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On Relation Modules

By

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Let $\overset{m}{\oplus}R$ and $\overset{n}{\oplus}R$ be free modules over a commutative ring R with canonical bases $T'_1,\dots, T'_m$ and $T_1,\dots, T_n$ respectively and m and n are positive integers. For any R-linear map $f: \overset{m}{\oplus}R \to \overset{n}{\oplus}R$, $f(T'_i)=\sum_{j=1}^{n} a_{ji}T_j$, $a_{ji}\in R$ $(i=1,\dots, m)$, we associate with an $n\times m$ matrix A defined by the coefficients of T'_js:

$$A=\begin{pmatrix} a_{11} \cdots\cdots\cdots a_{1m} \\ \cdots\cdots\cdots\cdots\cdots \\ a_{n1} \cdots\cdots\cdots a_{nm} \end{pmatrix}$$

Corresponding to f, we define an R-linear map ${}^tf$: $\overset{m}{\oplus}R \leftarrow \overset{n}{\oplus}R$, *transpose map* of f, by

$${}^tf(T_i)=\sum_{j=1}^{m} a_{ij}T'_j \qquad (i=1,\dots, n).$$

The matrix associated with ${}^tf$ is the transposed matrix ${}^tA$ of A.

Let $m_1,\dots, m_n$ be n elements of an R-module M. We denote by $\mathrm{Rel}\,(m_1,\dots, m_n)$ the set of sequences of n elements of R, $(r_1,\dots, r_n)\in \overset{n}{\oplus}R$, such that

$$r_1m_1+\cdots+r_nm_n=0.$$

Obviously $\mathrm{Rel}\,(m_1,\dots, m_n)$ is an R-module and we call it the *relation module* of $(m_1,\dots, m_n)$.

Lemma 1. *With the same notations as above,*

i) $(r_1,\dots, r_n)\in \mathrm{Rel}\,({}^tfT_1,\dots, {}^tfT_n)$ *if and only if* $(r_1,\dots, r_n)A=0$.

ii) *If* $g: \overset{n}{\oplus}R\to R$ *is an R-linear map, then* $(g(T_1),\dots, g(T_n))\in \mathrm{Rel}\,({}^tfT_1,\dots, {}^tfT_n)$ *if and only if* $g(\mathrm{Im}\,f)=0$.

PROOF. i) Assume $(r_1,\dots, r_n)\in \mathrm{Rel}\,({}^tfT_1,\dots, {}^tfT_n)$, then $r_1{}^tfT_1+\cdots+r_n{}^tfT_n=0$ so that $(T'_1\cdots T'_m)\,{}^tA\begin{pmatrix} r_1 \\ \vdots \\ r_n\end{pmatrix}=0$. Hence ${}^tA\begin{pmatrix} r_1 \\ \vdots \\ r_n\end{pmatrix}=0$, and whence $(r_1\cdots r_n)A=0$. Converse is clear.

ii) Let $g(T_i)=s_i$ $(i=1,\dots,n)$. Then, we have

$$g(\mathrm{Im}\,f)=0 \Longleftrightarrow g(f(T_i'))=0 \qquad (i=1,\dots,m)$$

$$\Longleftrightarrow g(\sum_{j=1}^{n} a_{ji}T_j)=0 \qquad (i=1,\dots,m)$$

$$\Longleftrightarrow 0=\sum_{j=1}^{n} a_{ji}g(T_j)=\sum_{j=1}^{n} a_{ji}s_j \qquad (i=1,\dots,m).$$

Hence $g(\mathrm{Im}\,f)=0$ if and only if $(s_1\cdots s_n)A=0$. Therefore we get ii) in view of i).

Lemma 2. $\mathrm{Rel}\,({}^tfT_1,\dots,{}^tfT_n)\simeq \mathrm{Hom}_R\,(\overset{n}{\oplus}R/\mathrm{Im}\,f,\ R)$.

PROOF. For any $(r_1,\dots,r_n)\in \mathrm{Rel}\,({}^tfT_1,\dots,{}^tfT_n)$, we define an R-linear map $g\colon \overset{n}{\oplus}R\to R$, $g(T_i)=r_i$ $(i=1,\dots,n)$.

By Lemma 1, ii), we have $g=0$ on $\mathrm{Im}\,f$, so that g induces an R-linear map $\bar{g}\colon \overset{n}{\oplus}R/\mathrm{Im}\,f\to R$. Thus we get a map $\lambda\colon \mathrm{Rel}\,({}^tfT_1,\dots,{}^tfT_n)\to \mathrm{Hom}_R\,(\overset{n}{\oplus}R/\mathrm{Im}\,f,\ R)$ such that

$$\lambda((r_1,\dots,r_n))=\bar{g}.$$

It is clear that λ is injective. Since $\mathrm{Hom}\,(\overset{n}{\oplus}R/\mathrm{Im}\,f,\ R)$ is identified with the set of R-linear maps $h\colon \overset{n}{\oplus}R\to R$ which vanish on $\mathrm{Im}\,f$, λ is surjective.

Summarizing the above consideration, we get

Theorem 1. *Let $f\colon \overset{m}{\oplus}R\to\overset{n}{\oplus}R$ be an R-linear map defined by the matrix A:*

$$A=\begin{pmatrix} a_{11} & \cdots\cdots\cdots & a_{1m} \\ & \cdots\cdots\cdots\cdots\cdots & \\ a_{n1} & \cdots\cdots\cdots & a_{nm} \end{pmatrix}$$

and let ${}^tf\colon \overset{m}{\oplus}R\leftarrow\overset{n}{\oplus}R$ be the transpose of f corresponding to ${}^tA$. Then,

$$\mathrm{Rel}\,({}^tfT_1,\dots,{}^tfT_n)\simeq(\mathrm{Coker}\,f)^*$$

where $M^*=\mathrm{Hom}_R\,(M,\ R)$ for an R-module M.

Remark. With the same notations as in Lemma 2, $\lambda^{-1}\colon (\mathrm{Coker}\,f)^*\to \mathrm{Rel}\,({}^tfT_1,\dots,{}^tfT_n)$ is given by

$$\bar{g}\longrightarrow(\bar{g}(T_1),\dots,\bar{g}(T_n)).$$

Corollary 1. *Let $\overset{l}{\oplus}R\xrightarrow{\ g\ }\overset{m}{\oplus}R\xrightarrow{\ f\ }\overset{n}{\oplus}R$ be an exact sequence of free R-*

modules. Then,

$$\mathrm{Rel}\,({}^{t}gT'_1,\dots,{}^{t}gT'_m)\simeq(\mathrm{Im}\,f)^*$$

where $T'_1,\dots,T'_m$ *is the canonical base of* $\overset{m}{\oplus}R$.

PROOF. By Theorem 1, we have

$$\mathrm{Rel}\,({}^{t}gT'_1,\dots,{}^{t}gT'_m)\simeq(\mathrm{Coker}\,g)^*.$$

On the other hand

$$(\mathrm{Coker}\,g)^*\simeq(\overset{m}{\oplus}R/\mathrm{Im}\,g)^*=(\overset{m}{\oplus}R/\mathrm{Ker}\,f)^*\simeq(\mathrm{Im}\,f)^*.$$

Thus we get our result.

Remark. The isomorphism $(\mathrm{Im}\,f)^*\mapsto\mathrm{Rel}\,({}^{t}gT'_1,\dots,{}^{t}gT'_m)$ obtained in the Corollary 1 is given by

$$\phi\longrightarrow((\phi\cdot f)(T'_1),\dots,(\phi\cdot f)(T'_m)).$$

Corollary 2. *Let* $f:\overset{m}{\oplus}R\to\overset{n}{\oplus}R$ *be an R-homomorphism and let* $\mathrm{Coker}f=Ru_1+\cdots+Ru_n$ *where* u_i *is the residue of* T_i *modulo* $\mathrm{Im}\,f\,(i=1,\dots,n)$. *Then, we have the following exact sequence of R-modules*:

$$0\longrightarrow\mathrm{Rel}\,(fT'_1,\dots,fT'_m)\longrightarrow\overset{m}{\oplus}R\longrightarrow\mathrm{Rel}\,(u_1,\dots,u_n)\longrightarrow 0.$$

PROOF. Clearly we have

$$(r_1,\dots,r_n)\in\mathrm{Rel}\,(u_1,\dots,u_n)\Longleftrightarrow r_1T_1+\cdots+r_nT_n\in\mathrm{Im}\,f$$

$$\Longleftrightarrow r_1T_1+\cdots+r_nT_n=r'_1f(T'_1)+\cdots+r'_mf(T'_m)\quad\text{for some elements}\quad r'_i\in R$$

$$\Longleftrightarrow\begin{pmatrix}r_1\\ \vdots\\ r_n\end{pmatrix}=A\begin{pmatrix}r'_1\\ \vdots\\ r'_m\end{pmatrix}\quad\text{for some elements}\quad r'_i\in R\ (i=1,\dots,m).$$

Now we define a map ϕ: $\mathrm{Hom}\,(\overset{m}{\oplus}R,R)\to\mathrm{Rel}\,(u_1,\dots,u_n)$ by

$$\phi(g)=A\begin{pmatrix}r'_1\\ \vdots\\ r'_m\end{pmatrix}$$

where $g\in\mathrm{Hom}\,(\overset{m}{\oplus}R,R)$ and $r'_i=g(T'_i)\ (i=1,\dots,m)$. Then, the first part of our proof shows that ϕ is surjective.

Consider an exact sequence

$$0\longrightarrow\mathrm{Ker}\,\phi\longrightarrow\mathrm{Hom}\,(\overset{m}{\oplus}R,R)\overset{\phi}{\longrightarrow}\mathrm{Rel}\,(u_1,\dots,u_n)\longrightarrow 0.$$

Since we have

$$g \in \operatorname{Ker} \phi \Longleftrightarrow A \begin{pmatrix} gT'_1 \\ \vdots \\ gT'_m \end{pmatrix} = 0$$

$$\Longleftrightarrow g = 0 \text{ on each component of } A \begin{pmatrix} T'_1 \\ \vdots \\ T'_m \end{pmatrix}$$

$$\Longleftrightarrow g = 0 \text{ on each component of } (T'_1 \cdots T'_m)\,{}^t\!A = ({}^t f T_1 \cdots {}^t f T_n)$$

$$\Longleftrightarrow g({}^t f T_i) = 0 \qquad \text{for} \quad i = 1, \ldots, n.$$

Hence, identifying g with $(g(T'_1), \ldots, g(T'_m))$ we get our assertion. q.e.d.

Theorem 2. *Let* $\overset{l}{\oplus} R \xrightarrow{g} \overset{m}{\oplus} R \xrightarrow{f} \overset{n}{\oplus} R$ *be an exact sequence of free R-modules where g and f are R-homomorphisms. Then,*

$$\operatorname{Ext}^1(\operatorname{Coker} f, R) \simeq \operatorname{Rel}({}^t g T'_1, \ldots, {}^t g T'_m)/\operatorname{Im} {}^t f$$

where $T'_1, \ldots, T'_m$ *is the canonical base of* $\overset{m}{\oplus} R$.

PROOF. From an exact sequence

$$0 \longrightarrow \operatorname{Im} f \xrightarrow{\phi} \overset{n}{\oplus} R \longrightarrow \operatorname{Coker} f \longrightarrow 0,$$

we get a long exact sequence of R-modules:

$$0 \longrightarrow \operatorname{Hom}(\operatorname{Coker} f, R) \longrightarrow \operatorname{Hom}(\overset{n}{\oplus} R, R) \xrightarrow{\phi^*} \operatorname{Hom}(\operatorname{Im} f, R)$$

$$\longrightarrow \operatorname{Ext}^1(\operatorname{Coker} f, R) \longrightarrow \operatorname{Ext}^1(\overset{n}{\oplus} R, R) \longrightarrow \cdots$$

Since $\operatorname{Ext}^1(\overset{n}{\oplus} R, R) = 0$, identifying $(\overset{n}{\oplus} R)^*$ with $(\overset{n}{\oplus} R)$, we have

$$\operatorname{Ext}^1(\operatorname{Coker} f, R) \simeq (\operatorname{Im} f)^* / \phi^*(\overset{n}{\oplus} R).$$

By the Corollary 1 of Theorem 1, we have

$$(\operatorname{Im} f)^* \simeq \operatorname{Rel}({}^t g T'_1, \ldots, {}^t g T'_m)$$

and $\phi^*(\overset{n}{\oplus} R)$ is generated by the restriction to the $\operatorname{Im} f$ of the projection map p_i: $\overset{n}{\oplus} R \to R$ $(i = 1, \ldots, n)$, i.e., $\phi^*(\overset{n}{\oplus} R)$ is generated by

$$((p_i f)(T'_1), \ldots, (p_i f)(T'_m)) \qquad (i = 1, \ldots, n).$$

Since $fT'_i = \sum_{j=1}^{n} a_{ji}T_j$ $(i=1,\dots, n)$ and $(p_j f)(T'_i)=a_{ji}$, we have

$$((p_i f)(T'_1),\dots, (p_i f)(T'_m))=(a_{i1},\dots, a_{im}) \qquad (i=1,\dots, n).$$

Identifying $(a_{i1},\dots, a_{im})$ with $a_{i1}T'_1+\cdots+a_{im}T'_m={}^t f T_i$ $(i=1,\dots, n)$, we see that $\phi^*(\overset{n}{\oplus}R)$ is generated by ${}^t f T_1,\dots, {}^t f T_n$, so that $\phi^*(\overset{n}{\oplus}R)=\mathrm{Im}\,{}^t f$. q.e.d.

Assume R is a Noetherian local ring and let M be a finitely generated R-module with minimal system of generators $u_1,\dots, u_n$. Then, it is well known that $\mathrm{Rel}\,(u_1,\dots, u_n)$ is determined uniquely up to isomorphism [1, theorem 26.1]. We call it the *relation module* of M and denote it $\mathrm{Rel}\,(M)$.

Now, let

$$\cdots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \cdots \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

be a minimal projective (free) resolution of M with augmentation ε. For any integer $n \geqq 1$, the n-th Syzygy module of M is defined to be $\mathrm{Im}\, d_n$ and is denoted by $\mathrm{Syz}_n(M)$.

Let N be a submodule of $\overset{n}{\oplus}R$, minimally generated by m elements, $N=Rv_1+\cdots+Rv_m$. Take a map $f\colon \overset{m}{\oplus}R\to\overset{n}{\oplus}R$ such that $f(T'_i)=v_i$, $(i=1,\dots, m)$. The submodule $\mathrm{Im}\,{}^t f$ of $\overset{m}{\oplus}R$ is called the *transpose* of N and is denoted by ${}^t N$.

Corollary. *If M is finitely generated over a Noetherian local ring R, then*

$$\mathrm{Ext}^n\,(M, R)\simeq \mathrm{Rel}\,{}^t(\mathrm{Syz}_{n+1}M)/{}^t(\mathrm{Syz}_n M)$$

for $n\geqq 1$.

PROOF. We can assume the sequence

$$\overset{l}{\oplus}R \xrightarrow{g} \overset{m}{\oplus}R \xrightarrow{f} \overset{n}{\oplus}R \longrightarrow M \longrightarrow 0$$

is the first three terms of a minimal free resolution of M. Hence, by Theorem 2, we have

$$\mathrm{Ext}^1\,(M, R)\simeq \mathrm{Rel}\,{}^t(\mathrm{Im}\,g)/{}^t(\mathrm{Im}\,f).$$

Since $\mathrm{Im}\,g=\mathrm{Syz}_2 M$ and $\mathrm{Im}\,f=\mathrm{Syz}_1 M$, we have our corollary in the case $n=1$.

In general we have

$$\begin{aligned}\mathrm{Ext}^n(M, R)&=\mathrm{Ext}^1(\mathrm{Syz}_{n-1}M, R)\\ &\simeq \mathrm{Rel}\,{}^t(\mathrm{Syz}_2(\mathrm{Syz}_{n-1}M)/{}^t(\mathrm{Syz}_1(\mathrm{Syz}_{n-1}M))\\ &=\mathrm{Rel}\,{}^t(\mathrm{Syz}_{n+1}M)/{}^t(\mathrm{Syz}_n M)),\end{aligned}$$

which finish our proof. q. e. d.

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References

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