J. Math. Tokushima Univ.
Vol. 13 (1979), 1–6

On the Poincaré Series of a Local Ring Reduced Modulo its Socle

Dedicated to Professor Yoshikazu Nakai on his sixtieth birthday

By

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(Received April 30, 1979)

Let $(R, \mathfrak{m})$ be a Noetherian local ring with residue field k. For any finitely generated R-module M, we let $P_R^M(Z)$ be the power series

$$P_R^M(Z)=\sum_{p=0}^{\infty} \dim_k \mathrm{Tor}_p^R(k, M)Z^p.$$

The Poincaré series of R is the power series $P_R(Z)=P_R^k(Z)$. In this note, we prove a slight generalization of a theorem of Gulliksen [2], which gives an unified approach to prove the rationality of Poincaré series for some classes of local rings. Especially, we present a simple proof of the rationality of Poincaré series of a Gorenstein local ring which satisfies $\mathfrak{m}^3=0$ [8], [9].

Throughout the paper all rings are commutative with identity and Noetherian. We shall use the same notations in [2].

Proposition 1. *Let $(R, \mathfrak{m})$ be a local ring and let $\bar{R}=R/\mathfrak{a}$ where $\mathfrak{a}$ is an ideal in R such that $\mathfrak{a}\subset 0: \mathfrak{m}$. Then,*

$$P_{\bar{R}}(Z)\leqq\frac{P_R(Z)}{1-(\dim_k\mathfrak{a})Z^2P_R(Z)}.$$

PROOF. Let $t=\dim_k \mathfrak{a}$ and let F be a minimal free R-algebra resolution of k. Put $\bar{F}=F\otimes_R\bar{R}$. Starting with $\bar{F}$, we construct the Eagon resolution X of k as follows: Take free graded $\bar{R}$-module N such that rank $N_p=\dim \tilde{H}_{p-1}(\bar{F})$ for $p\geqq 1$, where $\tilde{H}(\bar{F})$ is the kernel of $\bar{\varepsilon}_*: H(\bar{F})\to k$ induced by the augmentation $\bar{\varepsilon}: \bar{F}\to k$. Denote by T the tensor algebra generated by N over $\bar{R}$. Then, $X=\bar{F}\otimes_{\bar{R}}T$, which is a free $\bar{R}$-algebra resolution of k [3, Chap. 4, § 1].

Now, let $H_R(*)=\sum_{p=0}^{\infty}(\mathrm{rank}\, *_p)Z^p$ be the Hilbert series of graded R-module $*$. Then, we have

$$P_{\bar{R}}(Z)=\sum_{p=0}^{\infty}\dim_k H_p(X\otimes_{\bar{R}} k)Z^p\leqq\sum_{p=0}^{\infty}\dim_k (X\otimes_{\bar{R}} k)_pZ^p$$
$$=\sum_{p=0}^{\infty}(\mathrm{rank}\, X_p)Z^p$$

$$=(\sum_{p=0}^{\infty}(\text{rank } \bar{F}_p)Z^p)(\sum_{p=0}^{\infty}(\text{rank } T_p)Z^p)$$

$$=H_{\bar{R}}(\bar{F})H_{\bar{R}}(T)$$

$$=H_R(F)H_{\bar{R}}(T)$$

$$=P_R(Z)H_{\bar{R}}(T).$$

Hence

$$P_{\bar{R}}(Z)\leqq P_R(Z)H_{\bar{R}}(T).$$

On the other hand, from the long exact homology sequence of

$$0 \longrightarrow \mathfrak{a}F \longrightarrow F \longrightarrow \bar{F} \longrightarrow 0,$$

we have

$$H_q(\bar{F})\cong H_{q-1}(\mathfrak{a}F)\cong \overset{t}{\oplus} H_{q-1}(F/mF) \qquad (q\geqq 1),$$

since $\mathfrak{a}F\cong \overset{t}{\oplus}(F/\mathfrak{m}F)$. Hence

$$H_{\bar{R}}(N)=\sum_{p=0}^{\infty}(\text{rank } N_p)Z^p=\sum_{p=2}^{\infty}\dim H_{p-1}(\bar{F})Z^p$$

$$=\sum_{p=2}^{\infty} t\dim H_{p-2}(F/\mathfrak{m}F)Z^p$$

$$=tZ^2H_k(F/\mathfrak{m}F)$$

$$=tZ^2P_R(Z).$$

Consequently,

$$P_{\bar{R}}(Z)\leqq P_R(Z)H_{\bar{R}}(T)=P_R(Z)\frac{1}{1-H_{\bar{R}}(N)}$$

$$=P_R(Z)/1-tZ^2P_R(Z). \qquad \text{q.e.d.}$$

Let F be an augmented R-algebra in the sense of Tate with augmentation ε: $F\to k$. If S is a set of homogeneous cycles which represent a minimal set of generators for $\tilde{H}(F)$, the trivial Massey operation γ is a function defined on the set of finite sequences in S with values in F. For the detail of the definitions and results the reader is referred to [2].

Proposition 2. *Let F be a minimal free R-algebra resolution of k and let $\mathfrak{a}$ be an ideal in R such that $\mathfrak{a}\subset 0:\mathfrak{m}$. If $F/\mathfrak{a}F$ can be extended to a minimal free $R/\mathfrak{a}$-algebra resolution of k, then $F/\mathfrak{a}F$ has a trivial Massey operation.*

PROOF. Put $\bar{R}=R/\mathfrak{a}$, $\bar{\mathfrak{m}}=\mathfrak{m}/\mathfrak{a}$ and $\bar{F}=F/\mathfrak{a}F$. Then, $\bar{F}$ is a free $\bar{R}$-algebra with augmentation $\bar{\varepsilon}\colon \bar{F}\to k$. Let $\Psi\colon F\to\bar{F}$ be the canonical map and let $\bar{S}$ be a set of cycles representing a minimal set of generators of $\tilde{H}(\bar{F})$. To each $z\in\bar{S}$, we select $Z\in\Psi^{-1}(z)$ so that $dZ\in\mathfrak{a}F$. Then, as in [2], we can define the function $\Gamma(Z_1,\dots, Z_m)\in F$ for any $Z_1,\dots, Z_m\in\Psi^{-1}(\bar{S})$ inductively as follows:

i) $\Gamma(Z)=Z$

ii) $d(\Gamma(Z_1,\dots, Z_m))=\sum_{k=1}^{m-1}(-1)^{[1,k]}\Gamma(Z_1,\dots, Z_k)\Gamma(Z_{k+1},\dots, Z_m)$

where $[1, k]=\sum_{i=1}^{k}(1+\deg Z_i)$.

Then, it is clear that the function γ defined by $\gamma(z_1,\dots, z_m)=\Psi(\Gamma(Z_1,\dots, Z_m))$ becomes a Massey operation on $\bar{F}$ and we will complete the proof.

Now, by our assumption, $\bar{F}$ can be extended to a minimal $\bar{R}$-algebra resolution of k. Therefore, Proposition 2 of [2] shows that $\gamma(z_1,\dots, z_m)\in\bar{\mathfrak{m}}\bar{F}$ and hence $\Gamma(Z_1,\dots, Z_m)\in\mathfrak{m}F$. Thus, the inductive argument defining $\Gamma(Z_1,\dots, Z_m)$ presented in [2] works in our case since $(\mathfrak{a}F)(\mathfrak{m}F)=0$. q.e.d.

Theorem 1. *Let $(R, \mathfrak{m})$ be a local ring and let $\bar{R}=R/\mathfrak{a}$, $\mathfrak{a}\subset 0\colon\mathfrak{m}$. Assume $(R, \mathfrak{m})$ satisfies the following condition:*

(∗) *If F is a minimal R-algebra resolution of k, then $F/\mathfrak{a}F$ can be extended to a minimal $R/\mathfrak{a}$-algebra resolution of k.*

Then,

$$P_{\bar{R}}(Z)=\frac{P_R(Z)}{1-(\dim_k\mathfrak{a})Z^2P_R(Z)}.$$

PROOF. By Proposition 2, $\bar{F}=F\otimes_R\bar{R}$ has a trivial Massey operation. Hence, to prove the theorem, it is enough to see that the Eagon resolution $X=\bar{F}\otimes_{\bar{R}}T$ of k is minimal. But, since $dF\subset\mathfrak{m}F$, $\Gamma(Z_1,\dots, Z_m)\subset\mathfrak{m}F$ and $\gamma(z_1,\dots, z_m)=\Psi(\Gamma(Z_1,\dots, Z_m))$, we have

$$d\bar{F}\subset\bar{\mathfrak{m}}\bar{F}\quad\text{and}\quad \operatorname{Im}\gamma\subset\bar{\mathfrak{m}}\bar{F},$$

so that we can apply Proposition 1 in [2]. q.e.d.

Corollary 1. *Under the assumption (∗), $P_R(Z)$ is rational if and only if $P_{\bar{R}}(Z)$ is rational.*

We remark that if $R\to\bar{R}=R/\mathfrak{a}$, $\mathfrak{a}\subset 0\colon\mathfrak{m}$ is a Golod homomorphism in the sense of Levin [6] (or [7]), then, by Theorem 1.2 of [6] (or Theorem 1.5 of [7]), $\bar{F}$ is a direct summand of a minimal resolution of k over $\bar{R}$ so that our assumption (∗) is satisfied.

Therefore

Corollary 2. *Let* $(R, \mathfrak{m})$ *be a local ring and let* $\bar{R}=R/\mathfrak{a}$, $\mathfrak{a}\subset 0:\mathfrak{m}$. *If the canonical map* $R\to\bar{R}$ *is Golod, then*

$$P_{\bar{R}}(Z)=\frac{P_R(Z)}{1-(\dim_k \mathfrak{a})Z^2P_R(Z)}\,.$$

Theorem 2. *Let* $(R, \mathfrak{m})$ *be a Gorenstein local ring of embedding dimension* $n\geqq 1$, *which satisfies* $\mathfrak{m}^3=0$. *Then, R is a complete intersection or* $P_R(Z)$ *has the following form*:

$$P_R(Z)=\frac{1}{1-nZ+Z^2}\,.$$

PROOF. If $\mathfrak{m}^2=0$, then $\mathfrak{m}=0:\mathfrak{m}$ and it is a principal ideal. Hence, we must have $n=1$ and $P_R(Z)=1/1-Z$ [3, Prop. 3.4.4].

Assume $\mathfrak{m}^2\neq 0$. In this case we have $0:\mathfrak{m}=\mathfrak{m}^2$, since $\dim_k 0:\mathfrak{m}=1$. Put $\bar{R}=R/0:\mathfrak{m}=R/\mathfrak{m}^2$ and $\bar{\mathfrak{m}}=\mathfrak{m}/\mathfrak{m}^2$. As we showed in Theorem 1, $P_{\bar{R}}(Z)=P_R(Z)/1-Z^2P_R(Z)$. On the other hand, since $\bar{\mathfrak{m}}^2=0$, we have $P_{\bar{R}}(Z)=1/1-nZ$ [3, Prop. 3.4.4]. From these relations we get our formula of the Poincaré series. q.e.d.

In the following, we state some classes of local rings of embedding dimension n, to which we can apply our result.

Example 1 ([2]). *Assume R is a complete intersection and* $\bar{R}=R/0:\mathfrak{m}$. *Then,* $\bar{R}$ *is a complete intersection or* $P_{\bar{R}}(Z)$ *has the form*

$$P_{\bar{R}}(Z)=\frac{1}{(1-Z)^n-Z^2}\,,$$

since $R\to\bar{R}$ is a Golod homomorphism [7, Theorem 2.9].

Example 2 ([7]). *If there is an element* $x\in\mathfrak{m}\backslash\mathfrak{m}^2$ *which satisfies the conditions*:

i) $x^2=0$ and ii) $\mathfrak{m}^2=x\mathfrak{m}$.

Then,

$$P_R(Z)=\frac{1}{1-nZ+(\dim_k\mathfrak{m}^2)Z^2}\,.$$

PROOF. Under the conditions i) and ii), we see that $\mathfrak{m}^2\subset 0:\mathfrak{m}$ and, by Theorem 2.12 of [7], the canonical map $R\to\bar{R}=R/\mathfrak{m}^2$ is Golod so that

$$P_{\bar{R}}(Z)=\frac{P_R(Z)}{1-(\dim\mathfrak{m}^2)Z^2P_R(Z)}\,.$$

On the other hand $P_{\bar{R}}(Z)=1/1-nZ$. Hence we get our result.

Example 3 ([5]). *Suppose R is equicharacteristic and assume $\mathfrak{m}^3=0$ and* $\dim \mathfrak{m}^2=1$. *Then, $P_R(Z)$ has the form*

$$P_R(Z)=\frac{1}{1-nZ} \quad \text{or} \quad P_R(Z)=\frac{1}{1-nZ+Z^2}.$$

Proof. We can assume $n>1$ and we can choose a minimal system of generators $x_1,\dots, x_n$ of $\mathfrak{m}$ such that $x_1^2=0$ [5, Prop. 3.1]. If $x_1\mathfrak{m}=0$, the same argument as in [5] can apply. If $x_1\mathfrak{m}\neq 0$, we have $\mathfrak{m}^2=x_1\mathfrak{m}$ and hence R satisfies the conditions i) and ii) of Example 2.

Example 4 ([1], [5]). *Let $A=k[[X_{ij}]]$ $(i, j=1, 2, 3)$ be a formal power series ring over a field k with indeterminates X_{ij} and let $R=A/\Delta$ where Δ is the ideal of A generated by 2×2 subdeterminants of the matrix (X_{ij}). Then,*

$$P_R(Z)=\frac{(1+Z)^5}{1-4Z+Z^2}.$$

Proof. R is a Gorenstein local ring of Krull dimension 5 [4]. Denote x_{ij} the residue of X_{ij} in R. Then the sequence

$$\{x_{11}-x_{23},\ x_{12}-x_{31},\ x_{13}-x_{32},\ x_{21}-x_{33},\ x_{22}\}$$

is an R-sequence which consists of elements in $\mathfrak{m}\backslash\mathfrak{m}^2$, where $\mathfrak{m}$ is the maximal ideal of R. Dividing R by this sequence, we get

$$\bar{R}=k[[X_1, X_2, X_3, X_4]]/\mathfrak{a}$$

where $\mathfrak{a}=(X_1^2, X_2^2, X_3^2, X_4^2, X_1X_2, X_1X_3, X_1X_4-X_2X_3, X_2X_4, X_3X_4)$.

It is clear that $\bar{R}$ is a Gorenstein local ring with maximal ideal $\bar{\mathfrak{m}}$ such that $\bar{\mathfrak{m}}^3=0$. Hence, by Theorem 2,

$$P_{\bar{R}}(Z)=\frac{1}{1-4Z+Z^2}$$

so that

$$P_R(Z)=\frac{(1+Z)^5}{1-4Z+Z^2}.$$

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