac{dz}{dt}=\begin{pmatrix}A(t)\\ 0\cdots0\end{pmatrix}z \tag{2.5}$$

with the initial condition $\Psi(0)=E$ (unit matrix) and by $L[\Psi(t)]$ we denote the matrix whose column vectors are $\tilde{L}[\tilde{\phi}_i(t)]$ $(i=1,2,\dots,n+1)$, where we put $\tilde{\phi}_i=(\phi_{i1},\psi_{in+1})={}^t\phi_i$ and where $\phi_i=\mathrm{col}\,[\phi_{i1},\psi_{in+1}]$ are column vectors of the matrix

$\Psi(t)$.

Then we have the following theorem.

Theorem 1 (Urabe [5]).

If the $(n+1)\times(n+1)$ *matrix* $G\equiv L[\Psi(t)]$ *is non-singular, namely,*

$$\det G=\det[\tilde{L}[\tilde{\boldsymbol{\phi}}_1],\dots,\tilde{L}[\tilde{\boldsymbol{\phi}}_{n+1}])\neq 0, \tag{2.6}$$

then the operator T *defined by* (2.4) *has a linear inverse operator* T^{-1}. *That is, for any* $\boldsymbol{n}=(\boldsymbol{\varphi}(t),\ \boldsymbol{v})\in N$ *there exists one and only one solution* $\boldsymbol{h}=(\boldsymbol{h}_1(t),\ h_{n+1})\in M$ *satisfying the equation* $T\boldsymbol{h}=\boldsymbol{n}$. *The solution* $\boldsymbol{h}$ *can be written as follows*:

$$\operatorname{col}[\boldsymbol{h}_1(t),\ h_{n+1}]=H_1\boldsymbol{\varphi}+H_2\boldsymbol{v}, \tag{2.7}$$

where

$$\begin{cases} H_1\boldsymbol{\varphi}=\Psi(t)\displaystyle\int_0^t\Psi^{-1}(s)\boldsymbol{\phi}(s)ds-\Psi(t)G^{-1}L\left[\Psi(t)\displaystyle\int_0^t\Psi^{-1}(s)\boldsymbol{\phi}(s)ds\right], \\ H_2\boldsymbol{v}=\Psi(t)G^{-1}\boldsymbol{v} \end{cases} \tag{2.8}$$

and

$$\boldsymbol{\phi}(s)=\operatorname{col}[\boldsymbol{\varphi}(s),\ 0].$$

Theorem 1 tells us that we can define two operators T_1^{-1} with domain N and range $C^1[I]$ and T_2^{-1} with domain N and range $\boldsymbol{R}$ by

$$T_1^{-1}\boldsymbol{n}=\boldsymbol{h}_1(t),\quad T_2^{-1}\boldsymbol{n}=h_{n+1}, \tag{2.9}$$

where $\boldsymbol{n}=(\boldsymbol{\varphi}(t),\ \boldsymbol{v})\in N$. Moreover, we have the equality

$$T^{-1}\boldsymbol{n}=(T_1^{-1}\boldsymbol{n},\ T_2^{-1}\boldsymbol{n}). \tag{2.10}$$

Let us write

$$H_1\boldsymbol{\varphi}=\begin{pmatrix} H_{11}\boldsymbol{\varphi} \\ H_{12}\boldsymbol{\varphi}\end{pmatrix},\quad H_2\boldsymbol{v}=\begin{pmatrix} H_{21}\boldsymbol{v} \\ H_{22}\boldsymbol{v}\end{pmatrix} \tag{2.11}$$

in Theorem 1, where $H_{11}\boldsymbol{\varphi}$ and $H_{21}\boldsymbol{v}$ are n-dimensional vectors and $H_{12}\boldsymbol{\varphi}$ and $H_{22}\boldsymbol{v}$ are real numbers. Then we have the inequalities

$$\begin{aligned} &\|T_1^{-1}\|_c\leq\max(\|H_{11}\|_c,\ \|H_{21}\|_c), \\ &|T_2^{-1}|\leq\max(|H_{12}|,\ |H_{22}|), \end{aligned} \tag{2.12}$$

where $\|\cdot\|_c$ and $|\cdot|$ denote the induced operator norms.

Therefore T_1^{-1} and T_2^{-1} are bounded linear operators. From the fact and the relation

$$\|T^{-1}(\varphi, \boldsymbol{v})\|_\infty = \|T_1^{-1}(\varphi, \boldsymbol{v})\|_c + |T_2^{-1}(\varphi, \boldsymbol{v})|,$$

we have the inequality

$$\|T^{-1}\|_\infty \leq \|T_1^{-1}\|_c + |T_2^{-1}|, \tag{2.13}$$

where $\|\cdot\|_\infty$ denotes the induced operator norm.

The weak Fréchet differential of $\boldsymbol{F}(\boldsymbol{u})$ at $\boldsymbol{u}=\bar{\boldsymbol{u}}(t)=(\bar{\boldsymbol{x}}(t), \bar{\omega})$ can be written as follows:

$$\boldsymbol{F}'(\bar{\boldsymbol{u}})\boldsymbol{h} = \left[\frac{d\boldsymbol{h}_1}{dt} - \boldsymbol{X}_u(\bar{\boldsymbol{u}})\begin{pmatrix}\boldsymbol{h}_1\\ h_{n+1}\end{pmatrix}, \boldsymbol{f}'(\bar{\boldsymbol{u}})\boldsymbol{h}\right], \tag{2.14}$$

where $\boldsymbol{X}_u(\bar{\boldsymbol{u}}) = \left(\frac{\bar{\omega}}{2\pi}\boldsymbol{X}_x(\bar{\boldsymbol{x}})\ \frac{1}{2\pi}\boldsymbol{X}(\bar{\boldsymbol{x}})\right)$ is an $n\times(n+1)$ matrix, $\boldsymbol{X}_x(\bar{\boldsymbol{x}})$ denotes the Jacobian matrix at $\boldsymbol{x}=\bar{\boldsymbol{x}}$ and

$$\boldsymbol{f}'(\bar{\boldsymbol{u}})\boldsymbol{h} = \mathrm{col}\,[\boldsymbol{h}_1(0) - \boldsymbol{h}_1(2\pi), l(\boldsymbol{h})].$$

In (2.4) we take $A(t)$ and $\tilde{L}\boldsymbol{h}$ such that

$$A(t) = \boldsymbol{X}_u(\bar{\boldsymbol{u}}(t)),$$

$$\tilde{L}\boldsymbol{h} = \boldsymbol{f}'(\bar{\boldsymbol{u}}(t))\boldsymbol{h}.$$

Then we have $T=\boldsymbol{F}'(\bar{\boldsymbol{u}})$ and the following theorem.

Theorem 2.

Assume that the boundary value problem (2.3) *possesses an approximate solution* $\boldsymbol{u}=\bar{\boldsymbol{u}}(t)$ *in* S *such that* $\det G = \det \boldsymbol{f}'(\bar{\boldsymbol{u}})[\Psi(t)] \neq 0$, *where* $\Psi(t)$ *is the fundamental matrix of the linear homogeneous differential system*

$$\frac{d\boldsymbol{z}}{dt} = \begin{pmatrix}\boldsymbol{X}_u(\bar{\boldsymbol{u}})\\ 0\ \cdots\ 0\end{pmatrix}\boldsymbol{z}$$

satisfying the initial condition $\Psi(0)=E$ *(unit matrix).*

Let μ_1, μ_2 *and* r *be the positive numbers such that*

$$\mu_1 = \max(\|H_{11}\|_c, \|H_{21}\|_c),\ \mu_2 = \max(|H_{12}|, |H_{22}|), \tag{2.15}$$

$$r \geq |||\boldsymbol{F}(\bar{\boldsymbol{u}})||| = \left\|\frac{d\bar{\boldsymbol{x}}}{dt} - \frac{\bar{\omega}}{2\pi}\boldsymbol{X}(\bar{\boldsymbol{x}})\right\|_c + \|\boldsymbol{f}(\bar{\boldsymbol{u}})\|_{n+1}. \tag{2.16}$$

If there exist the positive numbers δ_1, δ_2 *and a non-negative number* $\kappa<1$ *such that*

$$D'_\delta \equiv \{\boldsymbol{u}(t);\ \|\boldsymbol{x}(t) - \bar{\boldsymbol{x}}(t)\|_c \leq \delta_1,\ |\omega - \bar{\omega}| \leq \delta_2,\ \boldsymbol{u}(t) \in C[I]\times\boldsymbol{R}\} \subset S', \tag{2.17}$$

$$\|\boldsymbol{X}_u(\boldsymbol{u}) - \boldsymbol{X}_u(\bar{\boldsymbol{u}})\|_c + \|\boldsymbol{f}'(\boldsymbol{u}) - \boldsymbol{f}'(\bar{\boldsymbol{u}})\|_{n+1} \leq \frac{\kappa}{\mu_1+\mu_2} \quad \textit{on } D'_\delta, \tag{2.18}$$

(2.19) $$\frac{\mu_1 r}{1-\kappa} \leq \delta_1, \quad \frac{\mu_2 r}{1-\kappa} \leq \delta_2,$$

then the boundary value problem (2.3) *has one and only one* 2π*-periodic solution* $\boldsymbol{u}=\hat{\boldsymbol{u}}(t)$ in

$$D_\delta \equiv \{\boldsymbol{u}(t)\,;\ \|\boldsymbol{x}(t)-\bar{\boldsymbol{x}}(t)\|_c \leq \delta_1,\ |\omega-\bar{\omega}| \leq \delta_2,\ \boldsymbol{u}(t) \in M\}$$

and for this solution $\hat{\boldsymbol{u}}(t)$ *we have*

$$\|\hat{\boldsymbol{x}}(t)-\bar{\boldsymbol{x}}(t)\|_c \leq \frac{\mu_1 r}{1-\kappa}, \quad |\hat{\omega}-\bar{\omega}| \leq \frac{\mu_2 r}{1-\kappa}. \tag{2.20}$$

PROOF. Since $T=\boldsymbol{F}'(\bar{\boldsymbol{u}})$, we have from (2.4), (2.12), (2.13), (2.14), (2.15) and (2.18) that

$$\text{(i)} \quad \begin{cases} \|T_1^{-1}\|_c \leq \mu_1 \\ |T_2^{-1}| \leq \mu_2 \\ \|T^{-1}\|_\infty \leq \|T_1^{-1}\|_c + |T_2^{-1}| \leq \mu_1+\mu_2 \\ |||\boldsymbol{F}'(\boldsymbol{u})-T||| \leq \dfrac{\kappa}{\mu_1+\mu_2} \qquad \text{on } D'_\delta. \end{cases}$$

Now let us show that the iterative process

$$\boldsymbol{u}_{p+1}=\boldsymbol{u}_p-T^{-1}\boldsymbol{F}(\boldsymbol{u}_p) \qquad (p=0,\,1,\,2,\dots;\ \boldsymbol{u}_0=\bar{\boldsymbol{u}}(t)) \tag{2.21}$$

can be practised indefinitely in D_δ. For this purpose we shall prove by the induction that

$$\text{(ii)} \quad \begin{cases} \|\boldsymbol{u}_{p+1}-\boldsymbol{u}_p\|_\infty \leq \kappa^p \|\boldsymbol{u}_1-\boldsymbol{u}_0\|_\infty \\ \|\boldsymbol{x}_{p+1}-\boldsymbol{x}_p\|_c \leq \kappa^p \mu_1 r \\ |\omega_{p+1}-\omega_p| \leq \kappa^p \mu_2 r \end{cases}$$

$$\text{(iii)} \quad \boldsymbol{u}_{p+1} \in D_\delta$$

where $p=0,\,1,\,2,\dots$.

For $p=0$, the inequality $\|\boldsymbol{u}_1-\boldsymbol{u}_0\|_\infty \leq \kappa^0 \|\boldsymbol{u}_1-\boldsymbol{u}_0\|_\infty$ is evident. For $\boldsymbol{u}_1$, we have

$$\boldsymbol{u}_1-\boldsymbol{u}_0=-T^{-1}\boldsymbol{F}(\boldsymbol{u}_0)=(-T_1^{-1}\boldsymbol{F}(\boldsymbol{u}_0),\ -T_2^{-1}\boldsymbol{F}(\boldsymbol{u}_0)).$$

Hence we have the relations

$$\boldsymbol{x}_1-\boldsymbol{x}_0=-T_1^{-1}\boldsymbol{F}(\boldsymbol{u}_0),$$

$$\omega_1-\omega_0=-T_2^{-1}\boldsymbol{F}(\boldsymbol{u}_0),$$

which imply the inequalities

$$\|\boldsymbol{x}_1-\boldsymbol{x}_0\|_c \le \|T_1^{-1}\|_c \|\!|\boldsymbol{F}(\boldsymbol{u}_0)\|\!| \le \mu_1 r \le (1-\kappa)\delta_1 \le \delta_1,$$

$$|\omega_1-\omega_0| \le |T_2^{-1}| \, \|\!|\boldsymbol{F}(\boldsymbol{u}_0)\|\!| \le \mu_2 r \le (1-\kappa)\delta_2 \le \delta_2.$$

The above inequalities prove (ii) and (iii) for $p=0$.

Assume that (ii) and (iii) hold up to $p-1$ $(p\ge 1)$. Then from (2.21) we have

$$(2.22) \qquad \boldsymbol{u}_{p+1}-\boldsymbol{u}_p=\boldsymbol{u}_p-\boldsymbol{u}_{p-1}-T^{-1}(\boldsymbol{F}(\boldsymbol{u}_p)-\boldsymbol{F}(\boldsymbol{u}_{p-1}))$$

$$=T^{-1}\int_0^1 [T-\boldsymbol{F}'(\boldsymbol{u}_{p-1}+\theta(\boldsymbol{u}_p-\boldsymbol{u}_{p-1}))](\boldsymbol{u}_p-\boldsymbol{u}_{p-1})\,d\theta$$

$$=\left(T_1^{-1}\int_0^1 [T-\boldsymbol{F}'(\cdot)](\boldsymbol{u}_p-\boldsymbol{u}_{p-1})\,d\theta,\ T_2^{-1}\int_0^1 [T-\boldsymbol{F}'(\cdot)](\boldsymbol{u}_p-\boldsymbol{u}_{p-1})\,d\theta\right).$$

Here $\boldsymbol{u}_{p-1}+\theta(\boldsymbol{u}_p-\boldsymbol{u}_{p-1})\in D_\delta$ for $0\le\theta\le 1$ since $\boldsymbol{u}_{p-1}, \boldsymbol{u}_p\in D_\delta$ by the assumption. Then by (2.18) we have

$$(2.23) \qquad \|\boldsymbol{u}_{p+1}-\boldsymbol{u}_p\|_\infty \le \|T^{-1}\|_\infty \frac{\kappa}{\mu_1+\mu_2}\|\boldsymbol{u}_p-\boldsymbol{u}_{p-1}\|_\infty \le \kappa\|\boldsymbol{u}_p-\boldsymbol{u}_{p-1}\|_\infty,$$

which proves the first inequality in (ii) for p because

$$\|\boldsymbol{u}_p-\boldsymbol{u}_{p-1}\|_\infty \le \kappa^{p-1}\|\boldsymbol{u}_1-\boldsymbol{u}_0\|_\infty$$

by the assumption. Since $\boldsymbol{u}_{p+1}-\boldsymbol{u}_p=(\boldsymbol{x}_{p+1}-\boldsymbol{x}_p,\ \omega_{p+1}-\omega_p)$, it follows from (2.22) and (i) that

$$\|\boldsymbol{x}_{p+1}-\boldsymbol{x}_p\|_c \le \|T_1^{-1}\|_c \frac{\kappa}{\mu_1+\mu_2}\|\boldsymbol{u}_p-\boldsymbol{u}_{p-1}\|_\infty$$

$$\le \frac{\mu_1\kappa}{\mu_1+\mu_2}\kappa^{p-1}\|\boldsymbol{u}_1-\boldsymbol{u}_0\|_\infty$$

$$\le \frac{\mu_1\kappa^p}{\mu_1+\mu_2}\|T^{-1}\|_\infty \|\!|\boldsymbol{F}(\boldsymbol{u}_0)\|\!|$$

$$\le \kappa^p\mu_1 r$$

and hence we have

$$(2.24) \qquad \|\boldsymbol{x}_{p+1}-\boldsymbol{x}_0\|_c \le \|\boldsymbol{x}_{p+1}-\boldsymbol{x}_p\|_c+\|\boldsymbol{x}_p-\boldsymbol{x}_{p-1}\|_c+\cdots+\|\boldsymbol{x}_1-\boldsymbol{x}_0\|_c$$

$$\le(\kappa^p+\kappa^{p-1}+\cdots+\kappa+1)\mu_1 r$$

$$\le\frac{\mu_1 r}{1-\kappa}\le\delta_1.$$

Similarly we have that

$$
\begin{aligned}
|\omega_{p+1}-\omega_p| &\leq |T_2^{-1}| \frac{\kappa}{\mu_1+\mu_2} \|\boldsymbol{u}_p-\boldsymbol{u}_{p-1}\|_\infty \\
&\leq \frac{\mu_2\kappa}{\mu_1+\mu_2} \kappa^{p-1} \|\boldsymbol{u}_1-\boldsymbol{u}_0\|_\infty \\
&\leq \frac{\mu_2\kappa^p}{\mu_1+\mu_2} \|T^{-1}\|_\infty |||\boldsymbol{F}(\boldsymbol{u}_0)||| \\
&\leq \kappa^p \mu_2 r
\end{aligned}
$$

and

$$
|\omega_{p+1}-\omega_0| \leq \frac{\mu_2 r}{1-\kappa} \leq \delta_2. \tag{2.25}
$$

These complete the induction.

By the iterative process (2.21), we thus have an infinite sequence $\{\boldsymbol{u}_p\}$ in $D_\delta \subset D'_\delta$, which is convergent by (ii) and the completeness of the space $C[I] \times \boldsymbol{R}$. Let

$$
\hat{\boldsymbol{u}} = \lim_{p\to+\infty} \boldsymbol{u}_p.
$$

On the other hand we see that

$$
\begin{aligned}
T\boldsymbol{u}_{p+1} &= T\boldsymbol{u}_p - \boldsymbol{F}(\boldsymbol{u}_p) \\
&= \left[\frac{\omega_p}{2\pi} \boldsymbol{X}(\boldsymbol{x}_p) - \boldsymbol{X}_{\boldsymbol{u}}(\bar{\boldsymbol{u}}) \begin{pmatrix} \boldsymbol{x}_p \\ \omega_p \end{pmatrix}, \boldsymbol{f}'(\bar{\boldsymbol{u}})\boldsymbol{u}_p - \boldsymbol{f}(\boldsymbol{u}_p) \right]
\end{aligned}
$$

which follows

$$
\boldsymbol{u}_{p+1} = T^{-1} \left[\frac{\omega_p}{2\pi} \boldsymbol{X}(\boldsymbol{x}_p) - \boldsymbol{X}_{\boldsymbol{u}}(\bar{\boldsymbol{u}}) \begin{pmatrix} \boldsymbol{x}_p \\ \omega_p \end{pmatrix}, \boldsymbol{f}'(\bar{\boldsymbol{u}})\boldsymbol{u}_p - \boldsymbol{f}(\boldsymbol{u}_p) \right].
$$

Letting $p \to +\infty$, we have

$$
\hat{\boldsymbol{u}} = T^{-1} \left[\frac{\hat{\omega}}{2\pi} \boldsymbol{X}(\hat{\boldsymbol{x}}) - \boldsymbol{X}_{\boldsymbol{u}}(\bar{\boldsymbol{u}}) \begin{pmatrix} \hat{\boldsymbol{x}} \\ \hat{\omega} \end{pmatrix}, \boldsymbol{f}'(\bar{\boldsymbol{u}})\hat{\boldsymbol{u}} - \boldsymbol{f}(\hat{\boldsymbol{u}}) \right].
$$

Hence we have $\hat{\boldsymbol{u}} \in M$, which implies that the limit $\hat{\boldsymbol{u}}$ is a solution of (2.3) lying in D_δ.

Inequality (2.20) readily follows from (2.24) and (2.25). Finally, we shall prove the uniqueness of the solution in D_δ. Let $\boldsymbol{u}'$ be an arbitrary solution of (2.3) lying in D_δ. Then we have

$$
\boldsymbol{u}' = \boldsymbol{u}' - T^{-1}\boldsymbol{F}(\boldsymbol{u}').
$$

Since $\hat{\boldsymbol{u}}$ satisfies

$$
\hat{\boldsymbol{u}} = \hat{\boldsymbol{u}} - T^{-1}\boldsymbol{F}(\hat{\boldsymbol{u}}),
$$

we have

$$\|\boldsymbol{u}'-\hat{\boldsymbol{u}}\|_\infty \leq \kappa\|\boldsymbol{u}'-\hat{\boldsymbol{u}}\|_\infty \tag{2.26}$$

in analogy to (2.23). Since $0\leq\kappa<1$, (2.26) clearly implies

$$\|\boldsymbol{u}'-\hat{\boldsymbol{u}}\|_\infty=0,$$

which proves the uniqueness of the solution in D_δ.

§3. Numerical Examples

In order to get a 2π-periodic approximate solution $\bar{\boldsymbol{u}}(t)$ in Theorem 2, let us consider a trigonometric polynomial of the form

$$\boldsymbol{x}_m(t)=\boldsymbol{a}_0+\sum_{n=1}^{m}(\boldsymbol{a}_{2n-1}\sin nt+\boldsymbol{a}_{2n}\cos nt) \tag{3.1}$$

and put

$$\boldsymbol{u}_m(t)=(\boldsymbol{x}_m(t),\ \omega)\,,$$

where $\boldsymbol{a}_i$ $(i=0, 1,\dots, 2m)$ are n-dimensional vectors and ω is a real number. Then, by Galerkin's method, we shall determine the unknown coefficients $\boldsymbol{a}_0$, $\boldsymbol{a}_1$, $\boldsymbol{a}_2,\dots$, $\boldsymbol{a}_{2m-1}$, $\boldsymbol{a}_{2m}$ and ω so that

$$\frac{d\boldsymbol{x}_m}{dt}=P_m\boldsymbol{X}[\boldsymbol{u}_m(t)] \tag{3.2}$$

and

$$\boldsymbol{f}(\boldsymbol{u}_m(t))=\boldsymbol{0} \tag{3.3}$$

may be valid, where P_m denotes a truncation of the Fourier series of the 2π-periodic operand function discarding all harmonic terms of the order higher than m and $\boldsymbol{X}[\boldsymbol{u}_m(t)]=\frac{\omega}{2\pi}\boldsymbol{X}(\boldsymbol{x}_m(t))$.

The equalities (3.2) and (3.3) are clearly equivalent to the system of $n(2m+1)+n+1$ equalities

$$\begin{cases} \boldsymbol{F}_0(\boldsymbol{a})\equiv\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\boldsymbol{X}[\boldsymbol{u}_m(s)]\,ds=\boldsymbol{0}, \\ \boldsymbol{F}_{2n-1}(\boldsymbol{a})\equiv\dfrac{1}{\pi}\displaystyle\int_0^{2\pi}\boldsymbol{X}[\boldsymbol{u}_m(s)]\sin ns\ ds+n\boldsymbol{a}_{2n}=\boldsymbol{0}, \\ \boldsymbol{F}_{2n}(\boldsymbol{a})\equiv\dfrac{1}{\pi}\displaystyle\int_0^{2\pi}\boldsymbol{X}[\boldsymbol{u}_m(s)]\cos ns\ ds-n\boldsymbol{a}_{2n-1}=\boldsymbol{0}, \\ \boldsymbol{F}_f(\boldsymbol{a})\equiv\boldsymbol{f}[\boldsymbol{u}_m(t)]=\boldsymbol{0} \qquad (n=1, 2,\dots, m), \end{cases} \tag{3.4}$$

where $\boldsymbol{a}=\text{col}\,[\boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{a}_2,\dots, \boldsymbol{a}_{2m-1}, \boldsymbol{a}_{2m}, \omega]$ is a $(n(2m+1)+1)$-dimensional vector. In (3.4), the system $\boldsymbol{F}_f(\boldsymbol{a})=\boldsymbol{0}$ essentially consists of an equation. Hence the determining equation (3.4) of Galerkin approximation can be solved by Newton method.

Example 1. In order to test our method, we first consider the equation

$$\frac{d^2x}{d\tau^2}-\lambda\left(1-\alpha x^2-\beta\left(\frac{dx}{d\tau}\right)^2\right)\frac{dx}{d\tau}+x=0 \tag{3.5}$$

with $\lambda=1$, $\alpha=1$, $\beta=1$ given by Shinohara Y..

By the transformation $\tau=\frac{\omega t}{2\pi}$, the equation (3.5) is rewritten in the form

$$\frac{d^2x}{dt^2}-\frac{\omega}{2\pi}\lambda(1-\alpha x^2)\frac{dx}{dt}+\beta\lambda\frac{2\pi}{\omega}\left(\frac{dx}{dt}\right)^3+\left(\frac{\omega}{2\pi}\right)^2x=0. \tag{3.6}$$

Hence the problem is to find a solution of the system

$$\begin{cases} \dfrac{dx}{dt}=y, \\ \dfrac{dy}{dt}=-\left(\dfrac{\omega}{2\pi}\right)^2x+\dfrac{\omega}{2\pi}\lambda(1-\alpha x^2)y-\beta\lambda\dfrac{2\pi}{\omega}y^3 \end{cases} \tag{3.7}$$

with the periodic boundary conditions

$$\begin{cases} x(0)-x(2\pi)=0, \\ y(0)-y(2\pi)=0 \end{cases} \tag{3.8}$$

and the additional condition

$$l(\boldsymbol{u})=\frac{1}{\pi}\int_0^{2\pi}x(t)\cos t\,dt=\frac{\sqrt{2}}{2}, \tag{3.9}$$

where $\boldsymbol{u}=(\boldsymbol{x}(t), \omega)$, $\boldsymbol{x}(t)=\text{col}\,[x(t), y(t)]$.

Note that the linear functional $l(\boldsymbol{u})$ satisfies the isolatedness condition $l\left[\left(\frac{\hat{\omega}}{2\pi}\boldsymbol{X}(\hat{\boldsymbol{x}}), 0\right)\right]\neq 0$, since $\hat{x}(t)=\frac{\sqrt{2}}{2}\sin t+\frac{\sqrt{2}}{2}\cos t$ is the exact solution of our boundary value problem (3.7)–(3.9).

By Galerkin's method proposed in the previous paper [12], we computed the Galerkin approximation to the above boundary value problem and gave a posteriori error estimation. Numerical results are shown in Table 1. Since $\hat{x}(t)=\frac{\sqrt{2}}{2}\sin t+\frac{\sqrt{2}}{2}\cos t$ is the exact solution and $\hat{\omega}=2\pi$ is the exact period, our error estimations

$$\|\hat{\boldsymbol{x}}(t)-\boldsymbol{x}_m(t)\|_c=\sup_{t\in I}[(\hat{x}(t)-x_m(t))^2+(\hat{y}(t)-y_m(t))^2]^{\frac{1}{2}}\leq\frac{\mu_1 r}{1-\kappa}=0.21\times10^{-12}$$

and

$$|\hat{\omega}-\bar{\omega}| \leq \frac{\mu_2 r}{1-\kappa}=0.351\times 10^{-13}$$

are very good, because of $\hat{\omega}-\bar{\omega}=0.752\cdots\times 10^{-14}$.

Example 2. As the second example we shall compute the periodic solution and its period to the equation (3.5) with $\alpha=1$, $\beta=0$. In the previous paper [12], Y. Shinohara and the author have given the Galerkin approximations and their error bounds of periodic solutions and their periods to van der Pol equations with $\lambda=1, 2, 3$. But the error bound for the approximation of period of periodic solution to van der Pol equation with $\lambda=3$ was not satisfactory. In the paper this error bound has been improved and given in Table 2.

As for the cases with $\lambda=4, 5$, the order of Galerkin approximations of periodic solutions becomes too high and it is difficult to solve the determining equation (3.4) only by Newton method. Hence we solve the determining equation of Galerkin approximation of some order $2m_0$ by Newton method and then we will also make use of the least square technique to determine the remainder unknown coefficients $a_{2m_0+1}, a_{2m_0+2},\dots, a_{2m_0+2l-1}, a_{2m_0+2l}$. Put $m'=m_0+l$ and

$$x_{m'}(t)=x_{m_0}(t)+\sum_{n=m_0+1}^{m'}\{a_{2n-1}\sin(2n-1)t+a_{2n}\cos(2n-1)t\},$$

then define the function

$$F(a_{2m_0+1}, a_{2m_0+2},\dots, a_{2m'-1}, a_{2m'})$$
$$=\frac{1}{2}\int_0^{2\pi}\left[\frac{d^2x_{m'}}{dt^2}-\frac{\omega_{m_0}}{2\pi}\lambda(1-x_{m'}^2)\frac{dx_{m'}}{dt}+\left(\frac{\omega_{m_0}}{2\pi}\right)^2 x_{m'}\right]^2 dt.$$

In order to minimize the function F, we solve by Newton method the following equations

$$\frac{\partial F}{\partial a_i}(a_{2m_0+1}, a_{2m_0+2},\dots, a_{2m'-1}, a_{2m'})=0 \qquad (i=2m_0+1,\dots, 2m').$$

Numerical results are shown in Tables 3 and 4.

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Table 1

$$\frac{d^2x}{dt^2}-\frac{\omega}{2\pi}\lambda(1-\alpha x^2)\frac{dx}{dt}+\beta\lambda\frac{2\pi}{\omega}\left(\frac{dx}{dt}\right)^3+\left(\frac{\omega}{2\pi}\right)^2x=0,$$

$$\lambda=1,\ \alpha=1,\ \beta=1,\ x_m(t)=a_0+\sum_{n=1}^{m}(a_{2n-1}\sin(nt)+a_{2n}\cos(nt)),$$

$$y_m(t)=\dot{x}_m(t)=a'_0+\sum_{n=1}^{m}(a'_{2n-1}\sin(nt)+a'_{2n}\cos(nt)),\ m=10.$$

$\bar{\omega}=6.28318\ 53071\ 79594$, $\det G=0.6869$, $r=0.5\times10^{-14}$, $\mu_1=40.0$, $\mu_2=7.0$,

$\kappa=10^{-10}$, $\delta_1=0.22\times10^{-12}$, $\delta_2=0.36\times10^{-13}$, $|\hat{\omega}-\bar{\omega}|\le\frac{\mu_2 r}{1-\kappa}=0.351\times10^{-13}$.

n	a_n	a'_n
0	0.00000 00000 00000	0.0
1	0.70710 67811 86549	−0.70710 67811 86548
2	0.70710 67811 86548	0.70710 67811 86549
3	−0.00000 00000 00000	−0.00000 00000 00000
4	0.00000 00000 00000	−0.00000 00000 00000
5	0.00000 00000 00000	−0.00000 00000 00000
6	0.00000 00000 00000	0.00000 00000 00000
7	−0.00000 00000 00000	0.00000 00000 00000
8	−0.00000 00000 00000	−0.00000 00000 00000
9	0.00000 00000 00000	−0.00000 00000 00000
10	0.00000 00000 00000	0.00000 00000 00000
11	−0.00000 00000 00000	0.00000 00000 00000
12	−0.00000 00000 00000	−0.00000 00000 00000
13	−0.00000 00000 00000	0.00000 00000 00000
14	−0.00000 00000 00000	−0.00000 00000 00000
15	−0.00000 00000 00000	−0.00000 00000 00000
16	0.00000 00000 00000	−0.00000 00000 00000
17	0.00000 00000 00000	0.00000 00000 00000
18	−0.00000 00000 00000	0.00000 00000 00000
19	−0.00000 00000 00000	−0.00000 00000 00000
20	0.00000 00000 00000	−0.00000 00000 00000

Table 2

$\lambda=3$, $\alpha=1$, $\beta=0$, $\bar{\omega}=8.85909\ 54997\ 19845$, $r=0.6\times10^{-10}$, $\mu_1=130.0$, $\mu_2=20.0$,

$\kappa=10^{-4}$, $\delta_1=0.80\times10^{-8}$, $\delta_2=0.13\times10^{-8}$, $|\hat{\omega}-\bar{\omega}|\le0.121\times10^{-8}$.

For the values of a_n and a', see Table 3 in the paper [12].

Table 3

$\lambda=4$, $\alpha=1$, $\beta=0$. Note that $x(t+\pi)=-x(t)$, $\dot{x}(t+\pi)=-\dot{x}(t)$.

$$x_m(t)=\sum_{n=1}^{m}(a_{2n-1}\sin(2n-1)t+a_{2n}\cos(2n-1)t),$$

$$y_m(t)=\dot{x}_m(t)=\sum_{n=1}^{m}(a'_{2n-1}\sin(2n-1)t+a'_{2n}\cos(2n-1)t),\ m=120.$$

$\bar{\omega}=10.20352\ 36909\ 9344$, $\det G=-1.2304$, $r=0.7\times10^{-10}$, $\mu_1=310.0$, $\mu_2=30.0$,

$\kappa=1.3\times10^{-3}$, $\delta_1=0.22\times10^{-7}$, $\delta_2=0.22\times10^{-8}$, $|\hat{\omega}-\bar{\omega}|\le\frac{\mu_2 r}{1-\kappa}=0.211\times10^{-8}$.

n	a_n	a'_n
1	−1.99806 98043 75547	0.625
2	−0.625	−1.99806 98043 75547
3	−0.53370 02957 60846	0.42708 17392 00050

Table 3 Continued.

n	a_n	a'_n
4	−0.14236 05797 33350	−1.60110 08872 82536
5	−0.26741 15522 02731	0.30907 95183 26469
6	−0.06181 59036 65294	−1.33705 77610 13656
7	−0.15920 94151 68156	0.19849 29304 90084
8	−0.02835 61329 27155	−1.11446 59061 77095
9	−0.10220 45239 81799	0.10239 63035 47275
10	−0.01137 73670 60808	−0.91984 07158 36189
11	−0.06812 51683 43828	0.02607 96720 42273
12	−0.00237 08792 76570	−0.74937 68517 82112
13	−0.04628 63891 01305	−0.02951 28880 77013
14	0.00227 02221 59770	−0.60172 30583 16968
15	−0.03172 72611 39261	−0.06627 91476 80572
16	0.00441 86098 45371	−0.47590 89170 88911
17	−0.02180 24825 52538	−0.08748 66806 55238
18	0.00514 62753 32661	−0.37064 22033 93145
19	−0.01495 66336 72738	−0.09676 20527 80150
20	0.00509 27396 20008	−0.28417 60397 82021
21	−0.01021 02937 77396	−0.09750 12525 06719
22	0.00464 29167 86034	−0.21441 61693 25317
23	−0.00691 73870 92511	−0.09258 56879 29621
24	0.00402 54646 92592	−0.15909 99031 27751
25	−0.00463 86055 31361	−0.08429 38151 09146
26	0.00337 17526 04366	−0.11596 51382 84020
27	−0.00306 95468 63476	−0.07432 42554 34215
28	0.00275 27502 01267	−0.08287 77653 13849
29	−0.00199 69692 08657	−0.06387 31692 60139
30	0.00220 25230 77936	−0.05791 21070 51054
31	−0.00127 06722 41839	−0.05373 04102 16011
32	0.00173 32390 39226	−0.03939 08394 97004
33	−0.00078 46848 33786	−0.04437 44813 91962
34	0.00134 46812 54302	−0.02589 45995 14939
35	−0.00046 43230 83205	−0.03605 64400 87862
36	0.00103 01840 02510	−0.01625 13079 12161
37	−0.00025 71212 88823	−0.02886 90455 14958
38	0.00078 02444 73377	−0.00951 34876 86469
39	−0.00012 64037 80740	−0.02280 08931 88666
40	0.00058 46382 86889	−0.00492 97474 48860
41	−0.00004 66993 64219	−0.01777 70124 20683
42	0.00043 35856 68797	−0.00191 46739 32995
43	−0.00000 04610 74681	−0.01368 81047 86530
44	0.00031 83280 18291	−0.00001 98262 11295
45	0.00002 42806 92395	−0.01041 07220 22505
46	0.00023 13493 78278	0.00109 26311 57764
47	0.00003 55942 16913	−0.00782 05036 98077
48	0.00016 63936 95704	0.00167 29281 94920
49	0.00003 88393 75390	−0.00580 02873 88514
50	0.00011 83732 12010	0.00190 31293 94105
51	0.00003 75127 31364	−0.00424 45632 23819
52	0.00008 32267 29879	0.00191 31492 99554
53	0.00003 38426 97265	−0.00306 14214 81521

Table 3 Continued.

n	a_n	a'_n
54	0.00005 77626 69463	0.00179 36629 55033
55	0.00002 92057 21491	−0.00217 28598 34098
56	0.00003 95065 42438	0.00160 63146 82020
57	0.00002 44148 94383	−0.00151 40836 63698
58	0.00002 65628 71293	0.00139 16489 79833
59	0.00001 99181 19577	−0.00103 22476 29291
60	0.00001 74957 22530	0.00117 51690 55021
61	0.00001 59326 08877	−0.00068 49443 13827
62	0.00001 12285 95309	0.00097 18891 41514
63	0.00001 25348 90128	−0.00043 86395 57592
64	0.00000 69625 32660	0.00078 96980 78051
65	0.00000 97200 15516	−0.00026 71770 74917
66	0.00000 41104 16537	0.00063 18010 08543
67	0.00000 74396 71523	−0.00015 04207 87283
68	0.00000 22450 86377	0.00049 84579 92046
69	0.00000 56260 16197	−0.00007 30665 02016
70	0.00000 10589 34812	0.00038 81951 17581
71	0.00000 42060 05944	−0.00002 36306 06248
72	0.00000 03328 25440	0.00029 86264 22042
73	0.00000 31095 02202	0.00000 63912 62174
74	−0.00000 00875 51537	0.00022 69936 60777
75	0.00000 22734 05778	0.00002 32108 38626
76	−0.00000 03094 77848	0.00017 05054 33362
77	0.00000 16433 33942	0.00003 12982 73200
78	−0.00000 04064 71081	0.00012 65367 13496
79	0.00000 11738 45034	0.00003 38177 18035
80	−0.00000 04280 72380	0.00009 27337 57714
81	0.00000 08278 63578	0.00003 29668 50182
82	−0.00000 04069 98150	0.00006 70569 49851
83	0.00000 05757 19289	0.00003 02380 04257
84	−0.00000 03643 13304	0.00004 77847 01015
85	0.00000 03940 52564	0.00002 66164 14200
86	−0.00000 03131 34285	0.00003 34944 67898
87	0.00000 02647 32894	0.00002 27287 64182
88	−0.00000 02612 50163	0.00002 30317 61757
89	0.00000 01738 68190	0.00001 89530 29816
90	−0.00000 02129 55391	0.00001 54742 68875
91	0.00000 01109 40035	0.00001 54985 66989
92	−0.00000 01703 13923	0.00001 00955 43222
93	0.00000 00680 74052	0.00001 24636 30814
94	−0.00000 01340 17536	0.00000 63308 86812
95	0.00000 00394 39855	0.00000 98759 96080
96	−0.00000 01039 57853	0.00000 37467 86244
97	0.00000 00207 67614	0.00000 77210 94088
98	−0.00000 00795 98908	0.00000 20144 58605
99	0.00000 00089 65154	0.00000 59610 56526
100	−0.00000 00602 12692	0.00000 08875 50295
101	0.00000 00018 19151	0.00000 45472 36222
102	−0.00000 00450 22141	0.00000 01837 34260
103	−0.00000 00022 34991	0.00000 34281 27498

Table 3 Continued.

n	a_n	a'_n
104	−0.00000 00332 82791	−0.00000 02302 04096
105	−0.00000 00042 88231	0.00000 25541 03985
106	−0.00000 00243 24800	−0.00000 04502 64235
107	−0.00000 00050 89596	0.00000 18800 07204
108	−0.00000 00175 70161	−0.00000 05445 86809
109	−0.00000 00051 44175	0.00000 13663 24750
110	−0.00000 00125 35089	−0.00000 05607 15054
111	−0.00000 00047 85224	0.00000 09794 79283
112	−0.00000 00088 24138	−0.00000 05311 59854
113	−0.00000 00042 26662	0.00000 06915 87989
114	−0.00000 00061 20248	−0.00000 04776 12777
115	−0.00000 00036 00842	0.00000 04799 35044
116	−0.00000 00041 73348	−0.00000 04140 96784
117	−0.00000 00029 85412	0.00000 03263 15799
118	−0.00000 00027 89024	−0.00000 03492 93214
119	−0.00000 00024 22169	0.00000 02163 52426
120	−0.00000 00018 18088	−0.00000 02882 38071
121	−0.00000 00019 30093	0.00000 01388 40133
122	−0.00000 00011 47439	−0.00000 02335 41308
123	−0.00000 00015 14226	0.00000 00851 55706
124	−0.00000 00006 92323	−0.00000 01862 49757
125	−0.00000 00011 71577	0.00000 00487 42272
126	−0.00000 00003 89938	−0.00000 01464 47159
127	−0.00000 00008 94983	0.00000 00246 73031
128	−0.00000 00001 94276	−0.00000 01136 62793
129	−0.00000 00006 75525	0.00000 00092 90069
130	−0.00000 00000 72016	−0.00000 00871 42708
131	−0.00000 00005 03999	−0.00000 00000 89083
132	0.00000 00000 00680	−0.00000 00660 23863
133	−0.00000 00003 71736	−0.00000 00054 06034
134	0.00000 00000 40647	−0.00000 00494 40883
135	−0.00000 00002 71015	−0.00000 00080 46362
136	0.00000 00000 59603	−0.00000 00365 87025
137	−0.00000 00001 95218	−0.00000 00089 83328
138	0.00000 00000 65572	−0.00000 00267 44827
139	−0.00000 00001 38830	−0.00000 00088 88220
140	0.00000 00000 63944	−0.00000 00192 97371
141	−0.00000 00000 97360	−0.00000 00082 14068
142	0.00000 00000 58256	−0.00000 00137 27729
143	−0.00000 00000 67215	−0.00000 00072 58395
144	0.00000 00000 50758	−0.00000 00096 11704
145	−0.00000 00000 45567	−0.00000 00062 09695
146	0.00000 00000 42825	−0.00000 00066 07277
147	−0.00000 00000 30224	−0.00000 00051 81423
148	0.00000 00000 35248	−0.00000 00044 42894
149	−0.00000 00000 19503	−0.00000 00042 36526
150	0.00000 00000 28433	−0.00000 00029 05912
151	−0.00000 00000 12132	−0.00000 00034 04898
152	0.00000 00000 22549	−0.00000 00018 31947
153	−0.00000 00000 07160	−0.00000 00026 95586

Table 3 Continued.

n	a_n	a'_n
154	0.00000 00000 17618	−0.00000 00010 95455
155	−0.00000 00000 03882	−0.00000 00021 05180
156	0.00000 00000 13582	−0.00000 00006 01686
157	−0.00000 00000 01783	−0.00000 00016 23441
158	0.00000 00000 10340	−0.00000 00002 79971
159	−0.00000 00000 00492	−0.00000 00012 36967
160	0.00000 00000 07780	−0.00000 00000 78228
161	0.00000 00000 00257	−0.00000 00009 31539
162	0.00000 00000 05786	0.00000 00000 41443
163	0.00000 00000 00652	−0.00000 00006 93339
164	0.00000 00000 04254	0.00000 00001 06266
165	0.00000 00000 00821	−0.00000 00005 09916
166	0.00000 00000 03090	0.00000 00001 35491
167	0.00000 00000 00853	−0.00000 00003 70356
168	0.00000 00000 02218	0.00000 00001 42516
169	0.00000 00000 00808	−0.00000 00002 65410
170	0.00000 00000 01570	0.00000 00001 36484
171	0.00000 00000 00722	−0.00000 00001 87413
172	0.00000 00000 01096	0.00000 00001 23487
173	0.00000 00000 00621	−0.00000 00001 30137
174	0.00000 00000 00752	0.00000 00001 07461
175	0.00000 00000 00519	−0.00000 00000 88605
176	0.00000 00000 00506	0.00000 00000 90847
177	0.00000 00000 00424	−0.00000 00000 58895
178	0.00000 00000 00333	0.00000 00000 75066
179	0.00000 00000 00340	−0.00000 00000 37959
180	0.00000 00000 00212	0.00000 00000 60866
181	0.00000 00000 00268	−0.00000 00000 23454
182	0.00000 00000 00130	0.00000 00000 48559
183	0.00000 00000 00209	−0.00000 00000 13607
184	0.00000 00000 00074	0.00000 00000 38186
185	0.00000 00000 00160	−0.00000 00000 07085
186	0.00000 00000 00038	0.00000 00000 29637
187	0.00000 00000 00121	−0.00000 00000 02901
188	0.00000 00000 00016	0.00000 00000 22719
189	0.00000 00000 00091	−0.00000 00000 00333
190	0.00000 00000 00002	0.00000 00000 17211
191	0.00000 00000 00067	0.00000 00000 01141
192	−0.00000 00000 00006	0.00000 00000 12887
193	0.00000 00000 00049	0.00000 00000 01893
194	−0.00000 00000 00010	0.00000 00000 09537
195	0.00000 00000 00036	0.00000 00000 02185
196	−0.00000 00000 00011	0.00000 00000 06972
197	0.00000 00000 00026	0.00000 00000 02195
198	−0.00000 00000 00011	0.00000 00000 05033
199	0.00000 00000 00018	0.00000 00000 02046
200	−0.00000 00000 00010	0.00000 00000 03582
201	0.00000 00000 00012	0.00000 00000 01818
202	−0.00000 00000 00009	0.00000 00000 02511
203	0.00000 00000 00009	0.00000 00000 01561

Table 3 Continued.

n	a_n	a'_n
204	−0.00000 00000 00008	0.00000 00000 01729
205	0.00000 00000 00006	0.00000 00000 01305
206	−0.00000 00000 00006	0.00000 00000 01165
207	0.00000 00000 00004	0.00000 00000 01069
208	−0.00000 00000 00005	0.00000 00000 00765
209	0.00000 00000 00002	0.00000 00000 00860
210	−0.00000 00000 00004	0.00000 00000 00485
211	0.00000 00000 00001	0.00000 00000 00681
212	−0.00000 00000 00003	0.00000 00000 00293
213	0.00000 00000 00001	0.00000 00000 00533
214	−0.00000 00000 00003	0.00000 00000 00164
215	0.00000 00000 00000	0.00000 00000 00411
216	−0.00000 00000 00002	0.00000 00000 00079
217	0.00000 00000 00000	0.00000 00000 00313
218	−0.00000 00000 00001	0.00000 00000 00026
219	−0.00000 00000 00000	0.00000 00000 00236
220	−0.00000 00000 00001	−0.00000 00000 00006
221	−0.00000 00000 00000	0.00000 00000 00176
222	−0.00000 00000 00001	−0.00000 00000 00023
223	−0.00000 00000 00000	0.00000 00000 00129
224	−0.00000 00000 00001	−0.00000 00000 00032
225	−0.00000 00000 00000	0.00000 00000 00094
226	−0.00000 00000 00000	−0.00000 00000 00034
227	−0.00000 00000 00000	0.00000 00000 00067
228	−0.00000 00000 00000	−0.00000 00000 00033
229	−0.00000 00000 00000	0.00000 00000 00048
230	−0.00000 00000 00000	−0.00000 00000 00030
231	−0.00000 00000 00000	0.00000 00000 00033
232	−0.00000 00000 00000	−0.00000 00000 00026
233	−0.00000 00000 00000	0.00000 00000 00023
234	−0.00000 00000 00000	−0.00000 00000 00022
235	−0.00000 00000 00000	0.00000 00000 00015
236	−0.00000 00000 00000	−0.00000 00000 00018
237	−0.00000 00000 00000	0.00000 00000 00010
238	−0.00000 00000 00000	−0.00000 00000 00015
239	−0.00000 00000 00000	0.00000 00000 00006
240	−0.00000 00000 00000	−0.00000 00000 00012

Table 4

$\lambda=5,\ \alpha=1,\ \beta=0.\quad x_m(t)=\sum_{n=1}^{m}(a_{2n-1}\sin(2n-1)t+a_{2n}\cos(2n-1)t),$

$y_m(t)=\dot{x}_m(t)=\sum_{n=1}^{m}(a'_{2n-1}\sin(2n-1)t+a'_{2n}\cos(2n-1)t),\ m=170.$

$\bar{\omega}=11.61223\ 06677\ 1902,\ \det G=-1.0882,\ r=0.6\times10^{-10},\ \mu_1=650.0,\ \mu_2=35.0,$

$\kappa=9.0\times10^{-3},\ \delta_1=0.40\times10^{-7},\ \delta_2=0.22\times10^{-8},\ |\hat{\omega}-\bar{\omega}|\leq\frac{\mu_2 r}{1-\kappa}=0.212\times10^{-8}.$

n	a_n	a'_n
1	−2.01120 21979 25835	0.625
2	−0.625	−2.01120 21979 25835

Table 4 Continued.

n	a_n	a'_n
3	−0.55512 44926 86670	0.54494 87396 43144
4	−0.18164 95798 81048	−1.66537 34780 60011
5	−0.28748 32362 41786	0.52973 37262 12096
6	−0.10594 67452 42419	−1.43741 61812 08928
7	−0.17809 80117 32222	0.49697 11830 89662
8	−0.07099 58832 98523	−1.24668 60821 25557
9	−0.12015 53449 71299	0.45043 80102 80341
10	−0.05004 86678 08927	−1.08139 81047 41693
11	−0.08515 88106 73790	0.39720 61232 77514
12	−0.03610 96475 70683	−0.93674 69174 11687
13	−0.06228 87111 01456	0.34263 38799 85376
14	−0.02635 64523 06567	−0.80975 32443 18922
15	−0.04654 81590 46129	0.29021 84357 69298
16	−0.01934 78957 17953	−0.69822 23856 91933
17	−0.03531 75125 09703	0.24201 28281 82774
18	−0.01423 60487 16634	−0.60039 77126 64953
19	−0.02709 45525 49289	0.19904 39260 09052
20	−0.01047 59961 05740	−0.51479 64984 36494
21	−0.02095 80114 96497	0.16164 68035 86899
22	−0.00769 74668 37471	−0.44011 82414 26446
23	−0.01631 27022 13054	0.12971 45233 78951
24	−0.00563 97618 86041	−0.37519 21509 00252
25	−0.01275 79355 12203	0.10287 75728 25351
26	−0.00411 51029 13014	−0.31894 83878 05085
27	−0.01001 49458 31643	0.08062 84655 76946
28	−0.00298 62394 65813	−0.27040 35374 54373
29	−0.00788 46183 80049	0.06240 51493 26515
30	−0.00215 19017 00914	−0.22865 39330 21410
31	−0.00622 16994 81179	0.04764 43772 29417
32	−0.00153 69153 94497	−0.19287 26839 16559
33	−0.00491 84217 01996	0.03581 37884 32140
34	−0.00108 52663 16125	−0.16230 79161 65856
35	−0.00389 37397 43420	0.02642 93318 03112
36	−0.00075 51237 65803	−0.13628 08910 19691
37	−0.00308 60384 00380	0.01906 29241 92256
38	−0.00051 52141 67358	−0.11418 34208 14066
39	−0.00244 80627 72416	0.01334 38568 08769
40	−0.00034 21501 74584	−0.09547 44481 24209
41	−0.00194 33143 92734	0.00895 64110 54741
42	−0.00021 84490 50116	−0.07967 58901 02104
43	−0.00154 34407 24533	0.00563 53642 15350
44	−0.00013 10549 81752	−0.06636 79511 54907
45	−0.00122 63139 89967	0.00316 04999 59489
46	−0.00007 02333 32433	−0.05518 41295 48523
47	−0.00097 45983 93736	0.00135 08381 67962
48	−0.00002 87412 37616	−0.04580 61245 05571
49	−0.00077 46696 58185	0.00005 90209 57142
50	−0.00000 12045 09329	−0.03795 88132 51081
51	−0.00061 57926 61987	−0.00083 38966 14072
52	0.00001 63509 14001	−0.03140 54257 61327

Table 4 Continued.

n	a_n	a'_n
53	−0.00048 94906 75606	−0.00142 31274 12150
54	0.00002 68514 60607	−0.02594 30058 07139
55	−0.00038 90584 26276	−0.00178 42611 09481
56	0.00003 24411 11081	−0.02139 82134 45182
57	−0.00030 91841 63841	−0.00197 68070 09948
58	0.00003 46808 24736	−0.01762 34973 38948
59	−0.00024 56549 98906	−0.00204 72054 91934
60	0.00003 46983 98168	−0.01449 36449 35465
61	−0.00019 51262 91558	−0.00203 13577 26893
62	0.00003 33009 46343	−0.01190 27037 85040
63	−0.00015 49405 90282	−0.00195 67294 30809
64	0.00003 10591 97314	−0.00976 12571 87768
65	−0.00012 29850 90947	−0.00184 40849 58983
66	0.00002 83705 37831	−0.00799 40309 11587
67	−0.00009 75791 65432	−0.00170 89053 20186
68	0.00002 55060 49555	−0.00653 78040 83923
69	−0.00007 73854 46119	−0.00156 25391 66768
70	0.00002 26454 95171	−0.00533 95957 82221
71	−0.00006 13394 18949	−0.00141 31304 89664
72	0.00001 99032 46333	−0.00435 50987 45384
73	−0.00004 85935 96360	−0.00126 63612 29047
74	0.00001 73474 14097	−0.00354 73325 34312
75	−0.00003 84732 00395	−0.00112 60416 76010
76	0.00001 50138 89013	−0.00288 54900 29591
77	−0.00003 04409 49378	−0.00099 45765 98475
78	0.00001 29165 79201	−0.00234 39531 02143
79	−0.00002 40690 56129	−0.00087 33305 79015
80	0.00001 10548 17456	−0.00190 14554 34194
81	−0.00001 90169 46486	−0.00076 29121 38214
82	0.00000 94186 68373	−0.00154 03726 65335
83	−0.00001 50135 20444	−0.00066 33928 27411
84	0.00000 79926 84668	−0.00124 61221 96891
85	−0.00001 18430 24058	−0.00057 44745 77149
86	0.00000 67585 24437	−0.00100 66570 44958
87	−0.00000 93337 93679	−0.00049 56161 37621
88	0.00000 56967 37214	−0.00081 20400 50084
89	−0.00000 73492 86619	−0.00042 61273 94702
90	0.00000 47879 48255	−0.00065 40865 09134
91	−0.00000 57809 32787	−0.00036 52386 38912
92	0.00000 40136 11417	−0.00052 60648 83623
93	−0.00000 45424 37209	−0.00031 21504 56800
94	0.00000 33564 56525	−0.00042 24466 60462
95	−0.00000 35652 38947	−0.00026 60687 59639
96	0.00000 28007 23786	−0.00033 86976 99937
97	−0.00000 27948 91891	−0.00022 62285 23532
98	0.00000 23322 52820	−0.00027 11045 13453
99	−0.00000 21881 80615	−0.00019 19090 52645
100	0.00000 19384 75279	−0.00021 66298 80915
101	−0.00000 17108 22354	−0.00016 24429 62877
102	0.00000 16083 46167	−0.00017 27930 57719

Table 4 Continued.

n	a_n	a'_n
103	−0.00000 13356 36382	−0.00013 72205 90947
104	0.00000 13322 38747	−0.00013 75705 47338
105	−0.00000 10410 86094	−0.00011 56911 41381
106	0.00000 11018 20394	−0.00010 93140 39848
107	−0.00000 08101 18240	−0.00009 73615 72853
108	0.00000 09099 21242	−0.00008 66826 51721
109	−0.00000 06292 39071	−0.00008 17939 80198
110	0.00000 07504 03488	−0.00006 85870 58702
111	−0.00000 04877 79297	−0.00006 86020 36473
112	0.00000 06180 36365	−0.00005 41435 02014
113	−0.00000 03773 09535	−0.00005 74469 10083
114	0.00000 05083 79735	−0.00004 26359 77418
115	−0.00000 02911 75610	−0.00004 80329 56657
116	0.00000 04176 77884	−0.00003 34851 95196
117	−0.00000 02241 29350	−0.00004 01033 96993
118	0.00000 03427 64077	−0.00002 62231 33992
119	−0.00000 01720 35370	−0.00003 34361 25346
120	0.00000 02809 75843	−0.00002 04722 09033
121	−0.00000 01316 38357	−0.00002 78397 41930
122	0.00000 02300 80512	−0.00001 59282 41152
123	−0.00000 01003 78471	−0.00002 31498 66244
124	0.00000 01882 10295	−0.00001 23465 51932
125	−0.00000 00762 45011	−0.00001 92257 60528
126	0.00000 01538 06084	−0.00000 95306 26388
127	−0.00000 00576 60484	−0.00001 59472 73245
128	0.00000 01255 69081	−0.00000 73228 81508
129	−0.00000 00433 88834	−0.00001 32120 98874
130	0.00000 01024 19371	−0.00000 55971 59568
131	−0.00000 00324 62841	−0.00001 09333 41245
132	0.00000 00834 60620	−0.00000 42526 32204
133	−0.00000 00241 26743	−0.00000 90373 61870
134	0.00000 00679 50089	−0.00000 32088 56820
135	−0.00000 00177 90911	−0.00000 74618 91415
136	0.00000 00552 73270	−0.00000 24017 72999
137	−0.00000 00129 96092	−0.00000 61543 80934
138	0.00000 00449 22489	−0.00000 17804 64654
139	−0.00000 00093 85217	−0.00000 50705 69210
140	0.00000 00364 78915	−0.00000 13045 45170
141	−0.00000 00066 81196	−0.00000 41732 43114
142	0.00000 00295 97469	−0.00000 09420 48667
143	−0.00000 00046 69454	−0.00000 34311 69076
144	0.00000 00239 94189	−0.00000 06677 31924
145	−0.00000 00031 84200	−0.00000 28181 75201
146	0.00000 00194 35691	−0.00000 04617 09037
147	−0.00000 00020 97655	−0.00000 23123 65316
148	0.00000 00157 30376	−0.00000 03083 55297
149	−0.00000 00013 11601	−0.00000 18954 47905
150	0.00000 00127 21127	−0.00000 01954 28607
151	−0.00000 00007 50771	−0.00000 15521 64703
152	0.00000 00102 79236	−0.00000 01133 66413

Table 4 Continued.

n	a_n	a'_n
153	−0.00000 00003 57674	−0.00000 12698 05338
154	0.00000 00082 99381	−0.00000 00547 24046
155	−0.00000 00000 88560	−0.00000 10377 96040
156	0.00000 00066 95458	−0.00000 00137 26841
157	0.00000 00000 89722	−0.00000 08473 51855
158	0.00000 00053 97146	0.00000 00140 86380
159	0.00000 00002 02179	−0.00000 06911 83176
160	0.00000 00043 47064	0.00000 00321 46512
161	0.00000 00002 67570	−0.00000 05632 48570
162	0.00000 00034 98438	0.00000 00430 78704
163	0.00000 00002 99877	−0.00000 04585 46991
164	0.00000 00028 13172	0.00000 00488 79977
165	0.00000 00003 09451	−0.00000 03729 43398
166	0.00000 00022 60263	0.00000 00510 59404
167	0.00000 00003 03883	−0.00000 03030 22649
168	0.00000 00018 14507	0.00000 00507 48485
169	0.00000 00002 88685	−0.00000 02459 67285
170	0.00000 00014 55428	0.00000 00487 87808
171	0.00000 00002 67806	−0.00000 01994 55445
172	0.00000 00011 66406	0.00000 00457 94901
173	0.00000 00002 44030	−0.00000 01615 75727
174	0.00000 00009 33964	0.00000 00422 17203
175	0.00000 00002 19276	−0.00000 01307 56269
176	0.00000 00007 47179	0.00000 00383 73265
177	0.00000 00001 94829	−0.00000 01057 05769
178	0.00000 00005 97208	0.00000 00344 84680
179	0.00000 00001 71512	−0.00000 00853 64492
180	0.00000 00004 76897	0.00000 00307 00734
181	0.00000 00001 49820	−0.00000 00688 63615
182	0.00000 00003 80462	0.00000 00271 17337
183	0.00000 00001 30008	−0.00000 00554 91548
184	0.00000 00003 03233	0.00000 00237 91499
185	0.00000 00001 12175	−0.00000 00446 66056
186	0.00000 00002 41438	0.00000 00207 52319
187	0.00000 00000 96306	−0.00000 00359 11215
188	0.00000 00001 92039	0.00000 00180 09286
189	0.00000 00000 82320	−0.00000 00288 38383
190	0.00000 00001 52584	0.00000 00155 58486
191	0.00000 00000 70090	−0.00000 00231 30499
192	0.00000 00001 21102	0.00000 00133 87204
193	0.00000 00000 59468	−0.00000 00185 29161
194	0.00000 00000 96006	0.00000 00114 77300
195	0.00000 00000 50296	−0.00000 00148 23971
196	0.00000 00000 76020	0.00000 00098 07642
197	0.00000 00000 42415	−0.00000 00118 43791
198	0.00000 00000 60121	0.00000 00083 55839
199	0.00000 00000 35676	−0.00000 00094 49554
200	0.00000 00000 47485	0.00000 00070 99431
201	0.00000 00000 29934	−0.00000 00075 28364
202	0.00000 00000 37455	0.00000 00060 16691

Table 4 Continued.

n	a_n	a'_n
203	0.00000 00000 25060	−0.00000 00059 88665
204	0.00000 00000 29501	0.00000 00050 87133
205	0.00000 00000 20936	−0.00000 00047 56283
206	0.00000 00000 23201	0.00000 00042 91802
207	0.00000 00000 17456	−0.00000 00037 71183
208	0.00000 00000 18218	0.00000 00036 13428
209	0.00000 00000 14528	−0.00000 00029 84831
210	0.00000 00000 14281	0.00000 00030 36451
211	0.00000 00000 12071	−0.00000 00023 58027
212	0.00000 00000 11175	0.00000 00025 46998
213	0.00000 00000 10013	−0.00000 00018 59147
214	0.00000 00000 08728	0.00000 00021 32795
215	0.00000 00000 08293	−0.00000 00014 62703
216	0.00000 00000 06803	0.00000 00017 83060
217	0.00000 00000 06859	−0.00000 00011 48181
218	0.00000 00000 05291	0.00000 00014 88377
219	0.00000 00000 05665	−0.00000 00008 99085
220	0.00000 00000 04105	0.00000 00012 40570
221	0.00000 00000 04672	−0.00000 00007 02166
222	0.00000 00000 03177	0.00000 00010 32565
223	0.00000 00000 03849	−0.00000 00005 46799
224	0.00000 00000 02452	0.00000 00008 58276
225	0.00000 00000 03167	−0.00000 00004 24469
226	0.00000 00000 01887	0.00000 00007 12478
227	0.00000 00000 02602	−0.00000 00003 28365
228	0.00000 00000 01447	0.00000 00005 90706
229	0.00000 00000 02136	−0.00000 00002 53044
230	0.00000 00000 01105	0.00000 00004 89153
231	0.00000 00000 01751	−0.00000 00001 94165
232	0.00000 00000 00841	0.00000 00004 04584
233	0.00000 00000 01435	−0.00000 00001 48265
234	0.00000 00000 00636	0.00000 00003 34254
235	0.00000 00000 01174	−0.00000 00001 12594
236	0.00000 00000 00479	0.00000 00002 75843
237	0.00000 00000 00959	−0.00000 00000 84965
238	0.00000 00000 00359	0.00000 00002 27393
239	0.00000 00000 00783	−0.00000 00000 63643
240	0.00000 00000 00266	0.00000 00001 87255
241	0.00000 00000 00639	−0.00000 00000 47257
242	0.00000 00000 00196	0.00000 00001 54042
243	0.00000 00000 00521	−0.00000 00000 34723
244	0.00000 00000 00143	0.00000 00001 26591
245	0.00000 00000 00424	−0.00000 00000 25185
246	0.00000 00000 00103	0.00000 00001 03929
247	0.00000 00000 00345	−0.00000 00000 17971
248	0.00000 00000 00073	0.00000 00000 85239
249	0.00000 00000 00280	−0.00000 00000 12552
250	0.00000 00000 00050	0.00000 00000 69843
251	0.00000 00000 00228	−0.00000 00000 08515
252	0.00000 00000 00034	0.00000 00000 57172

Table 4 Continued.

n	a_n	a'_n
253	0.00000 00000 00185	−0.00000 00000 05535
254	0.00000 00000 00022	0.00000 00000 46743
255	0.00000 00000 00150	−0.00000 00000 03366
256	0.00000 00000 00013	0.00000 00000 38190
257	0.00000 00000 00121	−0.00000 00000 01808
258	0.00000 00000 00007	0.00000 00000 31173
259	0.00000 00000 00098	−0.00000 00000 00711
260	0.00000 00000 00003	0.00000 00000 25421
261	0.00000 00000 00079	0.00000 00000 00041
262	−0.00000 00000 00000	0.00000 00000 20712
263	0.00000 00000 00064	0.00000 00000 00539
264	−0.00000 00000 00002	0.00000 00000 16859
265	0.00000 00000 00052	0.00000 00000 00850
266	−0.00000 00000 00003	0.00000 00000 13710
267	0.00000 00000 00042	0.00000 00000 01027
268	−0.00000 00000 00004	0.00000 00000 11139
269	0.00000 00000 00034	0.00000 00000 01107
270	−0.00000 00000 00004	0.00000 00000 09042
271	0.00000 00000 00027	0.00000 00000 01122
272	−0.00000 00000 00004	0.00000 00000 07332
273	0.00000 00000 00022	0.00000 00000 01092
274	−0.00000 00000 00004	0.00000 00000 05941
275	0.00000 00000 00017	0.00000 00000 01034
276	−0.00000 00000 00004	0.00000 00000 04808
277	0.00000 00000 00014	0.00000 00000 00959
278	−0.00000 00000 00003	0.00000 00000 03888
279	0.00000 00000 00011	0.00000 00000 00875
280	−0.00000 00000 00003	0.00000 00000 03141
281	0.00000 00000 00009	0.00000 00000 00790
282	−0.00000 00000 00003	0.00000 00000 02537
283	0.00000 00000 00007	0.00000 00000 00704
284	−0.00000 00000 00002	0.00000 00000 02045
285	0.00000 00000 00006	0.00000 00000 00623
286	−0.00000 00000 00002	0.00000 00000 01647
287	0.00000 00000 00005	0.00000 00000 00547
288	−0.00000 00000 00002	0.00000 00000 01325
289	0.00000 00000 00004	0.00000 00000 00478
290	−0.00000 00000 00002	0.00000 00000 01065
291	0.00000 00000 00003	0.00000 00000 00415
292	−0.00000 00000 00001	0.00000 00000 00855
293	0.00000 00000 00002	0.00000 00000 00358
294	−0.00000 00000 00001	0.00000 00000 00686
295	0.00000 00000 00002	0.00000 00000 00308
296	−0.00000 00000 00001	0.00000 00000 00549
297	0.00000 00000 00001	0.00000 00000 00264
298	−0.00000 00000 00001	0.00000 00000 00439
299	0.00000 00000 00001	0.00000 00000 00226
300	−0.00000 00000 00001	0.00000 00000 00351
301	0.00000 00000 00001	0.00000 00000 00192
302	−0.00000 00000 00001	0.00000 00000 00280

Table 4 Continued.

n	a_n	a'_n
303	0.00000 00000 00001	0.00000 00000 00163
304	−0.00000 00000 00001	0.00000 00000 00223
305	0.00000 00000 00001	0.00000 00000 00138
306	−0.00000 00000 00000	0.00000 00000 00178
307	0.00000 00000 00000	0.00000 00000 00117
308	−0.00000 00000 00000	0.00000 00000 00141
309	0.00000 00000 00000	0.00000 00000 00099
310	−0.00000 00000 00000	0.00000 00000 00112
311	0.00000 00000 00000	0.00000 00000 00083
312	−0.00000 00000 00000	0.00000 00000 00089
313	0.00000 00000 00000	0.00000 00000 00070
314	−0.00000 00000 00000	0.00000 00000 00070
315	0.00000 00000 00000	0.00000 00000 00058
316	−0.00000 00000 00000	0.00000 00000 00055
317	0.00000 00000 00000	0.00000 00000 00049
318	−0.00000 00000 00000	0.00000 00000 00044
319	0.00000 00000 00000	0.00000 00000 00041
320	−0.00000 00000 00000	0.00000 00000 00034
321	0.00000 00000 00000	0.00000 00000 00034
322	−0.00000 00000 00000	0.00000 00000 00027
323	0.00000 00000 00000	0.00000 00000 00028
324	−0.00000 00000 00000	0.00000 00000 00021
325	0.00000 00000 00000	0.00000 00000 00024
326	−0.00000 00000 00000	0.00000 00000 00016
327	0.00000 00000 00000	0.00000 00000 00020
328	−0.00000 00000 00000	0.00000 00000 00013
329	0.00000 00000 00000	0.00000 00000 00016
330	−0.00000 00000 00000	0.00000 00000 00010
331	0.00000 00000 00000	0.00000 00000 00013
332	−0.00000 00000 00000	0.00000 00000 00008
333	0.00000 00000 00000	0.00000 00000 00011
334	−0.00000 00000 00000	0.00000 00000 00006
335	0.00000 00000 00000	0.00000 00000 00009
336	−0.00000 00000 00000	0.00000 00000 00005
337	0.00000 00000 00000	0.00000 00000 00008
338	−0.00000 00000 00000	0.00000 00000 00003
339	0.00000 00000 00000	0.00000 00000 00006
340	−0.00000 00000 00000	0.00000 00000 00003

Table 5 *Comparison of periods*

λ	Urabe et al.'s results [3]	Krogdahl's results [4]	Strasberg's results [11]
3	8.8613	8.85946	8.85909 5500
4	10.2072	10.20368	10.20352 3690
5	11.6055	11.61242	11.61223 0668

λ	Our results	Error bound given in [12]	Our error bounds
3	8.85909 54997 19845	0.37×10^{-7}	0.121×10^{-8}
4	10.20352 36909 9344	/	0.211×10^{-8}
5	11.61223 06677 1902	/	0.212×10^{-8}