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On the Finsler Connection Associated with a Linear Connection Satisfying $P^h_{ikj}=0$

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In a Finsler manifold provided with a linear connection $\Gamma^i_{jk}(x)$, a Finsler connection such as $\Gamma^*=\{\Gamma^i_{jk}(x),\ \Gamma^i_{lk}(x)y^l,\ C^i_{jk}\}$ can be considered, where $C^i_{jk}=\frac{1}{2}g^{im}\dot{\partial}_m g_{jk}$. The connection Γ^* is called the Finsler connection associated with $\Gamma^i_{jk}(x)$. In this case, the h-covariant derivative ∇_k, v-covariant derivative $\dot{\nabla}_k$ and the hv-curvature tensor P^h_{ikj}, which have been defined by Matsumoto [8], can be also considered.

In his paper [1], Hashiguchi has treated a Finsler manifold admitting a linear connection with respect to which $\nabla_k g_{ij}=0$ holds good. He has called the manifold as a generalized Berwald space and investigated it in detail.

On the other hand, in the previous paper [3], the present author has introduced a notion of a Finsler manifold modeled on a Minkowski space. This is a Finsler manifold with the property such that the tangent Minkowski spaces at arbitrary points are congruent to a unique Minkowski space. As this example, he has created a notion of a $\{V, H\}$-manifold and proved that the $\{V, H\}$-manifold is a generalized Berwald space.

Next, in the paper [4], he has proved that the standard generalized Berwald space is a $\{V, H\}$-manifold. He has also treated a Finsler manifold with a linear connection satisfying $\nabla_k C^h_{ij}=0$. The condition $\nabla_k C^h_{ij}=0$ is equivalent to the condition $P^h_{ikj}=0$. It is apparent that $\nabla_k g_{ij}=0$ implies $\nabla_k C^h_{ij}=0$ but the converse is not true. Even so, it is an unsolved problem whether the Finsler manifold satisfying $\nabla_k C^h_{ij}=0$ is a generalized Berwald space or not. In other words, the question is whether the Finsler manifold with a linear connection $\Gamma^i_{jk}(x)$ satisfying $P^h_{ikj}=0$ admits another linear connection $\tilde{\Gamma}^i_{jk}(x)$ with respect to which $\tilde{\nabla}_k g_{ij}=0$ holds good.

The main purpose of the present paper is to solve this problem. It will be shown that the problem is solved affirmatively under some condition. For the present aim, it is sufficient only to show that the Finsler manifold with a linear connection satisfying $P^h_{ikj}=0$ is a $\{V, H\}$-manifold. In the paper [5], the present author has clarified the geometrical significance of the condition $P^h_{ikj}=0$. The result in [5] and Kobayashi-Nomizu's well-known theorem [7] will play an important role in the development of the present discussion. The main result will be shown as Theorem 4 in the last section.

§ 1. First we introduce the well-known result which is shown in the text written by Kobayashi-Nomizu [7].

LEMMA 1. *Let G be a subgroup of the orthogonal group $O(n)$ which acts irreducibly on an n-dimensional real vector space R^n. Then every symmetric bilinear form on R^n which is invariant by G is a multiple of the standard inner product* $(x, y)=\sum_{i=1}^{n} x^i y^i$.

The proof has been shown by Kobayashi and Nomizu in the appendix of [7]. By virtue of this lemma we can prove the following lemma by the similar method as Kobayashi-Nomizu's.

LEMMA 2. *Let M be a manifold endowed with two Riemannian metrics g and g^*. If the Riemannian connection of g coincides with that of g^* and its linear holonomy group is irreducible, then there exists a positive constant c such that $g^*=cg$.*

PROOF. Let Γ be the Riemannian connection, and let x be any point of M. The linear holonomy group $H(x)$ of Γ with reference point x is a irreducible subgroup of $O(n)$, and $H(x)$ leaves both g and g^* invariant. Hence, Lemma 1 shows us that there exists a positive constant C_x such that $g^*(X,Y)=C_x g(X,Y)$ for all $X,Y\in T_x(M)$, that is, $g^*_x=C_x g_x$. Since both g^* and g are parallel tensor fields with respect to Γ, C_x is constant.

§2. Let V be a Minkowski space. That is to say, V is an n-dimensional real vector space on which a Minkowski norm is defined. In the sequel, we assume that a Minkowski norm is a real valued function on V, whose value at $\xi\in V$ we denote by $\|\xi\|$, with properties:

(1) $\|\xi\|$ can be represented explicitly by $\|\xi\|=f(\xi^1, \xi^2, \cdots, \xi^n)$ for any vector $\xi=\xi^1 e_1+\xi^2 e_2+\cdots+\xi^n e_n(=\xi^\alpha e_\alpha)$ where $\{e_\alpha\}$ is a fixed basis of V, and the function $f(\xi^1, \xi^2, \cdots, \xi^n)$ is 3-times continuously differentiable at $\xi\neq 0$. For brevity we write $f(\xi^1, \xi^2, \cdots, \xi^n)$ as $f(\xi^\alpha)$ or $f(\xi)$.

(2) $\|\xi\|\geqq 0$.

(3) $\|\xi\|=0$ if and only if $\xi=0$.

(4) $\|k\xi\|=k\|\xi\|$ for $k>0$.

(5) The quadratic form $\dfrac{\partial^2 f^2(\xi)}{\partial\xi^\alpha\partial\xi^\beta}\eta^\alpha\eta^\beta$ is positive definite for all values of η^α.

Now, if we put $g_{\alpha\beta}=\dfrac{1}{2}\dfrac{\partial^2 f^2(\xi)}{\partial\xi^\alpha\partial\xi^\beta}$ then $g_{\alpha\beta}(\xi)d\xi^\alpha d\xi^\beta$ gives V a positive definite Riemannian metric. The coefficients of the Riemannian connection are given by $C^{\alpha}_{\beta\gamma}=\frac{1}{2}g^{\alpha\sigma}\frac{\partial g_{\beta\gamma}}{\partial\xi^{\sigma}}$ where $(g^{\alpha\beta})$ is the inverse of $(g_{\alpha\beta})$.

We call, hereafter, that the Minkowski space V is irreducible when the linear holonomy group with respect to the Riemannian connection $C^{\alpha}_{\beta\gamma}$ is irreducible. This is well-defined because of

THEOREM 1. *Let $\{e_{\alpha}\}$ and $\{\bar{e}_{\alpha}\}$ be two bases of a Minkowski space V. Let us express the Minkowski norm, with respect to the bases $\{e_{\alpha}\}$ and $\{\bar{e}_{\alpha}\}$, by the function $f(\xi)$ and $\bar{f}(\bar{\xi})$ respectively. Then the Riemannian metrics derived from $f(\xi)$ and $\bar{f}(\bar{\xi})$ are mutually equivalent.*

PROOF. As a matter of course, there exist n^2-constants A^{α}_{β} such that $\bar{e}_{\beta}=A^{\alpha}_{\beta}e_{\alpha}$. Then, for a vector $\xi=\xi^{\alpha}e_{\alpha}=\bar{\xi}^{\beta}\bar{e}_{\beta}$, we have $\xi^{\alpha}=A^{\alpha}_{\beta}\bar{\xi}^{\beta}$. Since $\|\xi\|=f(\xi^{\alpha})=\bar{f}(\bar{\xi}^{\alpha})$, we have $\bar{g}_{\alpha\beta}=g_{\sigma\tau}A^{\sigma}_{\alpha}A^{\tau}_{\beta}$. Hence, for all $X=X^{\alpha}e_{\alpha}=\bar{X}^{\alpha}\bar{e}_{\alpha}$ and $Y=Y^{\alpha}e_{\alpha}=\bar{Y}^{\alpha}\bar{e}_{\alpha}$, we obtain $\bar{g}(X, Y)=g(X, Y)$.

Q. E. D.

Now we shall prove

THEOREM 2. *Let V and $\tilde{V}$ be n-dimensional Minkowski spaces and let g and $\tilde{g}$ be the Riemannian metrics derived from the Minkowski norm of V and $\tilde{V}$ respectively. Let φ be a linear isomorphic mapping from V onto $\tilde{V}$, and g^{*} be the Riemannian metric of V induced by φ from $\tilde{V}$. If the Minkowski space V is irreducible and the Riemannian metrics g and g^{*} determine the same Riemannian connection, then there exists a positive constant k such that $\|\varphi(\xi)\|_{\tilde{V}}=k\|\xi\|_{V}$ for any $\xi\in V$.*

PROOF. Let f be the norm function of V with respect to a basis $\{e_{\alpha}\}$. Due to Theorem 1, we may adopt $\{\varphi(e_{\alpha})\}$ as a basis of $\tilde{V}$ without loss of generality. Denote by $\tilde{f}$ the norm function of $\tilde{V}$ with respect to $\{\varphi(e_{\alpha})\}$, then we have, for any $\xi\in V$,

$$\|\varphi(\xi)\|_{\tilde{V}}=\|\varphi(\xi^{\alpha}e_{\alpha})\|_{\tilde{V}}=\|\xi^{\alpha}\varphi(e_{\alpha})\|_{\tilde{V}}=\tilde{f}(\xi).$$

That is, we may adopt $\{\xi^{\alpha}\}$ as a current coordinate system of V and $\tilde{V}$ in common. Now let $\tilde{g}_{\alpha\beta}(\xi)$ be the Riemannian metric derived from $\tilde{f}(\xi)$ and let $\tilde{C}^{\alpha}_{\beta\gamma}(\xi)$ be the Riemanian connecttion of $\tilde{g}_{\alpha\beta}(\xi)$. For any $p=\xi=\xi^{\alpha}e_{\alpha}\in V$, we put $q=\varphi(p)=\xi^{\alpha}\varphi(e_{\alpha})$. Then we have $\varphi((\frac{\partial}{\partial\xi^{\alpha}})_p)=(\frac{\partial}{\partial\xi^{\alpha}})_q$. Now, by virtue of Lemma 2 and our assumption, we see that there exists a positive constant k such that $g^{*}=k^2g$. From the definition of g^{*}, we have

$$k^2g_{\alpha\beta}(\xi)=g^{*}_{\alpha\beta}(\xi)=g^{*}_{p}((\frac{\partial}{\partial\xi^{\alpha}})_p, (\frac{\partial}{\partial\xi^{\beta}})_p)=\tilde{g}_q(\varphi((\frac{\partial}{\partial\xi^{\alpha}})_p), \varphi((\frac{\partial}{\partial\xi^{\beta}})_p))$$

$$=\tilde{g}_q((\frac{\partial}{\partial\xi^\alpha})_q, (\frac{\partial}{\partial\xi^\beta})_q)=\tilde{g}_{\alpha\beta}(\xi).$$

Thus we have $k^2 g_{\alpha\beta}(\xi)\xi^\alpha\xi^\beta=\tilde{g}_{\alpha\beta}(\xi)\xi^\alpha\xi^\beta$, that is, $k^2 f^2(\xi)=\tilde{f}^2(\xi)$. Hence we obtain $k\|\xi\|_V=\|\varphi(\xi)\|_{\tilde{V}}$.

Q. E. D.

§3. Let M be an n-dimensinal connected Finsler manifold whose fundamental function is $F(x, y)$. We denote the metric tensor by $g_{ij}(x, y)$, and we put $C^i_{jk}=\frac{1}{2}g^{im}\frac{\partial g_{jk}}{\partial y^m}$. We assume moreover that a linear connection $\Gamma^i_{jk}(x)$ is endowed globally on M. Then we can introduce a Finsler connection such as $\Gamma^*=\{\Gamma^i_{jk}(x), \Gamma^i_{lk}(x)y^l, C^i_{jk}\}$. This connection Γ^* is called the Finsler connection associated with the linear connection $\Gamma^i_{jk}(x)$ ([2], [4]). With respect to Γ^*, the h- and v-covariant derivatives are expressed respectively as follows:

$$\nabla_k K^i_j=\partial_k K^i_j-\dot{\partial}_m K^i_j\Gamma^m_{lk}y^l+\Gamma^i_{mk}K^m_j-K^i_m\Gamma^m_{jk},$$
$$\dot{\nabla}_k K^i_j=\dot{\partial}_k K^i_j+C^i_{mk}K^m_j-K^i_m C^m_{jk},$$

where K^i_j is a Finsler tensor of type (1, 1) and ∂_k and $\dot{\partial}_k$ stand for $\frac{\partial}{\partial x^k}$ and $\frac{\partial}{\partial y^k}$ respectively. The hv-curvature tensor P^i_{jrk} with respect to a Finsler connection $\{N^i_{jk}, N^i_j, C^i_{jk}\}$, which has been defined by Matsumoto [8], is written in the form

$$P^i_{jrk}=\dot{\partial}_k N^i_{jr}-\nabla_r C^i_{jk}+C^i_{js}(\dot{\partial}_k N^s_r-N^s_{kr}).$$

Since $N^i_{jk}=\Gamma^i_{jk}(x)$ and $N^i_j=\Gamma^i_{lj}(x)y^l$ for the associated Finsler connection under consideration, it follows that $P^i_{jrk}=-\nabla_r C^i_{jk}$.

In the paper [5], the geometrical significance for the condition $P^i_{jrk}=0$ has been found.

Now, in a Finsler manifold M, the tangent space $T_p(M)$ at each point $P=(x^i_0)$ of M can be regarded as a Minkowski space, where the norm of any vector $y=y^i(\frac{\partial}{\partial x^i})_p$ is given by $\|y\|=F(x^i_0, y^i)$. Therefore $T_p(M)$ is called the tangent Minkowski space at p. But the tangent Minkowski spaces at two distinct points are, in general, not congruent (isometrically linearly isomorphic). Especially, a Finsler manifold with the property that the tangent Minkowski spaces at arbitrary points are congruent to a unique Minkowski space is called a Finsler manifold modeled on a Minkowski space [3].

Now we shall prove

THEOREM 3. *Let M be a Finsler manifold provided with a linear connection* $\Gamma^i_{jk}(x)$. *We assume that M is connected and there exists in M such a point p that the tangent Minkowski space at p is irreducible, and we assume moreover that the hv-curvature tensor* P^h_{ikj} *with respect to the Finsler connection associated with* $\Gamma^i_{jk}(x)$ *vanishes (or equivalently* $\nabla_k C^h_{ij}=0$). *Then M is a Finsler manifold modeled on a Minkowski space.*

PROOF. Since M is connected, any point q of M can be joined with p by a piecewise differentiable curve l. Let us denote l by $l=\{x(t)\,|\,x(0)=p, x(1)=q\}$. For any vector $X_p \in T_p(M)$, we consider the ordinary parallel displacement of X_p along l with respect to the linear connection $\Gamma^i_{jk}(x)$. Let Y_q be a tangent vector at q given by the above parallel displacement of X_p. Then the correspondence $X_p \rightarrow Y_q$ defines a linear isomorphic mapping $\varphi : T_p(M) \rightarrow T_q(M)$.

On the other hand, in [5], the following has been proved:

The condition $P^i_{jrk}=0$ implies that, relating to the mapping $\varphi : T_p(M) \rightarrow T_q(M)$, the induced connection of the Riemannian connection $C^i_{jk}(x(1), \varphi(y))$ coincides with the Riemannian connection $C^i_{jk}(x(0), y)$.

Now, let g and $\tilde{g}$ be the Riemannian metrics derived from the tangent Minkowski metrics of $T_p(M)$ and $T_q(M)$ respectively. Let g^* be the induced metric of $\tilde{g}$ by the mapping φ. Then the mapping φ is an isometry from the Riemannian space $\{T_p(M), g^*\}$ to the Riemannian space $\{T_q(M), \tilde{g}\}$. Since an isometry is an affine mapping, the Riemannian connection of g^* coincides with the induced connection of $C^i_{jk}(x(1), \varphi(y))$. It follows, therefore, that the Riemannian connection of g^* coincides with $C^i_{jk}(x(0), y)$. By virtue of our assumption and Theorem 2, we have that there exists a positive constant k such that $\|\varphi(y)\|_{T_q(M)} = k\|y\|_{T_p(M)}$. Then we have $F(x(0), y)=F(x(1), \frac{1}{k}\varphi(y))$. Now, let us consider the mapping $\psi: T_p(M) \rightarrow T_q(M)$ $(\psi(y)=\frac{1}{k}\varphi(y), \forall y \in T_p(M))$. Then ψ is a linear isomorphic mapping from $T_p(M)$ onto $T_q(M)$ and is an isometry. Thus, for any $q \in M$, the tangent Minkowski space $T_q(M)$ is congruent to the tangent Minkowski space $T_p(M)$. That is to say, M is a Finsler manifold modeled on a Minkowski space.

Q. E. D.

§4. Let V be a Minkowski space and f be the norm function of V. Let us put

$$G=\{T\,|\,T \in GL(n, R),\ \|T\xi\|=\|\xi\| \text{ for any } \xi \in V\}.$$

Then G is a linear Lie group [3]. Let H be a Lie subgroup of G. If a manifold M admits the H-structure in the sence of a G-structure, then M admits a Finsler metric such as $F(x^i, y^i) = f(\mu^a_i(x)\, y^i)$ where $\mu^a_i(x)$ are n linearly independent local covariant vectors. This is called a $\{V, H\}$-Finsler metric and M is called a $\{V, H\}$-manifold [3]. Moreover, in [3], it has been shown that a $\{V, H\}$-manifold is a generalized Berwald space with respect to a G-connection relating to the H-structure. The generalized Berwald space, which has been defined by Hashiguchi [1], is a Finsler space admitting a linear connection $\Gamma(x)$ satisfying the condition $\nabla_k g_{ij}=0$ with respect to the Finsler connection associated with the $\Gamma(x)$. It has been also proved in [4] that a generalized Berwald space is a $\{V, H\}$-manifold.

Now, we shall prove

THEOREM 4. *Let M be the manifold under the same assumption as in Theorem 3. Let V be the tangent Minkowski space at p and let G be the linear Lie group which consists of all linear transformations leaving the norm of V invariant. Then M is a $\{V, G\}$-manifold.*

PROOF. Owing to Theorem 3, M is a Finsler manifold modeled on a Minkowski space V. Let $\{U, x^i\}$ be a coordinate neighbourhood such that $p\in U$. If we put $p=(x_0^i)$ and $F(x_0^i, y^i)=f(y^i)$, then $f(y^i)$ is the norm function of V relating to the basis $\{(\frac{\partial}{\partial x^i})_p\}$.

First, we consider a piecewise differentiable closed curve l starting at p. Let ψ_p be the congruent mapping from $T_p(M)$ to $T_p(M)$ along l as shown in the proof of Theorem 3. If we put $\psi_p((\frac{\partial}{\partial x^i})_p)=(X_i)_p$, then we can re-write it as $(X_i)_p=g_i^j(\frac{\partial}{\partial x^j})_p$. And we see that $\psi_p(y^i(\frac{\partial}{\partial x^i})_p)=g_j^i y^j(\frac{\partial}{\partial x^i})_p$ and $F(x_0^i, y^i)=F(x_0^i, g_j^i y^j)$, i.e., $f(g_j^i y^j)=f(y^i)$. Hence we have

$$\psi_p((\frac{\partial}{\partial x^i})_p)=g_i^j(\frac{\partial}{\partial x^j})_p \quad \text{where } (g_i^j)\in G.$$

Next, we consider a point q in M different from p. Let l_1 and l_2 be two distinct piecewise differentiable curve joining p with q. Let ψ_1 and ψ_2 be the congruent mappings from $T_p(M)$ to $T_q(M)$ along l_1 and l_2 respectively which are defined in the proof of Theorem 3. If we put $\psi_1((\frac{\partial}{\partial x^i})_p)=(X_i)_q$ and $\psi_2((\frac{\partial}{\partial x^i})_p)=(\tilde{X}_i)_q$, then we have $(\tilde{X}_i)_q=\tilde{g}_i^j(X_j)_q$ where $(\tilde{g}_i^j)\in GL(n, R)$. Let $\{\bar{U}, \bar{x}^i\}$ be a coordinate neighbourhood such that $q=(\bar{x}_1^i)\in\bar{U}$, and let us express $(X_i)_q$ by $(X_i)_q=p_i^k(\frac{\partial}{\partial\bar{x}^k})_q$. Then we have that $\psi_1(y^i(\frac{\partial}{\partial x^i})_p)=p_j^i y^j(\frac{\partial}{\partial\bar{x}^i})_q$ and $\psi_2(y^i(\frac{\partial}{\partial x^i})_p)=p_j^i\tilde{g}_k^j y^k(\frac{\partial}{\partial\bar{x}^i})_q$. Hence we have

$$f(y^i)=F(x_0^i, y^i)=\bar{F}(\bar{x}_1^i, p_j^i y^j)=\bar{F}(\bar{x}_1^i, p_j^i\tilde{g}_k^j y^k)=f(\tilde{g}_k^i y^k),$$

that is, $(\tilde{g}_i^j)\in G$. Thus we have

$$\psi_2((\frac{\partial}{\partial x^i})_p)=\tilde{g}^j_i\,\psi_1((\frac{\partial}{\partial x^j})_p) \quad \text{where } (\tilde{g}^j_i)\in G.$$

Hence we obtain that the manifold M admits the G-structure.

Moreover, for any $\bar{y}^i(\frac{\partial}{\partial \bar{x}^i})_q$, we see that

$$\bar{F}(\bar{x}^i,\ \bar{y}^i)=\bar{F}(\bar{x}^i,\ p^i_j\,\overset{-1}{p}{}^j_k\,\bar{y}^k)=f(\overset{-1}{p}{}^i_j\,\bar{y}^j).$$

Hence we obtain that the Finsler metric of M is the $\{V,\ G\}$-Finsler metric.

Q. E. D.

Since any $\{V,\ H\}$-manifold is a generalized Berwald space, this theorem can be rewritten as

COROLLARY. *Let M be the manifold under the same assumption as in Theorem 3. Then there exists globally in M such a linear connection $\tilde{\Gamma}^i_{jk}(x)$ that $\tilde{\nabla}_h g_{jk}=0$ holds good with respect to the Finsler connection associated with $\tilde{\Gamma}^i_{jk}(x)$.*

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