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Finsler Manifolds with a Linear Connection

By

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In the previous paper [5], the present author has treated Finsler manifolds with such a property that the tangent spaces at arbitrary points are congruent (isometrically linearly isomorphic) to a single Minkowski space, and introduced, as a typical example of such a space, the notion of $\{V, H\}$-manifolds. At the same time, it has been shown that the $\{V, H\}$-manifolds are generalized Berwald spaces defined by Hashiguchi [3] and Wagner [9].

Now, the present paper has two main purposes. One is to consider the converse of the above-mentioned result. After some preparation, it will be proved, in §3, that if a generalized Berwald space is connected, it is actually a $\{V, H\}$-manifold.

Next, in a Minkowski space is presented a Riemann metric, which is different from the Minkowski norm. Therefore, geodesic lines with respect to this Riemann metric can be introduced in the Minkowski space, which we call C-geodesics.

A connected Finsler manifold M with a linear connection $\Gamma^i_{jk}(x)$ is to be considered. With regard to arbitrary two points p and q in M and any piecewise differentiable curve C joining p and q, we can define a linear isomorphic mapping σ between the tangent Minkowski spaces $T_p(M)$ and $T_q(M)$ by parallel displacement with respect to $\Gamma^i_{jk}(x)$ along the curve C.

Now, the other main purpose of the present paper is to find a necessary and sufficient condition for σ to map any C-geodesic in $T_p(M)$ to a C-geodesic in $T_q(M)$. It will be shown, in the last section, that the condition is $C^i_{jk|h}=0$ or equivalently $P^i_{jkh}=0$ with respect to the Finser connection associated with the linear connection $\Gamma^i_{jk}(x)$.

§1. Minkowski spaces

Let V be an n-dimensional Minkowski space, that is to say, an n-dimensional linear space on which a *Minkowski norm* is defined. In this paper, a Minkowski norm on a linear space V means a real valued function on V, whose value at $\xi \epsilon V$ we denote by $\|\xi\|$, with properties:

(1) $\|\xi\|$ can be represented explicitly by

$$\|\xi\|=f(\xi^1, \xi^2, \cdots\cdots, \xi^n)$$

for any vector $\xi=\xi^1 e_1+\xi^2 e_2+\cdots\cdots+\xi^n e_n(=\xi^\alpha e_\alpha)$ where $\{e_\alpha\}$ is a fixed basis of V, and the function $f(\xi^1, \xi^2, \cdots\cdots, \xi^n)$ is 3-times continuously differentiable at $\xi \neq 0$. For brevity we write $f(\xi^1, \xi^2, \cdots\cdots, \xi^n)$ as $f(\xi)$ or $f(\xi^\alpha)$.

(2) $\|\xi\| \geqq 0$.

(3) $\|\xi\|=0$ if and only if $\xi=0$.

(4) $\|k\xi\|=k\|\xi\|$ for $k>0$.

(5) The quadratic form

$$\frac{\partial^2 f^2(\xi)}{\partial \xi^\alpha \partial \xi^\beta} \zeta^\alpha \zeta^\beta$$

is positive definite for all values of ζ^α.

It is to be remarked here that the condition (5) leads us to

$$\|\xi_1+\xi_2\| \leqq \|\xi_1\|+\|\xi_2\| ,$$

but the converse is not true [8].

Now, if we put

$$G=\{T \mid T \epsilon GL(n, R),\ \|T\xi\|=\|\xi\| \text{ for any } \xi \epsilon V\},$$

then G is a Lie group [5].

As a matter of course, the Minkowski space V admits a Minkowski metric $\|\xi\|$, and furthermore, it admits a Riemann metric like the following:

If we put

$$g_{\alpha\beta}=\frac{1}{2} \frac{\partial^2 f^2(\xi)}{\partial \xi^\alpha \partial \xi^\beta} \tag{1.1}$$

then $g_{\alpha\beta}(\xi)$ is a tensor field on V and $g_{\alpha\beta}(\xi) d\xi^\alpha d\xi^\beta$ is a positive definite Riemann metric on V.

The distance $\overline{AB}_{(R)}$ between two points A and B with respect to the Riemann metric, however, does not coincide with the Minkowski norm $\|\overrightarrow{AB}\|$ in general.

Now we put

$$C_{\alpha\beta\gamma}=\frac{1}{2} \frac{\partial g_{\alpha\beta}}{\partial \xi^\gamma} \quad \text{and} \quad C^\alpha_{\beta\gamma}=g^{\alpha\sigma} C_{\sigma\beta\gamma}, \tag{1.2}$$

where $(g^{\alpha\beta})$ is the inverse of $(g_{\alpha\beta})$, then $C^{\ \alpha}_{\beta\gamma}$ is the Christoffel symbol with respect to the Riemann metric $g_{\alpha\beta}$.

With respect to this Riemann metric, we may consider geodesic lines, which we call *C-geodesics* in a Minkowski space. Generally speaking, a C-geodesic is not always a straight line.

Let V and $\tilde{V}$ be n-dimensional Minkowski spaces and σ be a linear isomorphic mapping from V to $\tilde{V}$. We shall take $\{\sigma(e_\alpha)\}$ as a basis of $\tilde{V}$ and denote the norm function of $\tilde{V}$ with respect to this basis by $\tilde{f}$. Then it is easily seen that

$$\|\sigma(\xi)\|_{\tilde{V}}=\|\sigma(\xi^\alpha e_\alpha)\|_{\tilde{V}}=\|\xi^\alpha\sigma(e_\alpha)\|_{\tilde{V}}=\tilde{f}(\xi).$$

Hence we can take $\{\xi^\alpha\}$ a current coordinate system of V and $\tilde{V}$ in common.

If the relation $f=\tilde{f}$ holds good, then the mapping σ is an isometry in accordance with $\|\sigma(\xi)\|_{\tilde{V}}=\|\xi\|_V$. Hence V is congruent to $\tilde{V}$ in this case.

If the linear mapping σ maps every C-geodesic in V to the C-geodesic in $\tilde{V}$, then the mapping σ is said to be a *C-projective mapping.* Two Minkowski spaces V and $\tilde{V}$ are said to be *C-projective* mutually if there exists a C-projective isomorphic mapping from V to $\tilde{V}$. It is obvious that an isometry is C-projective.

Now, let us seek for a condition that a linear isomorphic mapping σ from V to $\tilde{V}$ is C-projective. For this purpose, we denote by $\tilde{g}_{\alpha\beta}$ and $\tilde{C}^\alpha_{\beta\gamma}$ the metric tensor and the Christoffel symbol of $\tilde{V}$ with respect to the above mentioned current coordinate system $\{\xi^\alpha\}$, namely, $g_{\alpha\beta}=\frac{1}{2}\frac{\partial^2 f^2(\xi)}{\partial\xi^\alpha\partial\xi^\beta}$ and $C^\alpha_{\beta\gamma}=\frac{1}{2}g^{\alpha\sigma}\frac{\partial g_{\beta\gamma}}{\partial\xi^\sigma}$. In such a case, a necessary and sufficient condition for σ to be C-projective is, as is well known, that $\tilde{C}^\alpha_{\beta\gamma}$ has the form

$$\tilde{C}^\alpha_{\beta\gamma}=C^\alpha_{\beta\gamma}+\delta^\alpha_\beta\varphi_\gamma+\delta^\alpha_\gamma\varphi_\beta, \tag{1.3}$$

for a suitable covariant vector field φ_γ on V. In our case, by virtue of the homogenity condition, the identities

$$C^\alpha_{\beta\gamma}\xi^\gamma=\tilde{C}^\alpha_{\beta\gamma}\xi^\gamma=0$$

hold true. Hence, from (1.3), we have

$$\delta^\alpha_\beta\varphi_\gamma\xi^\gamma+\xi^\alpha\varphi_\beta=0. \tag{1.4}$$

Summing for α and β, we get $\varphi_\gamma\xi^\gamma=0$. Then (1.4) leads us to $\xi^\alpha\varphi_\beta=0$. Thus we get $\varphi_\alpha=0$, that is,

$$\tilde{C}^\alpha_{\beta\gamma}=C^\alpha_{\beta\gamma}. \tag{1.5}$$

Conversely, if $\tilde{C}^\alpha_{\beta\gamma}=C^\alpha_{\beta\gamma}$, it is obvious that V is C-projective to $\tilde{V}$. Thus we obtain

THEOREM 1. *Let V and $\tilde{V}$ be n-dimensional Minkowski spaces, σ be a linear isomorphic mapping from V to $\tilde{V}$, and $\{e_\alpha\}$ be the basis of V. Let us take*

$\{\sigma(e_\alpha)\}$ as a basis of $\widetilde{V}$. Then the mapping σ is C-projective if and only if the relation $\widetilde{C}^\alpha_{\beta\gamma}=C^\alpha_{\beta\gamma}$ holds good with respect to these bases.

§2. Finsler manifolds with a linear connection

Let M be an n-dimensional connected Finsler manifold whose fundamental function is $F(x, y)$. We denote the metric tensor and C-tensor by

$$(2.1)\qquad g_{ij}=\frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j} \text{ and } C_{ijk}=\frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}.$$

In this section, we assume moreover that M admits a linear connection $\Gamma^i_{jk}(x)$. Then we can introduce such a Finsler connection $\Gamma^*=\{F^i_{jk}, N^i_j, C^i_{jk}\}$ ([3], [4], [7]) as

$$(2.2)\qquad F^i_{jk}=\Gamma^i_{jk}(x),\quad N^i_j=\Gamma^i_{lj}(x)y^l,\quad C^i_{jk}=g^{il}C_{ljk}.$$

The connection Γ^* is called the *Finsler connection associated with the linear connection* $\Gamma^i_{jk}(x)$. Then h- and v-covariant derivatives with respect to Γ^* are expressed respectively as follows:

$$T^i_{j|k}=\partial_k T^i_j-\dot{\partial}_m T^i_j \Gamma^m_{lk} y^l+T^m_j \Gamma^i_{mk}-T^i_m \Gamma^m_{jk},$$
$$T^i_j|_k=\dot{\partial}_k T^i_j+T^m_j C^i_{mk}-T^i_m C^m_{jk},$$

where T^i_j is a Finsler tensor of type (1,1), and ∂_k and $\dot{\partial}_k$ mean $\frac{\partial}{\partial x^k}$ and $\frac{\partial}{\partial y^k}$ respectively.

If T^i_j is a tensor field of M, namely, T^i_j depends on position alone, then $\dot{\partial}_k T^i_j$ vanishes, and it holds that

$$T^i_{j|k}=\nabla_k T^i_j,$$

where ∇_k denotes the ordinary covariant differentiation with respect to $\Gamma^i_{jk}(x)$.

Let $C=\{x(t)\}$ be a piecewise differentiable curve in M. We pick up arbitrarily two points $x(a)$ and $x(b)$ on C, and consider a tangent vector $\xi_{(a)}$ at $x(a)$. Let $\xi_{(b)}$ be a tangent vector at $x(b)$ given by parallel displacement of $\xi_{(a)}$ along the curve C with respect to $\Gamma^i_{jk}(x)$. Then the correspondence $\xi_{(a)}\rightarrow\xi_{(b)}$ defines a differentiable mapping

$$(2.3)\qquad \sigma_{ab}: T_{x(a)}(M)\rightarrow T_{x(b)}(M)\quad (\sigma_{ab}(\xi_{(a)})=\xi_{(b)}),$$

where $T_{x(a)}(M)$ and $T_{x(b)}(M)$ are tangent spaces of M at $x(a)$ and $x(b)$ respectively. It follows apparently that σ_{ab} is a bijective linear mapping. In the sequel, we simply call σ_{ab} a *parallel displacement along* C.

We set, for brevity, $a=0$, $b=t$ and $Y(0)=y^i(0)\frac{\partial}{\partial x^i}$. On putting $Y(t)=\sigma_{0t}Y(0)$, $Y(t)$ is a vector field on the curve C. If we put $x'(t)=\frac{dx^i}{dt}\frac{\partial}{\partial x^i}$, then

$x'(t)$ is a tangent vector of the curve C.

For the covariant derivative of a vector field $v(t)$ along C with respect to $x'(t)$, we have, as is well-known, that

$$\nabla_{x'} v(0)=\lim_{t\to 0} \frac{1}{t}\{\sigma_{0t}^{-1} v(t)-v(0)\}=\{(\frac{dv^i(t)}{dt}+\Gamma^i_{jk}(x(t))\, v^j(t)\frac{dx^k}{dt})\frac{\partial}{\partial x^i}\}_{x(0)}.$$

In regard to $Y(t)$, however, it follows clearly that $\nabla_{x'} Y(t)=0$, i.e.,

$$\frac{dY^i(t)}{dt}+\Gamma^i_{jk}(x(t))\, Y^j(t)\frac{dx^k}{dt}=0. \tag{2.4}$$

Next, we shall be concerned with the covariant derivative of a Finsler tensor field T. For instance, we assume T is of type $(1,2)$, i.e.,

$$T(x, y)=T^i_{jk}(x, y)\frac{\partial}{\partial x^i}\otimes dx^j\otimes dx^k. \tag{2.5}$$

Let $T^*_{x(t)}(M)$ be the dual space of $T_{x(t)}(M)$ and σ^*_{0t} be the dual mapping of σ_{0t}. As $Y(t)=\sigma_{0t} Y(0)$, we can define a mapping $\tilde{\sigma}_{t0}$ by

$$\tilde{\sigma}_{t0}\, T(x(t), Y(t))=T^i_{jk}(x(t), Y(t))\,\sigma_{0t}^{-1}(\frac{\partial}{\partial x^i})_{x(t)}\otimes\sigma^*_{0t}(dx^j)_{x(t)}\otimes\sigma^*_{0t}(dx^k)_{x(t)}. \tag{2.6}$$

Then, with respect to the vector $x'(t)$, the covariant derivative of T can be defined by

$$(\nabla_{x'} T)_{x(0)}=\lim_{t\to 0} \frac{1}{t}(\tilde{\sigma}_{t0} T(x(t), Y(t))-T(x(0), Y(0))). \tag{2.7}$$

Using (2.4), we can rewrite (2.7) as

$$\begin{aligned}(\nabla_{x'} T)_{x(0)}=\Big\{&(\frac{d}{dt}T^i_{jk}(x(t), Y(t)))_{t=0}\\ &+\Gamma^i_{ml}(x(0))\,T^m_{jk}(x(0), Y(0))\left(\frac{dx^l}{dt}\right)_{x=0}\\ &-\Gamma^m_{jl}(x(0))\,T^i_{mk}(x(0), Y(0))\left(\frac{dx^l}{dt}\right)_{t=0}\\ &-\Gamma^m_{kl}(x(0))\,T^i_{jm}(x(0), Y(0))\left(\frac{dx^l}{dt}\right)_{t=0}\Big\}\left(\frac{\partial}{\partial x^i}\right)_{x(0)}\otimes(dx^j)_{x(0)}\otimes(dx^k)_{x(0)}\end{aligned}$$

Hence we obtain

$$\nabla_{x'} T=(T^i_{jk|l}\;\frac{dx^l}{dt})\frac{\partial}{\partial x^i}\otimes dx^j\otimes dx^k. \tag{2.8}$$

Following Hashiguchi's definition [3], a Finsler manifold is said to be a *generalized Berwald space* if it is possible to introduce such a linear connection $\Gamma^i_{jk}(x)$ that the metric tensor g_{ij} is h-covariant constant with respect to the Finsler connection associated with the linear connection. As to the generalized Berwald space, we have obtained in the paper [4] that

THEOREM 2. *A Finsler manifold with a linear connection* $\Gamma^i_{jk}(x)$ *is a generalized Berwald space with respect to the Finsler connection* Γ^* *associated with* $\Gamma^i_{jk}(x)$ *if and only if* $F_{|k}=0$ *holds.*

PROOF. Since components of the linear connection Γ^{i}_{jk} are functions of position alone, the relation $(\dot{\partial}_h T)_{|k}=\dot{\partial}_h(T_{|k})$ holds true. Therefore we have $g_{ij|k}=\dot{\partial}_i\dot{\partial}_j(F\,F_{|k})$. Hence, $F_{|k}=0$ implies $g_{ij|k}=0$. Conversely, by virtue of $y^i{}_{|k}=0$, we have $2FF_{|k}=g_{ij|k}\,y^iy^j$, which completes the proof.

In a Finsler manifold with a linear connection $\Gamma^{i}_{jk}(x)$, if the Finsler connection Γ^* associated with $\Gamma^{i}_{jk}(x)$ satisfies the condition $g_{ij|k}=0$ or $F_{|k}=0$, we say that the manifold admits a (F, Γ, g)-*structure*. If Γ^* satisfies the condition $C^{i}_{jk|h}=0$, we say that the manifold admits a (F, Γ, C)-*structure*.

If a Finsler manifold with a linear connection $\Gamma^{i}_{jk}(x)$ admits a (F, Γ, g) structure, then the manifold is a generalized Berwald space and also admits a (F, Γ, C)-structure obviously.

Now, in a Finsler manifold M, whose metric function is $F(x, y)$, the tangent space $T_p(M)$ at each point $p=(x^i_0)$ of M can be regarded as a Minkowski space, where the norm of any vector $v\in T_p(M)$ is given by $\|v\|=F(x_0, v)$. Therefore $T_p(M)$ can be called a *tangent Minkowski space* at p.

THEOREM 3. *If a connected manifold M admits a (F, Γ, g)-structure, then the tangent Minkowski spaces $T_p(M)$ and $T_q(M)$ at arbitrary distinct two points p and q in M are congruent mutually.*

PROOF. Since M is connected, we can take a piecewise differentiable curve C joining p and q. We represent it by $C=\{x(t)\,|\,x(0)=p,\ x(1)=q\}$. Let v be a vector in $T_p(M)$. Here we denote by $Y(t)$ the vector field on C given by $Y(t)=\sigma_{0t}v$, where σ_{0t} is the parallel displacement along C defined by (2.3). Of course, we put $Y(0)=v$. Then, as is shown above, the mapping σ_{01} is a linear isomorphism from $T_p(M)$ to $T_q(M)$. And we see, according to (2.4), that

$$\begin{aligned}\frac{d}{dt}F(x(t), Y(t)) &= (\partial_i F(x, Y)-\dot{\partial}_m F(x, Y)\Gamma^{m}_{li}Y^l)\frac{dx^i}{dt}\\ &=0.\end{aligned}$$

This shows us that σ_{01} is an isometry, namely, $T_p(M)$ and $T_q(M)$ are congruent mutually.

REMARK. Zaguskin has obtained this result using another expression and terminology [12].

§3. $\{V, H\}$-manifolds

Let V be an n-dimensional Minkowski space, whose norm function is denoted by f, and H be a linear Lie group leaving the Minkowski norm invariant. Let M be an n-dimensional connected C^{∞}-manifold.

Here, we assume that M admits the H-structure in the sense of a G-structure.

Let $\{U\}$ be a coordinate neighbourhood system, $\{X_1, X_2, \cdots\cdots, X_n\}$ be an n-frame of U adapted to the H-structure, and y be any vector in $T_p(M)$ with the expression $y=y^i(\frac{\partial}{\partial x^i})=\xi^{\alpha}X_{\alpha}$

Now, we can express

$$(3.1) \qquad \frac{\partial}{\partial x^i}=\mu_i^{\alpha}(x)X_{\alpha} \text{ and } X_{\alpha}=\lambda_{\alpha}^{i}(x)\frac{\partial}{\partial x^i}.$$

We have proved in the preceding paper [5] that the function

$$(3.2) \qquad F(x, y)=f(\xi^{\alpha})=f(\mu_i^{\alpha}(x)y^i)$$

gives M globally a Finsler metric. This Finsler metric is called a $\{V, H\}$-*Finsler metric.* When M admits a $\{V, H\}$-Finsler metric, we say that M is a $\{V, H\}$-*manifold.* In the $\{V, H\}$-manifold, the tangent Minkowski space $T_p(M)$ at any point $p\epsilon M$ is congruent to the given Minkowski space V. In the paper [5], we have also obtained

THEOREM 4. *Let M be a $\{V, H\}$-manifold, $F(x, y)$ be the $\{V, H\}$-Finsler metric function given by* (3.2), *and $\Gamma_{jk}^{i}(x)$ be a G-connection relative to the H-structure. Then M admits a (F, Γ, g)-structure.*

We are now in a position to consider the converse of Theorem 4. For the sake of this aim, we assume that M admits a (F, Γ, g)-structure. Then, according to Theorem 3, the tangent Minkowski space $T_p(M)$ at any point $p\epsilon M$ is congruent to a certain Minkowski space V. If M is assumed to be connected and to admit a linear connection $\Gamma_{jk}^{i}(x)$, then the holonomy group H of the linear connection $\Gamma_{jk}^{i}(x)$ is a Lie group [10]. Using these, we shall now prove

THEOREM 5. *If a manifold M is connected and admits a (F, Γ, g)-structure, then M is a $\{V, H\}$-manifold, whose $\{V, H\}$-Finsler metric coincides with the given Finsler metric.*

PROOF. It follows from the proof of Theorem 3 that the holonomy group H_p at any $p\epsilon M$ is an isometry of the tangent Minkowski space $T_p(M)$, that is,

$$(3.3) \qquad F(x, y)=F(x, Ty), \quad {}^{\forall}T\epsilon H_p.$$

According to the well-known holonomy theorem (e.g. see [10]), M admits an H-structure and the connection Γ is a G-connection relative to the H-

structure. Then, in any coordinate neighbourhood system U, there exists an n-frame $\{X_1, X_2, \cdots\cdots, X_n\}$ adapted to the H-structure. Apparently we can set $X_\alpha=\lambda^i_\alpha(x)\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial x^i}=\mu^\alpha_i(x)X_\alpha$. Denote the Lie algebra of the holonomy group H by $\mathscr{H}$. Then for any $A=(A^\alpha_\beta)\in\mathscr{H}$, we have $(\exp\ tA)\in H$. Therefore we obtain, from (3.3),

$$F(x, (\exp\ tA)y)=F(x, y). \tag{3.4}$$

We can set also $y=y^i\frac{\partial}{\partial x^i}=\xi^\alpha X_\alpha$, then we have $y^i=\lambda^i_\alpha\xi^\alpha$ and

$$\{\frac{d}{dt}(\exp\ tA)y\}_{t=0}=A^\alpha_\beta\xi^\beta X_\alpha=\lambda^i_\alpha A^\alpha_\beta\mu^\beta_j y^j\frac{\partial}{\partial x^i}.$$

Differenting (3.4) with respect to t and putting $t=0$, we have

$$\dot\partial_j F(x^i, y^i)\lambda^j_\alpha A^\alpha_\beta\mu^\beta_m y^m=0. \tag{3.5}$$

We denote by $\Gamma^\alpha_{\beta\gamma}$ the components of Γ in terms of the n-frame $\{X_\alpha\}$. Since $\{X_\alpha\}$ is an adapted frame of the H-structure [11], we have, for any vector $v=v^\alpha X_\alpha=v^i\frac{\partial}{\partial x^i}$,

$$\nabla_v X_\beta=v^\gamma\Gamma^\alpha_{\beta\gamma}X_\alpha \text{ where } v^\gamma\Gamma^\alpha_{\beta\gamma}\in\mathscr{H}. \tag{3.6}$$

By virtue of $\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^i}=\Gamma^k_{ij}\frac{\partial}{\partial x^k}$, we have

$$\nabla_v X_\beta=v^j\nabla_j\lambda^i_\beta\frac{\partial}{\partial x^i}.$$

Moreover, according to the identity $v^\gamma\Gamma^\alpha_{\gamma\beta}X_\alpha=v^j\mu^\gamma_j\Gamma^\alpha_{\gamma\beta}\lambda^i_\alpha\frac{\partial}{\partial x^i}$, (3.6)leads us to $\Gamma^\alpha_{\beta\gamma}=\mu^\alpha_r\lambda^m_\gamma\nabla_m\lambda^r_\beta$. Then (3.5) shows us

$$\dot\partial_j F(x, y)\lambda^j_\alpha v^\gamma\mu^\alpha_r\lambda^m_\gamma\nabla_m\lambda^r_\beta\mu^\beta_k y^k=0$$

for any v. Hence we have

$$\dot\partial_m F\nabla_k\lambda^m_\beta\mu^\beta_j y^j=0. \tag{3.7}$$

Now, we set

$$F(x^i, y^i)=F(x^i, \lambda^i_\alpha(x)\xi^\alpha)=f(x^i, \xi^\alpha).$$

Then f can be regarded as a function of x^i and ξ^α. And, we have

$$\frac{\partial f}{\partial x^k}=\partial_k F+\dot\partial_m F\,\partial_k\lambda^m_\alpha\xi^\alpha.$$

From the definition of (F, Γ, g)-structure, it follows that

$$\partial_k F-\dot\partial_m F\,\Gamma^m_{lk}y^l=0,$$

from which it is easily seen, according to (3.7), that $\frac{\partial f}{\partial x^k}=0$. Hence f is a function of ξ^α alone and has a form

$$(3.8) \qquad F(x^i, y^i)=f(\xi^\alpha)=f(\mu_i^\alpha(x)y^i).$$

As a consequence of Theorem 3 and (3.8), we obtain that the tangent Minkowski space at any point is congruent to a Minkowski space V whose norm function is given by (3.8).

For any element T of the Lie group H, we can express it by (T_β^α) in terms of the frame $\{X_\alpha\}$ adapted to H, at the same time, we can express it, in terms of the canonical frame $\{\frac{\partial}{\partial x^i}\}$, by (T_j^i). By virtue of $T(y)=T_\beta^\alpha\xi^\beta X_\alpha= T_j^i y^j\frac{\partial}{\partial x^i}$, we have a relation $T_\beta^\alpha=\mu_i^\alpha T_j^i\lambda_\beta^j$. Then we see, from (3.3) and (3.8),

$$\begin{aligned} f(\xi^\alpha) &= f(\mu_i^\alpha y^i)=F(x^i, y^i)=F(x^i, T_j^i y^j) \\ &= f(\mu_i^\alpha T_j^i y^j)=f(\mu_i^\alpha T_j^i \lambda_\beta^j \xi^\beta) \\ &= f(T_\beta^\alpha \xi^\beta). \end{aligned}$$

Hence the Lie group H leaves the Minkowski norm of V invariant. Thus M is a $\{V, H\}$-manifold whose $\{V, H\}$-Finsler metric is the originally given Finsler metric.

Q.E.D.

From the definition of a Berwald space, it follows immediately that

COROLLARY. *If a connected manifold M is a Berwald space, then M is a $\{V, H\}$-manifold.*

§4. (F, Γ, C)-structures

With regard to a general Finsler connection $\Gamma^*=\{F_{jk}^i, N_j^i, C_{jk}^i\}$, there exists the *hv*-curvature tensor P (see e.g. [7]), which is defined by

$$(4.1) \qquad P_{rjk}^i=\dot\partial_k F_{rj}^i - C_{rk|j}^i + C_{rs}^i(\dot\partial_k N_j^s - F_{kj}^s).$$

In our case, where we treat a Finsler manifold with a linear connection $\Gamma_{jk}^i(x)$, we take as the Γ^* a Finsler connection associated with $\Gamma_{jk}^i(x)$, which is defined by (2.2). In this case, we have directly from (2.2) that

$$(4.2) \qquad P_{rjk}^i=-C_{rk|j}^i \quad .$$

Henceforce we shall prove

THEOREM 6. *Let M be a connected Finsler manifold with a linear connection $\Gamma_{jk}^i(x)$. Concerning the Finsler connection associated with $\Gamma_{jk}^i(x)$, the following three conditions are mutually equivalent:*

(1) $P_{r\,jk}^{\;i}=0$.

(2) *M admits the* (F,Γ,C)*-structure.*

(3) *Let p and q be arbitrary two points of M, C be any piecewise differentiable curve joining p and q. Then the parallel displacement along C from* $T_p(M)$ *to* $T_q(M)$ *is C-projective.*

PROOF. It is obvious that the condition (1) is equivalent to the condition (2). So, it is sufficient to show that the condition (2) is equivalent to the condition (3).

For this purpose, we treat the Finsler tensor field

$$C=C_{jk}^{i}(x,y)\frac{\partial}{\partial x^i}\otimes dx^j\otimes dx^k.$$

Denote the curve C by $C=\{x(t)\,|\,x(0)=p, x(1)=q\}$ and consider the tangent vector $x'=\frac{dx^i}{dt}\frac{\partial}{\partial x^i}$ of the curve C. Then, as is shown in §2, we have

$$\nabla_{x'}C=\lim_{t\to 0}\frac{1}{t}\{\tilde{\sigma}_{t0}C(x(t),Y(t))-C(x(0),y(0))\}=\{C_{jk|h}^{i}\frac{dx^h}{dt}\}\frac{\partial}{\partial x^i}\otimes dx^j\otimes dx^k,$$

where we put $\sigma_{0t}y(0)=Y(t)$. Of course, the parallel displacement σ_{01} is a linear isomorphic mapping from $T_p(M)$ to $T_q(M)$.

If the condition (2) is satisfied, then $\tilde{\sigma}_{10}C(x(1),Y(1))=C(x(0),y(0))$ is easily proved by well-known method. Consequently this result shows us that $T_q(M)$ is C-projective to $T_p(M)$.

Conversely, if the condition (3) is satisfied, then Theorem 1 leads us to $\nabla_{x'}C=0$. Hence we have that $C_{jk|h}^{i}\frac{dx^h}{dt}=0$ for any C. Thus we obtain $C_{jk|h}^{i}=0$.

Q.E.D.

COROLLARY *If a connected manifold M admits a* (F,Γ,C)*-structure, then the holonomy group of* Γ *at p carries C-geodesics into C-geodesics in* $T_p(M)$.

As an example of (F,Γ,C)-structure, we shall consider the following special Randers space.

Assume that M admits such a Finsler metric as

$$F(x,y)=\sqrt{a_{ij}(x)y^iy^j}+kb_i(x)y^i,$$

where $\sqrt{a_{ij}(x)\,y^iy^j}$ is a Riemann metric, $b_i(x)$ is a covariant vector field satisfying $a_{ij}b^ib^j=1$, and k is a constant of $0<k<1$.

Let $\{^{\,i}_{jk}\}$ be the Christoffel symbol with respect to a_{ij}. We assume moreover that $b_i(x)$ is parallel with respect to $\{^{\,i}_{jk}\}$.

In this case, it has been shown in the paper [4] that M becomes a Berwald space, namely, M admits a (F,Γ,g)-structure where $\Gamma_{jk}^{i}=\{^{\,i}_{jk}\}$. And, of course, this offers an example of a (F,Γ,C)-structure.

In the sequel, however, we shall treat a linear connection different from $\{^{\,i}_{jk}\}$.

We take a linear connection Γ such that $\Gamma^i_{jk}(x)=\{^{\,i}_{jk}\}-\frac{1}{2}\delta^i_j b_k$. Then we may consider the Finsler connection Γ^* associated with this linear connection Γ. By direct calculation, we can get, with respect to the Γ^*, that

$$a_{ij|k}=b_k a_{ij},\quad b_{i|k}=\frac{1}{2}b_i b_k,\quad \text{and}\quad F_{|k}=\frac{1}{2}Fb_k.$$

Using the fact that Γ depends on position only, we obtain $g_{ij|k}=b_k g_{ij}$ and $C_{ijk|h}=b_h C_{ijk}$. It then follows, from these, that $C^h_{ij|k}=0$. So, we have shown that the Randers space under consideration admits a (F,Γ,C)-structure.

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