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A Note on Bi-sequential Spaces

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§1. Introduction

For the following properties of topological spaces, it is well known that (a) implies (b), and (b) implies (c).

(a) X is a Hausdorff space.

(b) Each compact subset in X is closed.

(c) Each convergent sequence in X has exactly one sequential limit.

H.F.Cullen showed that these three properties are equivalent if X is a first countable space ([1] Theorem 2, Corollaries 1,2).

In this paper, we shall prove the following:

THEOREM. *Let X be a bi-sequential space. Then the properties* (a), (b), (c) *are equivalent.*

It is easily shown that a first countable space is a bi-sequential space. Since there is a bi-sequential space which is not first countable ([3] Example 10.4), our theorem is a generalization of Cullen's theorem.

§2. Definitions

A *filter base* $\mathscr{F}$ is a non-empty collection of non-empty sets such that F_1, $F_2 \in \mathscr{F}$ implies that $F_3 \subset F_1 \cap F_2$ for some $F_3 \in \mathscr{F}$. In a topological space Y, a filter base $\mathscr{F}$ *accumulates* at a point y if $y \in \overline{F}$ for every $F \in \mathscr{F}$, and $\mathscr{F}$ *converges* to y if every neighborhood of y in Y contains some $F \in \mathscr{F}$. We say that two filter bases $\mathscr{F}$ and $\mathscr{G}$ *mesh* if every $F \in \mathscr{F}$ intersects every $G \in \mathscr{G}$. An *ultrafilter* is a maximal filter base. It should be noted that every filter base accumulating at a point y is contained in an ultrafilter converging to y.

A space Y is *bi-sequential* if, whenever a filter base $\mathscr{F}$ accumulates at y in Y, then there is a decreasing sequence $\{A_n\}$ in Y which meshes with $\mathscr{F}$ and converges to y. It is easily verified that Y is bi-sequential if and only if every ultrafilter $\mathscr{F}$ converging to a point $y \in Y$ contains a decreasing sequence $\{A_n\}$ converging to y ([3]).

§3. Proof of the theorem.

It is sufficient to prove that (c) implies (a). Note that a space in which every filter base converges to at most one point is Hausdorff. Suppose that X is not Hausdorff. Then there exist two distinct points x, y and a filter base $\mathscr{F}'$ which converges to both x and y. Let $\mathscr{F}$ be an ultrafilter which contains $\mathscr{F}'$. Since X is bi-sequential, there exist two decreasing sequences $\{A_n\}$ and $\{B_n\}$ which are contained in $\mathscr{F}$ and converges to x and y, respectively. Then $A_n \cap B_n \neq \phi$ for each n by the definition of the filter. Choosing x_n in $A_n \cap B_n$ for each n, we obtain a sequence $\{x_n\}$. Since $\{A_n\}$ converges to x, any neighborhood U of x contains $A_{n(0)}$ for some natural number $n(0)$. Since $\{A_n\}$ is a decreasing sequence, $A_n \subset A_{n(0)}$ and $x_n \in A_n \subset A_{n(0)} \subset U$ for each $n \geqq n(0)$. Thus the sequence $\{x_n\}$ converges to x. Similarly $\{x_n\}$ converges to y. This is a contradiction. Thus X is Hausdorff.

Remarks.

1. It is known that a locally countably compact (locally sequentially compact, locally compact) sequential space X with unique sequential limits is Hausdorff ([2] 5.6 Proposition). But the following example of a bi-sequential space with unique sequential limits which is not locally countably compact shows that the above-cited proposition does not imply our theorem.

Example. ([4] Example 58) Let $X = \boldsymbol{Z}$ be the set of all integers with the topology generated by the sets of the form $a + k\boldsymbol{Z} = \{a + km \mid a, m \in \boldsymbol{Z}, k \in N\}$, where N is the set of all natural numbers. The base sets are simply the cosets of factor groups of $\boldsymbol{Z}$ and each base set is closed. Then X is a first countable Hausborff space and hence, X is a bi-sequential space with unique sequential limits. The set of primes is an infinite set without a limit point, so X is not countably compact. The function $a + km \longrightarrow m$ defines a homeomorphism between any base element and the entire space X. Therefore no neighborhood of any point is countably compact and X is not locally countably compact.

2. A bi-sequential space is a Fréchet space and a Fréchet space is a sequential space. However we cannot replace in our theorem the term "bi-sequential space" by "sequential space" or "Fréchet space", since there exist a sequential space and a Fréchet space with unique sequential limits which are not Hausdorff ([2] 5.3 Example, 6.2 Example).

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