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On the Theory of Vector Valued Fourier Hyperfunctions

By

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Contents

§ 0. Introduction.

Recently the theory of vector valued hyperfunctions has been developed by Ion, P. D. F. and T. Kawai [5]. It has been done by the method of 'soft analysis' in parallel with Sato's theory of hyperfunctions. The extension of Sato's theory to the theory of vector valued hyperfunctions is worthy to be considered not only for its own sake but also for applications. It would be also worthwhile that we extend the theory of Fourier hyperfunctions in this direction. In fact, one of the coauthors has found the applications of the theory of vector valued Fourier hyperfunctions to some problems in the quantum field theory (see Nagamachi, S. and N. Mugibayashi [12]). We have done these extensions by the method analogous to Kawai's method of constructing the theory of Fourier hyperfunctions. We construct the sheaf ${}^{H}\mathcal{R}$ of H-valued Fourier hyperfunctions over $\boldsymbol{D}^n$, as the n-th derived sheaf of ${}^{H}\tilde{\mathcal{O}}$ with support in $\boldsymbol{D}^n$, where $\boldsymbol{D}^n$ is the radial compactification of $\boldsymbol{R}^n$ and H is a separable complex Hilbert space and ${}^{H}\tilde{\mathcal{O}}$ is the sheaf of slowly increasing H-valued holomorphic functions over $\boldsymbol{D}^n\times\sqrt{-1}\,\boldsymbol{R}^n$. Next we realize H-valued Fourier hyperfunctions as continuous linear operators from $\underset{\sim}{\mathcal{O}}(K)$ to H. Namely, we show that the space $H^n_K(V, {}^H\widetilde{\mathcal{O}})$ of H-valued Fourier hyperfunctions with supports in a compact subset K of $\boldsymbol{D}^n$ is isomorphic to the space $L_b(\underset{\sim}{\mathcal{O}}(K);H)$ of all continuous linear operators from $\underset{\sim}{\mathcal{O}}(K)$ to H equipped with the topology of bounded convergence. We also show that the space $H^n_K(V, {}^H\widetilde{\mathcal{O}})$ is isomorphic to the tensor product $H^n_K(V,\widetilde{\mathcal{O}})\hat{\otimes} H$ of the space $H^n_K(V,\widetilde{\mathcal{O}})$ of scalar valued Fourier hyperfunctions with supports in K and the Hilbert space H. The sheaf ${}^H\mathscr{R}$ is a flabby sheaf and its restriction to $\boldsymbol{R}^n$ coincides with the sheaf ${}^H\mathscr{B}$ of H-valued hyperfunctions over $\boldsymbol{R}^n$, and its global sections are stable under Fourier transformation. Hence any H-valued hyperfunction on $\boldsymbol{R}^n$ can be extended to an H-valued Fourier hyperfunction on $\boldsymbol{D}^n$ and then we can consider its Fourier transformation. §1 is devoted to the preliminaries from functional analysis, especially, the extensions of Hörmander's existence theorems for the Cauchy-Riemann operator to the case of H-valued functions. In §2, we define the sheaf ${}^H\widetilde{\mathcal{O}}$ and the sheaf ${}^H\underset{\sim}{\mathcal{O}}$ of rapidly decreasing H-valued holomorphic functions over $\boldsymbol{D}^n\times\sqrt{-1}\,\boldsymbol{R}^n$ and construct their soft resolutions. In §3, we mention the generalizations of Oka-Cartan theorem B and Malgrange's theorem and Martineau-Harvey's theorem. In §4, we prove the analogue of Runge's theorem. In §5, we prove the pure-codimensionality of $\boldsymbol{D}^n$ with respect to ${}^H\widetilde{\mathcal{O}}$ and define the sheaf ${}^H\mathscr{R}$ and then study some properties of H-valued Fourier hyperfunctions. In §6, we prove two isomorphism theorems $H^n_K(V, {}^H\widetilde{\mathcal{O}})\cong L_b(\underset{\sim}{\mathcal{O}}(K);H)\cong H^n_K(V,\widetilde{\mathcal{O}})\hat{\otimes} H$. In §7, we define the Fourier transformation of H-valued Fourier hyperfunctions on $\boldsymbol{D}^n$ and study its properties, especially, we give an operation formula of an H-valued Fourier hyperfunction on $\mathscr{P}_*$ as a continuous linear operator. Lastly we prove an analogue of the Paley-Wiener theorem.

§1. Preliminaries

In this section, we prepare some fundamental facts from functional analysis.

First, we mention sevral properties of tensor product of Hilbert spaces.

DEFINITION 1.1. *Let X and Y be complex Hilbert spaces. We denote the algebraic tensor product of X and Y by $X\otimes Y$, which is composed of all the elements of the from $\sum_{j=1}^n x_j\otimes y_j$ where $x_j\in X$, $y_j\in Y$ for $j=1.2,\cdots,n$. Then $X\otimes Y$ is a pre-Hilbert space with the norm $\|\sum_{j=1}^n x_j\otimes y_j\|=[\sum_{i,j}(x_i,x_j)_X\,(y_i,y_j)_Y]^{1/2}$, where $(\ ,\ )_X$ and $(\ ,\ )_Y$ denote the inner products in X and Y, respectively. We denote the completion of*

$X \otimes Y$ *by* $X \hat{\otimes} Y$.

Now, we assume that Y is separable. Then Y has a countable orthonormal basis $\{e_1, e_2, \cdots, e_n, \cdots\}$, so that in the expression $z = \sum_{j=1}^{n} x_j \otimes y_j$ $(x_j \in X, y_j \in Y)$, each y_j can be written in the form $y_j = \sum_{k=1}^{\infty} \alpha_{jk} e_k$ with $\sum_{k=1}^{\infty} |\alpha_{jk}|^2 < \infty$. Hence

$$\|z\|^2 = \sum_{i,j=1}^{n} (x_i, x_j)_X (y_i, y_j)_Y = \sum_{k=1}^{\infty} \|\sum_{j=1}^{n} \alpha_{jk} x_j\|^2 < \infty.$$

Thus, every element $z = \sum_{j=1}^{n} x_j \otimes y_j$ of $X \otimes Y$ can be written in the form $z = \sum_{k=1}^{\infty} z_k \otimes e_k$, where z_k's $\in X$ satisfy the following conditions,

(F) (i) $\sum_{k=1}^{\infty} \|z_k\|_X^2 < \infty$

(ii) each z_k is a linear combination $\sum_{j=1}^{n} \alpha_{jk} x_j$ of a finite number of elements $x_1, x_2, \cdots, x_n$ in X with coefficients satisfying the properties $\sum_{k=1}^{\infty} |\alpha_{jk}|^2 < \infty$ for $j = 1, 2, \cdots, n$.

Conversely, any element z in $X \hat{\otimes} Y$ of the form $z = \sum_{k=1}^{\infty} z_k \otimes e_k$ $(z_k \in X)$ is contained in $X \otimes Y$ whenever z_k's satisfy the condition (F). Hence, in order that an element z in $X \hat{\otimes} Y$ of the form $z = \sum_{k=1}^{\infty} z_k \otimes e_k$ $(z_k \in X)$ is contained in $X \otimes Y$, it is necessary and sufficient that $\{z_k\}$ satisfies the above condition (F).

Now, we put $Z = \{\sum_{k=1}^{\infty} z_k \otimes e_k; z_k \in X \ (k = 1, 2, \cdots), \ \sum_{k=1}^{\infty} \|z_k\|_X^2 < \infty\}$. Then it is easy to see that $X \hat{\otimes} Y$ is isomorphic to Z, and the latter is isomorphic to the Hilbert direct sum $\bigoplus_{n=1}^{\infty} X$ of countable copies of X.

Thus we have the following proposition.

PROPOSITION 1.2. *Let* X *and* Y *be complex Hilbert spaces. Then, if* Y *is separable,* $X \hat{\otimes} Y$ *is isomorphic to the Hilbert direct sum* $\bigoplus_{n=1}^{\infty} X$ *of countable copies of* X.

DEFINITION 1.3. *Let* X, Y *and* H *be complex Hilbert spaces. For a linear operator* T *from* X *to* Y *with domain* D(T), *we can define a linear operator* $\mathrm{T} \otimes \mathrm{I}$ *from* $X \hat{\otimes} H$ *to* $Y \hat{\otimes} H$ *whose domain is* $\mathrm{D}(\mathrm{T} \otimes \mathrm{I}) = \mathrm{D}(\mathrm{T}) \otimes H$, *putting* $\mathrm{T} \otimes \mathrm{I}(\sum_{j=1}^{n} x_j \otimes h_j) = \sum_{j=1}^{n} \mathrm{T} x_j \otimes h_j$ *for* $x_j \in \mathrm{D}(\mathrm{T})$, $h_j \in H$ $(j = 1, 2, \cdots, n)$. *Here* I *denotes the identity operator of* H. *If* $\mathrm{T} \otimes \mathrm{I}$ *is closable, we denote its closure by* $\mathrm{T} \hat{\otimes} \mathrm{I}$.

In the above definition, if H is separable, each element $\sum_{j=1}^{n} x_j \otimes h_j$ in $\mathrm{D}(\mathrm{T} \otimes \mathrm{I}) = \mathrm{D}(\mathrm{T}) \otimes H$ can be written in the form $\sum_{j=1}^{n} x_j \otimes h_j = \sum_{k=1}^{\infty} z_k \otimes e_k$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis in H and z_k's $\in \mathrm{D}(\mathrm{T})$ satisfy the condition (F) for D(T) and $\sum_{k=1}^{\infty} \|Tz_k\|_Y^2 < \infty$, and $T\otimes I(\sum_{j=1}^{n} x_j \otimes h_j) = \sum_{k=1}^{\infty} Tz_k \otimes e_k$. Then, if T is closable, it is easy to see that $T\otimes I$ is closable. If T is closed, $D(T\hat{\otimes}I) = \{z = \sum_{k=1}^{\infty} z_k \otimes e_k;\ \{z_k\} \in \bigoplus_{k=1}^{\infty} D(T),\ \sum_{k=1}^{\infty} \|Tz_k\|_Y^2 < \infty\}$, where $\bigoplus_{k=1}^{\infty} D(T)$ is the Hilbert direct sum of countable copies of D(T). In fact, it is trivial that the right hand side is contained in $D(T\hat{\otimes}I)$. Conversely, if $z = \sum_{k=1}^{\infty} z_k \otimes e_k \in D(T\hat{\otimes}I)$, there exists a sequence $\{z_n = \sum_{k=1}^{n} z_k^n \otimes e_k\} \subset D(T\otimes I)$ such that $\lim_{n\to\infty} z_n = z$ and $\lim_{n\to\infty} (T\otimes I)(z_n) = \lim_{n\to\infty} \sum_{k=1}^{\infty} Tz_k^n \otimes e_k = (T\hat{\otimes}I)(z) = \sum_{k=1}^{\infty} u_k \otimes e_k$. Since T is closed, $z_k \in D(T)$ and $Tz_k = u_k$. Hence $\{z_k\} \in \bigoplus_{k=1}^{\infty} D(T)$ and $\sum_{k=1}^{\infty} \|Tz_k\|_Y^2 < \infty$, and $(T\hat{\otimes}I)(z) = \sum_{k=1}^{\infty} Tz_k \otimes e_k$.

LEMMA 1.4. *Let X, Y and H be complex Hilbert spaces and let* T *be a linear, closed, densely defined operator from X to Y, whose range is closed. Assume that H is separable. Then* $Im(T\hat{\otimes}I) = (Im T)\hat{\otimes} H$ *and* $Ker(T\hat{\otimes}I) = (Ker T)\hat{\otimes} H$.

PROOF. We take an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ in H. Then it was seen above that $Im(T\hat{\otimes}I) = \{\sum_{k=1}^{\infty} Tx_k \otimes e_k;\ x_k \in D(T),\ \sum_{k=1}^{\infty} \|x_k\|_X^2 < \infty.\ \sum_{k=1}^{\infty} \|Tx_k\|_Y^2 < \infty\}$ and $(Im T)\hat{\otimes} H = \{\sum_{k=1}^{\infty} y_k \otimes e_k;\ y_k \in Im T,\ \sum_{k=1}^{\infty} \|y_k\|_Y^2 < \infty\}$. Thus we see that $Im(T\hat{\otimes}I)$ is contained in $(Im T)\hat{\otimes} H$. Hence in order to prove the lemma, it is sufficent to show that $(Im T)\hat{\otimes} H$ is contained in $Im(T\hat{\otimes}I)$. However, by the theorem in Hörmander [3] (p. 91, Theorem 1.1.1), for each $y_k \in Im T$ we can find an element $x_k \in D(T)$ such that $Tx_k = y_k$ and $\|x_k\|_X \leqq C\|y_k\|_Y$ where C is some constant. Thus, the last inequality shows that $\sum_{k=1}^{\infty} \|x_k\|_X^2 < \infty$ when $\sum_{k=1}^{\infty} \|y_k\|_Y^2 < \infty$. Therefore $Im(T\hat{\otimes}I) = (Im T)\hat{\otimes} H$.

Next we will prove that $Ker(T\hat{\otimes}I) = (Ker T)\hat{\otimes} H$. We note that, for a closed operator T, $Ker T$ is a closed subspace of X. Then $(Ker T)\hat{\otimes} H = \{\sum_{k=1}^{\infty} x_k \otimes e_k;\ x_k \in Ker\ T,\ \sum_{k=1}^{\infty} \|x_k\|_X^2 < \infty\}$. On the other hand,

$$Ker(T\hat{\otimes}I) = \{\sum_{k=1}^{\infty} x_k \otimes e_k;\ x_k \in D(T),\ \sum_{k=1}^{\infty} \|x_k\|_X^2 < \infty,\ \sum_{k=1}^{\infty} \|Tx_k\|_Y^2 < \infty,\ (T\hat{\otimes}I)(\sum_{k=1}^{\infty} x_k \otimes e_k) = 0\}$$
$$= \{\sum_{k=1}^{\infty} x_k \otimes e_k;\ x_k \in Ker\ T,\ \sum_{k=1}^{\infty} \|x_k\|_X^2 < \infty\}.$$

Therefore $Ker(T\hat{\otimes}I) = (Ker T)\hat{\otimes} H$. This completes the proof.

PROPOSITION 1.5. *Let X, Y, Z and H be complex Hilbert spaces and let H be separable. In the sequence* $X \xrightarrow{T} Y \xrightarrow{S} Z$, *let* T *and* S *be two liner, closed, densely defined operators. If* $X \xrightarrow{T} Y \xrightarrow{S} Z$ *is an exact sequence, then* $X\hat{\otimes}H \xrightarrow{T\hat{\otimes}I} Y\hat{\otimes}H \xrightarrow{S\hat{\otimes}I} Z\hat{\otimes}H$ *is also an exact* *sequence.*

PROOF. Since $Im\mathrm{T} = Ker\,\mathrm{S}$ is closed, by the lemma 1.4, $Im(\mathrm{T}\hat{\otimes}\mathrm{I}) = (Im\mathrm{T})\hat{\otimes}H = (Ker\,\mathrm{S})\hat{\otimes}H = Ker(\mathrm{S}\hat{\otimes}\mathrm{I})$. This completes the proof.

DEFINITION 1.6. *Let H be a complex Hilbert space and let Ω be a measurable subset of $\boldsymbol{R}^n$ with respect to the Lebesgue measure $d\lambda$ in $\boldsymbol{R}^n$. We define the space $L^2(\Omega, H) \equiv {}^HL^2(\Omega)$ of H-valued L^2-functions as a Hilbert space of all H-valued measurable functions f defined on Ω which satisfy the condition*

$$\|f\| = \left(\int \|f(x)\|_H^2 d\lambda\right)^{1/2} < \infty.$$

PROPOSITION 1.7. *Let H and Ω be as in the definition 1.6. Further we assume that H is separable. Then $L^2(\Omega, H)$ is isomorphic to $L^2(\Omega)\hat{\otimes}H$ where $L^2(\Omega)$ is the Hilbert space of complex valued square integrable functions on Ω with the norm*

$$\|f\|_2 = \left(\int |f(x)|^2 d\lambda\right)^{1/2}.$$

PROOF. We take some orthonormal basis $\{e_k\}_{k=1}^{\infty}$ in H. Then each element $f(x)$ in $L^2(\Omega, H)$ can be written in the form $f(x) = \sum_{k=1}^{\infty} f_k(x)\, e_k$ where $f_k(x) \in L^2(\Omega)$. Since $\|f(x)\|_H^2 = \sum_{k=1}^{\infty} |f(x)|^2$, we have $\int_\Omega \|f(x)\|_H^2 d\lambda = \sum_{k=1}^{\infty} \|f_k\|_2^2 < \infty$. Thus, to $f(x)$, we can associate an element $\tilde{f}(x) = \sum_{k=1}^{\infty} f_k(x)\otimes e_k$ in $L^2(\Omega)\hat{\otimes}H$. Conversely, to any element $\tilde{f}(x) = \sum_{k=1}^{\infty} f_k(x)\otimes e_k$ in $L^2(\Omega)\hat{\otimes}H$, we assign an H-valued function $f(x) = \sum_{k=1}^{\infty} f_k(x)\, e_k$. Then $f(x)$ is in $L^2(\Omega, H)$ and $\tilde{f}(x)$ corresponds to $f(x)$ by the above correspondence. Since the above correspondence is isometry, this shows that $L^2(\Omega, H)$ is unitarily isomorphic to $L^2(\Omega)\hat{\otimes}H$. The proof is completed.

We shall now mention the existence theorem for the ${}^H\bar{\partial}$ operator. In the following we assume that H is a separable complex Hilbert space. Let Ω be an open set in $\boldsymbol{C}^n$. For a continuous function φ in Ω, we denote by ${}^HL^2(\Omega, \varphi)$ the space of H-valued measurable funktions in Ω whose H-norm is square integrable with respect to the measure $e^{-\varphi}d\lambda$, where $d\lambda$ is the Lebesgue measure. Then, by the proposition 1.7, ${}^HL^2(\Omega, \varphi)$ is isomorphic to $L^2(\Omega, \varphi)\hat{\otimes}H$, where $L^2(\Omega, \varphi)$ is the space of complex valued functions in Ω which are square integrable with respect to the measure

$e^{-\varphi}d\lambda$. We put ${}^{H}L^{2}_{(p,q)}(\Omega,\varphi) = {}^{H}L^{2}(\Omega,\varphi)\otimes\Lambda_{(p,q)}$ where $\Lambda_{(p,q)}$ is the space of forms of type (p,q) with coefficients in $\boldsymbol{C}$. We assume that ${}^{H}L^{2}_{(p,q)}(\Omega,\varphi)$ has the norm of the Hilbert direct sum $\oplus {}^{H}L^{2}(\Omega,\varphi)$ of $\binom{n}{p}\binom{n}{q}$ copies of ${}^{H}L^{2}(\Omega,\varphi)$ and we denote this norm by $\|\ \|_{H}$ in abuse of the notation. Then ${}^{H}L^{2}_{(p,q)}(\Omega,\varphi)$ is isomorphic to $L^{2}_{(p,q)}(\Omega,\varphi)\hat{\otimes}H$, where $L^{2}_{(p,q)}(\Omega,\varphi)$ is the space of all forms of type (p,q) with coefficients in $L^{2}(\Omega,\varphi)$.

For two continuous functions φ_1 and φ_2 in Ω, we can define the linear, closed, densely defined operator $\bar{\partial}$ from $L^{2}_{(p,q)}(\Omega,\varphi_1)$ to $L^{2}_{(p,q+1)}(\Omega,\varphi_2)$. We call this operator the Cauchy-Riemann operator. We can define the generalized Cauchy-Riemann operator ${}^{H}\bar{\partial}$ putting ${}^{H}\bar{\partial} = \bar{\partial}\hat{\otimes}\mathrm{I}$. From now on, we also call this operator ${}^{H}\bar{\partial}$ the Cauchy-Riemann operator.

Now we assume that Ω is a pseudoconvex domain in $\boldsymbol{C}^{n}$ such that $\sup_{z\in\Omega}|\mathrm{Im}z|\leqq \mathrm{M}<\infty$. Let $\psi(z)$ be a plurisubharmonic function in Ω. By $\eta(z)$ we denote the modification of $\sum_{j=1}^{n}|z_j|$ near $\{z_j=0$ for some $j\}$ so that it becomes C^{∞} and convex. Put

$$
\begin{aligned}
{}^{H}X_j &= {}^{H}L^{2}_{(p,q-1)}(\Omega;\ (1/j)\,\eta(z)+4\log(1+|z|^{2})+\psi(z))\\
{}^{H}Y_j &= {}^{H}L^{2}_{(p,q)}(\Omega;\ (1/j)\,\eta(z)+2\log(1+|z|^{2})+\psi(z))\\
{}^{H}Z_j &= {}^{H}L^{2}_{(p,q+1)}(\Omega;\ (1/j)\,\eta(z)+\psi(z)).
\end{aligned}
$$

Then $\{{}^{H}X_j\}$, $\{{}^{H}Y_j\}$ and $\{{}^{H}Z_j\}$ become weakly compact projective sequences, and we define FS*-spaces, $X=\varprojlim_{j}{}^{H}X_j$, $Y=\varprojlim_{j}{}^{H}Y_j$ and $Z=\varprojlim_{j}{}^{H}Z_j$ (see Komatsu [8]).

LEMMA 1.8. *Let ${}^{H}\bar{\partial}$ be the Cauchy-Riemann operator. Then* $X\overset{{}^{H}\bar{\partial}}{\to}Y\overset{{}^{H}\bar{\partial}}{\to}Z$ *is exact.*

PROOF. Put

$$
\begin{aligned}
X_j &= L^{2}_{(p,q-1)}(\Omega;\ (1/j)\,\eta(z)+4\log(1+|z|^{2})+\psi(z))\\
Y_j &= L^{2}_{(p,q)}(\Omega;\ (1/j)\,\eta(z)+2\log(1+|z|^{2})+\psi(z))\\
Z_j &= L^{2}_{(p,q+1)}(\ \ ;\ (1/j)\,\eta(z)+\psi(z)).
\end{aligned}
$$

Then $X_j\overset{\bar{\partial}}{\to}Y_j\overset{\bar{\partial}}{\to}Z_j$ is exact (see Kawai [7] p.470, Lemma 2.1.1). Since ${}^{H}X_j = X_j\hat{\otimes}H$, ${}^{H}Y_j = Y_j\hat{\otimes}H$ and ${}^{H}Z_j = Z_j\hat{\otimes}H$, by the proposition 1.5, the sequence ${}^{H}X_j\overset{{}^{H}\bar{\partial}}{\to}{}^{H}Y_j\overset{{}^{H}\bar{\partial}}{\to}{}^{H}Z_j$ is exact. Hence $({}^{H}X_j)'\overset{{}^{H}\vartheta}{\leftarrow}({}^{H}Y_j)'\overset{{}^{H}\vartheta}{\leftarrow}({}^{H}Z_j)'$ is exact, where $({}^{H}X_j)'$, $({}^{H}Y_j)'$ and $({}^{H}Z_j)'$ are the dual spaces of ${}^{H}X_j$, ${}^{H}Y_j$ and ${}^{H}Z_j$, respectively, and ${}^{H}\vartheta = ({}^{H}\bar{\partial})'$. Therefore, since $X'=\varinjlim_{j}({}^{H}X_j)'$, $Y'=\varinjlim_{j}({}^{H}Y_j)'$ and $Z'=\varinjlim_{j}({}^{H}Z_j)'$, we have the exact sequence $X'\overset{{}^{H}\vartheta}{\leftarrow}Y'\overset{{}^{H}\vartheta}{\leftarrow}Z'$. X', Y' and Z', being injective limits of Hilbert spaces, are DFS*-spaces. Hence, if $q>1$, it is trivial that the theorem is valid by the so-called Serre-Komatsu duality theorem (see Komatsu [8], p.381, Theorem 19). When $q=1$, in the exact sequence

$X' \overset{{}^H\vartheta}{\leftarrow} Y' \overset{{}^H\vartheta}{\leftarrow} Z'$, we need to prove the closed rangeness of $X' \overset{{}^H\vartheta}{\leftarrow} Y'$. We shall now prove this. Since DFS*-space, being the strong dual space of a reflexive Fréchet space, is fully complete, so, in order to prove the closed rangeness of ${}^H\vartheta$, it is sufficient to prove that $Im\,({}^H\vartheta) \cap V^\circ$ is closed in V° for the polar set V° of any neighbourhood V of 0 in X. Then $Im({}^H\vartheta) \cap V^\circ$ is bounded in X', so that, for some $k, Im({}^H\vartheta) \cap V^\circ = u_k(B_k)$ is a bounded set in $({}^HX_k)'$ and u_k is a weak homeomorphism (see Komatsu [8], p. 372, Theorem 6). Now assume that ${}^H\vartheta u_\nu \to f \in V^\circ$. Then ${}^H\vartheta u_\nu$ converges weakly to f in $({}^HX_k)'$. Here we need the following lemma.

LEMMA 1.9. *If* $u \in ({}^HY_j)'$ *and* ${}^H\vartheta u \in ({}^HX_k)'$ $(j>k)$, then there exists some v *in* $({}^HY_k)'$ *and* ${}^H\vartheta u = {}^H\vartheta v$ *holds.*

PROOF OF THE LEMMA 1.9. We take a sequence of C^∞ functions $\varphi_m(z) = \exp\left(-(1/m)\sum_{j=1}^{n}\bar{z}_j{}^2\right)$. Then, by the definition of ${}^H\vartheta$, ${}^H\vartheta(\varphi_m u) = \varphi_m{}^H\vartheta u$. Since we have assumed that $\sup_{z\in\Omega}|\mathrm{Im} z| = M<\infty$, $\varphi_m u$ belongs to $({}^HY_k)'$ for all m. On the other hand, $\varphi_m{}^H\vartheta u$ converges to ${}^H\vartheta u$ in $({}^HX_k)'$ by Lebesgue's theorem because φ_m is bounded and converges to 1 pointwise in Ω. However, the sequence ${}^HX_k \overset{{}^H\bar\partial}{\to} {}^HY_k \overset{{}^H\bar\partial}{\to} {}^HZ_k$ is exact. Hence ${}^HX_k \overset{{}^H\bar\partial}{\to} {}^HY_k$ is of closed range, so that, by the closed range theorem, ${}^H\vartheta$ is a closed range operetor from $({}^HY_k)'$ to $({}^HX_k)'$. Therefore ${}^H\vartheta u = \lim_{m\to\infty}\varphi_m{}^H\vartheta u = \lim_{m\to\infty}{}^H\vartheta(\varphi_m u) \in {}^H\vartheta(({}^HY_k)')$. This means that there exists some $v \in ({}^HY_k)'$ such that ${}^H\vartheta u = {}^H\vartheta v$. This completes the proof.

We now continue the proof of the lemma 1.8 in the case $q=1$. By the lemma 1.9, we may assume not only ${}^H\vartheta u_\nu \in ({}^HX_k)'$ but also $u_\nu \in ({}^HY_k)'$. However, since $Im\,({}^H\vartheta)$ is closed and convex in $({}^HX_k)'$, it is weakly closed in $({}^HX_k)'$. Hence there exists some $v \in ({}^HY_k)'$ such that $f = {}^H\vartheta v$, which shows that $Im\,({}^H\vartheta) \cap V^\circ$ is closed in V°. Therefore this means that $Im\,({}^H\vartheta)$ is closed in X'. Because DFS*-space is the strong dual space of FS*-space which is a Fréchet space, then $Im\,({}^H\bar\partial)$ is closed in Y (see Komatsu [8], p. 381, Theorem 19). Hence, by the Serre-Komatsu duality theorem, the lemma is proved in the case $q=1$ (see Komatsu [8], p. 381, Theorem 19). This completes the proof.

Next we prepare the H-valued version of the theorem of Hörmander [3] (p. 109, Proposition 2.3.2). In the following, we use that ${}^HL^2_{(p,q-1)}(\Omega, -\varphi)$ and ${}^HL^2_{(p,q-1)}(\Omega, \varphi)$ are antiduals of each other with respect to the sesquilinear form

$$\langle u, v\rangle = \int_{\Omega} \Sigma'_{I,K}(u_{I,K}, v_{I,K})_H d\lambda$$

where

$$u = \Sigma'_{|I|=p,|K|=q-1} u_{I,K} dz^I \wedge d\bar{z}^K \in {}^H L^2_{(p,q-1)}(\Omega, -\varphi)$$

and

$$v = \Sigma'_{|I|=p,|K|=q-1} v_{I,K} dz^I \wedge d\bar{z}^K \in {}^H L^2_{(p,q-1)}(\Omega, \varphi).$$

For the terminlogy, we refer to Hörmander [3].

PROPOSITION 1.10. *Let Ω be a pseudoconvex domain in C^n with C^2 boundary. Let φ and $\psi \in C^2(\bar{\Omega})$ be strictly plurisubharmonic in Ω, and let u be a form in ${}^H L^2_{(p,q-1)}(\Omega, -\varphi)$, let $u=0$ where $\psi > 0$ and assume that $\langle u, v\rangle = 0$ for every solution v such that ${}^H\bar{\partial} v = 0$ and $v \in {}^H L^2_{(p,q-1)}(\Omega, \varphi + \lambda\psi^+)$ for some λ; here $\psi^+ = \sup(\psi, 0)$. Then thre is a form $f \in {}^H L^2_{(p,q)}(\Omega, \mathrm{loc})$ such that f vanishes where $\psi > 0$ and satisfies ${}^H\vartheta f = u$ and*

$$\int_{\Omega} \Sigma'_{I,K} \Sigma_{j,k} (f_{I,jK}, f_{I,kK})_H (\partial^2\varphi/\partial z_j \partial\bar{z}_k) e^{\varphi} d\lambda \leqq \int_{\Omega} \| u \|^2 e^{\varphi} d\lambda.$$

PROOF. Let $\{e_m\}_{m=1}^{\infty}$ be an orthonormal basis in H. Then we have $u = \sum_{m=1}^{\infty} u^{(m)} e_m$, $u^{(m)} \in L^2_{(p,q-1)}(\Omega, -\varphi)$. Since $Ker({}^H\bar{\partial}) = (Ker\,\bar{\partial}) \hat{\otimes} H$ by the lemma 1.4, for any $v \in L^2_{(p,q-1)}(\Omega, \varphi + \lambda\psi^+)$ such that $\bar{\partial} v = 0$ which is identified with $ve_m \in {}^H L^2_{(p,q-1)}(\Omega, \varphi + \lambda\psi^+)$, we have

$$\langle u, v\rangle = \int_{\Omega} \Sigma'_{I,K} u^{(m)}_{I\,K} \bar{v}_{I,K} d\lambda \qquad \text{and } u^{(m)} = 0 \text{ where } \psi > 0.$$

Thus each coefficients $u^{(m)}$ satisfies the conditions of the theorem of Hörmander [3] (p. 109, Proposition 2.3.2). Hence there is a form $f^{(m)} \in L^2_{(p,q)}(\Omega, \mathrm{loc})$ such that $f^{(m)} = 0$ where $\psi > 0$ and $\vartheta f^{(m)} = u^{(m)}$.

$$\int_{\Omega} \Sigma'_{I,K} \Sigma_{jk} f^{(m)}_{I,jK} \bar{f}^{(m)}_{I,kK} (\partial^2\varphi/\partial z_j \partial\bar{z}_k) e^{\varphi} d\lambda \leqq \int_{\Omega} | u^{(m)} |^2 e^{\varphi} d\lambda.$$

Hence, if we put $f = \sum_{m=1}^{\infty} f^{(m)} e_m$, we have $f \in {}^H L^2_{(p,q)}(\Omega, \mathrm{loc})$ which vanishes where $\psi > 0$, and, since ${}^H\vartheta = \vartheta \hat{\otimes} \mathrm{I}$, we have ${}^H\vartheta f = u$ and

$$\int_{\Omega} \Sigma'_{I,K} \Sigma_{j,k} (f_{I,jK}, f_{I,kK})_H (\partial^2\varphi/\partial z_j \partial\bar{z}_k) e^{\varphi} d\lambda \leqq \int_{\Omega} \| u \|_H^2 e^{\varphi} d\lambda.$$

This completes the proof.

§2. The sheaves ${}^H\widetilde{\mathcal{O}}$ and ${}^H\underset{\sim}{\mathcal{O}}$ and their resolutions.

In the following sections, we assume that H is a separable complex Hilbert space. In this section, we give definitions and soft resolutions of sheaves ${}^H\widetilde{\mathcal{O}}$ and ${}^H\underset{\sim}{\mathcal{O}}$ over $\boldsymbol{D}^n \times \sqrt{-1}\boldsymbol{R}^n$ of some kinds of H-valued holomorphic functions, where $\boldsymbol{D}^n$ is the radial compactification of $\boldsymbol{R}^n$(see the following definition 2.1).

DEFINITION 2.1 (Kawai). *We denote by* $\boldsymbol{D}^n$ *the compactification* $\boldsymbol{R}^n \sqcup S_\infty^{n-1}$ *of* $\boldsymbol{R}^n$, *where* $\boldsymbol{R}^n \sqcup S_\infty^{n-1}$ *denotes the disjoint union of* $\boldsymbol{R}^n$ *and an* $(n-1)$*-dimensional sphere* S_∞^{n-1} *at infinity. When* x *is a vector in* $\boldsymbol{R}^n - \{0\}$, *we denote by* $x\infty$ *the point on* S_∞^{n-1} *which is represented by* x *in the identification of* S_∞^{n-1} *with* $\boldsymbol{R}^n - \{0\}/\boldsymbol{R}^+$. *The space* $\boldsymbol{D}^n$ *is given the natural topology, that is:* (i) *If a point* x *of* $\boldsymbol{D}^n$ *belongs to* $\boldsymbol{R}^n$, *a fundamental system of neighbourhoods of* x *is the family of all open balls containing the point* x. (ii) *If a point* x *of* $\boldsymbol{D}^n$ *belongs to* S_∞^{n-1}, *a fundamental system of neighbourhoods of* x $(=y\infty)$ *is given by* $\{(C+a) \cup C_\infty \mid C_\infty \ni y\infty\}$ *where* C *is an open cone generated by some open neighbourhood of* y *with its vertex at the origin,* a *is some vector in* $\boldsymbol{R}^n$, *namely* $C+a$ *is a cone with its vertex at* a, *and* C_∞ *denotes the points at infinity of that cone.*

We shall mainly consider the space $\boldsymbol{D}^n \times \sqrt{-1}\,\boldsymbol{R}^n$ from now on. Here we give the definitions of sheaves ${}^H\tilde{\mathcal{O}}$ and ${}^H\underset{\sim}{\mathcal{O}}$ over $\boldsymbol{D}^n \times \sqrt{-1}\,\boldsymbol{R}^n$. In the following sections, we denote by ${}^H\mathcal{O}$ the sheaf over $\boldsymbol{C}^n$ of germs of H-valued holomorphic functions.

DEFINITION 2.2. (The sheaf of slowly increasing H-valued holomorphic functions.) *We define* ${}^H\tilde{\mathcal{O}}$ *to be the sheaf subordinate to the presheaf* $\{{}^H\tilde{\mathcal{O}}(\Omega)\}$, *where, for an open set* $\Omega \subset \boldsymbol{D}^n \times \sqrt{-1}\,\boldsymbol{R}^n$, *the section module* ${}^H\tilde{\mathcal{O}}(\Omega)$ *is the set of all* H*-valued holomorphic functions* $f(z)$ $(\in {}^H\mathcal{O}(\Omega \cap \boldsymbol{C}^n))$ *such that, for any positive* ε *and any compact set* K *in* Ω, $f(z)$ *satisfies the condition* $\sup_{z \in K \cap \boldsymbol{C}^n} \| f(z) e^{-\varepsilon |z|} \|_H < \infty$. *It is easy to see that the presheaf* $\{{}^H\tilde{\mathcal{O}}(\Omega)\}$ *is a sheaf over* $\boldsymbol{D}^n \times \sqrt{-1}\,\boldsymbol{R}^n$.

REMARK. By the above definition, it is easy to see that ${}^H\tilde{\mathcal{O}} \mid \boldsymbol{C}^n = {}^H\mathcal{O}$ holds.

DEFINITION 2.3. (The sheaf of rapidly decreasing H-valued holomorphic functions.) *We define* ${}^H\underset{\sim}{\mathcal{O}}$ *to be the sheaf subordinate to the presheaf* $\{{}^H\underset{\sim}{\mathcal{O}}(\Omega)\}$, *where, for an open set* $\Omega \subset \boldsymbol{D}^n \times \sqrt{-1}\,\boldsymbol{R}^n$, *the section module* ${}^H\underset{\sim}{\mathcal{O}}(\Omega)$ *is the set of all* H*-valued holomorphic functions* $f(z)$ $(\in {}^H\mathcal{O}(\Omega \cap \boldsymbol{C}^n))$ *such that, for any compact set* K *in* Ω, *there exists some constant* δ_K *so that it satisfies the condition* $\sup_{z \in K \cap \boldsymbol{C}^n} \| f(z) e^{\delta_K |z|} \|_H < \infty$. *It is easy to see that the presheaf* $\{{}^H\underset{\sim}{\mathcal{O}}(\Omega)\}$ *is a sheaf over* $\boldsymbol{D}^n \times \sqrt{-1}\,\boldsymbol{R}^n$.

REMARK. By the avove definition, it is easy to see that ${}^H\underset{\sim}{\mathcal{O}} \mid \boldsymbol{C}^n = {}^H\mathcal{O}$ holds.

DEFINTION 2.4. (Topology of ${}^{H}\underset{\sim}{\mathcal{Q}}(K)$.) *Let K be a compact set in $\boldsymbol{D}^n\times\sqrt{-1}\,\boldsymbol{R}^n$ and $\{U_m\}$ be a fundamental system of neighbourhood of K satisfying $U_m \supset\supset U_{m+1}$, (where $U_m\supset\supset U_{m+1}$ means that U_{m+1} has a compact neighbourhood in U_m with respect to the topology of $\boldsymbol{D}^n\times\sqrt{-1}\,\boldsymbol{R}^n$). Let ${}^{H}\mathcal{O}_c^m(U_m)$ be the Banach space of all H-valued holomorphic functions $f(z)$ $(\in {}^{H}\mathcal{O}(U_m\cap \boldsymbol{C}^n))$ which are continuous in $\bar{U}_m\cap\boldsymbol{C}^n$ and there exists some A such that $\|f(z)\|_H\leqq Ae^{-(1/m)|z|}$. The norm of ${}^{H}\mathcal{O}_c^m(U_m)$ is $|\!|\!|f|\!|\!|=\sup_{z\in K\cap \boldsymbol{C}^n}\|f(z)e^{(1/m)|z|}\|_H$. Then we give ${}^{H}\underset{\sim}{\mathcal{Q}}(K)$ the inductive limit topology $\varinjlim_{m} {}^{H}\mathcal{O}_c^m(U_m)$. The topology of ${}^{H}\underset{\sim}{\mathcal{Q}}(K)$ is well-defind and it becomes* DFS*-*space* (see Komatsu [8]). *Especially, when $K=\boldsymbol{D}^n$, we denote ${}^{H}\underset{\sim}{\mathcal{Q}}(\boldsymbol{D}^n)$ by ${}^{H}\mathcal{F}_*$.*

Next we construct some soft resolutions of the sheaves ${}^{H}\tilde{\mathcal{O}}$ and ${}^{H}\underset{\sim}{\mathcal{Q}}$.

DEFINITION 2.5. *Let Ω be an open set in $\boldsymbol{D}^n\times\sqrt{-1}\,\boldsymbol{R}^n$. We define ${}^{H}\mathcal{X}_j(\Omega)$ to be the set of all H-valued $(0,j)$ forms u on $\Omega\cap\boldsymbol{C}^n$ which satisfy the following conditions: for any positive ε and any compact set K in Ω,*

$$\int_{K\cap \boldsymbol{C}^n}\|u\|_H^2e^{-\varepsilon\eta(z)}d\lambda<\infty \quad\text{and}\quad \int_{K\cap \boldsymbol{C}^n}\|{}^{H}\bar{\partial}u\|_H^2e^{-\varepsilon\eta(z)}d\lambda<\infty.$$

We denote by ${}^{H}\mathcal{X}_j$ the sheaf subordinate to the presheaf $\{{}^{H}\mathcal{X}_j(\Omega)\}$.

DEFINITION 2.6. *Let Ω be an open set in $\boldsymbol{D}^n\times\sqrt{-1}\,\boldsymbol{R}^n$. We define ${}^{H}\mathcal{Y}_j(\Omega)$ to be the set of all H-valued $(0,j)$ forms u on $\Omega\cap\boldsymbol{C}^n$ which satisfy the following conditions: for any compact set K in Ω, there exists some positive constant δ_K such that*

$$\int_{K\cap \boldsymbol{C}^n}\|u\|_H^2e^{\delta_K\eta(z)}d\lambda<\infty \quad\text{and}\quad \int_{K\cap \boldsymbol{C}^n}\|{}^{H}\bar{\partial}u\|_H^2e^{\delta_K\eta(z)}d\lambda<\infty.$$

We denote by ${}^{H}\mathcal{Y}_j$ the sheaf subordinate to the presheaf $\{{}^{H}\mathcal{Y}_j(\Omega)\}$.

Then we have the following proposition.

PROPOSITION 2.7. *The sheaves ${}^{H}\mathcal{X}_j$ and ${}^{H}\mathcal{Y}_j$ are soft.*

PROOF. Let Ω be an open set in $\boldsymbol{D}^n\times\sqrt{-1}\,\boldsymbol{R}^n$. We define $\mathcal{A}(\Omega)$ to be the set of all functions $\varphi(z)$ in $C^\infty(\Omega\cap\boldsymbol{C}^n)$ which satisfy the following conditions: for any compact set K in Ω, $\sup_{z\in K\cap\boldsymbol{C}^n}|\varphi(z)|<\infty$ and $\sup_{z\in K\cap\boldsymbol{C}^n}|\nabla\varphi(z)|<\infty$. We denote by $\mathcal{A}$ the sheaf subordinate to the presheaf $\{\mathcal{A}(\Omega)\}$. Then $\mathcal{A}$ is a sheaf of ring with unit and the sheaves ${}^{H}\mathcal{X}_j$ and ${}^{H}\mathcal{Y}_j$ are $\mathcal{A}$-modules. Hence, in order to prove that ${}^{H}\mathcal{X}_j$ and ${}^{H}\mathcal{Y}_j$ are soft, it is sufficient to prove that $\mathcal{A}$ is soft. However, for any compact set K in Ω, we can find $C_j=K_j\times\sqrt{-1}\,I_j$ such that $K\subset\bigcup_{j=1}^{m}C_j\subset\subset\Omega$, where K_j is a relatively compact open set in $\boldsymbol{R}^n$ or an open convex cone and I_j is a relatively compact open set in $\boldsymbol{R}^n$. Hence we can find a C^∞ function $\varphi(z)$ on $\boldsymbol{C}^n$ which is equal to 1 on some neighbourhood of $K\cap \boldsymbol{C}^n$ and vanishes outside Ω with $\sup|\varphi(z)|<\infty$, $\sup|\nabla\varphi(z)|<\infty$. Hence $\mathscr{A}$ is soft. This completes the proof.

Proposition 2.8. $\{{}^H\mathscr{X}_j\}$ and $\{{}^H\mathscr{Y}_j\}$ *give the soft resolutions of the sheaves* ${}^H\tilde{\mathscr{O}}$ *and* ${}^H\underset{\sim}{\mathscr{O}}$, *respectively. That is,*

$$0\to{}^H\tilde{\mathscr{O}}\to{}^H\mathscr{X}_0\overset{{}^H\bar\partial}{\to}{}^H\mathscr{X}_1\overset{{}^H\bar\partial}{\to}\cdots\overset{{}^H\bar\partial}{\to}{}^H\mathscr{X}_{n-1}\overset{{}^H\bar\partial}{\to}{}^H\mathscr{X}_n\to 0\quad\text{(exact)}$$

and

$$0\to{}^H\underset{\sim}{\mathscr{O}}\to{}^H\mathscr{Y}_0\overset{{}^H\bar\partial}{\to}{}^H\mathscr{Y}_1\overset{{}^H\bar\partial}{\to}\cdots\overset{{}^H\bar\partial}{\to}{}^H\mathscr{Y}_{n-1}\overset{{}^H\bar\partial}{\to}{}^H\mathscr{Y}_n\to 0\quad\text{(exact)},$$

Let Ω *be an open set in* $\boldsymbol{D}^n\times\sqrt{-1}\,\boldsymbol{R}^n$. *Then we have the Dolbeault isomorphisms*:

$$H^p(\Omega,{}^H\tilde{\mathscr{O}})=\{u\in{}^H\mathscr{X}_p(\Omega)\,;{}^H\bar\partial u=0\}/{}^H\bar\partial({}^H\mathscr{X}_{p-1}(\Omega)),$$
$$H^p_c(\Omega,{}^H\underset{\sim}{\mathscr{O}})=\{u\in{}^H\mathscr{Y}_p(\Omega)_c;{}^H\bar\partial u=0\}/{}^H\bar\partial({}^H\mathscr{Y}_{p-1}(\Omega)_c).$$

(By $H^p_c(\Omega,{}^H\underset{\sim}{\mathscr{O}})$ *we mean the* p*-th cohomology group with compact support.)*

Proof. By the definitions of ${}^H\tilde{\mathscr{O}}$ and ${}^H\underset{\sim}{\mathscr{O}}$ and the existence theorem for ${}^H\bar\partial u=f$ with bounds (see lemma 1.8), we obtain the above soft resolutions of ${}^H\tilde{\mathscr{O}}$ and ${}^H\underset{\sim}{\mathscr{O}}$. (We can use Cauchy's integral formula to change the L^2-norm to the sup-norm for holomorphic functions.) This completes the proof.

Now we give another representations of the cohomology groups $H^p(\Omega,{}^H\tilde{\mathscr{O}})$ and $H^p_c(\Omega,{}^H\underset{\sim}{\mathscr{O}})$. For that purpose, we need some more function spaces.

Definition 2.9. *Let* Ω *be an open set in* $\boldsymbol{D}^n\times\sqrt{-1}\,\boldsymbol{R}^n$. *We define* ${}^HX_j(\Omega)$ *to be the set of all H-valued* $(0,j)$ *forms* u *on* $\Omega\cap\boldsymbol{C}^n$ *satisfying the following condition: for any compact set* K *in* Ω *and any positive* ε,

$$\int_{K\cap C^n}\|u\|_H^2e^{-\varepsilon\eta(z)}\,d\lambda<\infty.$$

Definition 2.10. *Let* Ω *be an open set in* $\boldsymbol{D}^n\times\sqrt{-1}\,\boldsymbol{R}^n$. *We define* ${}^HY_j(\Omega)$ *to be the set of all H-valued* $(0,j)$ *forms* u *with compact support in* Ω *satisfying the condition: for some positive* δ,

$$\int_{C^n}\|u\|_H^2e^{\delta\eta(z)}\,d\lambda<\infty.$$

From the above definitions, it is evident that ${}^HX_j(\Omega)$ has the natural FS*-space structure and ${}^HY_j(\Omega)$ has the natural DFS*-space structure and ${}^HY_{n-j}(\Omega)=[{}^HX_j(\Omega)]'$. Then obviously $H^p(\Omega,{}^H\tilde{\mathscr{O}})$ is isomorphic to the p-th cohomology group of the complex

$$\cdots \to {}^H X_{p-1}(\Omega) \overset{{}^H\bar\partial}{\to} {}^H X_p(\Omega) \overset{{}^H\bar\partial}{\to} {}^H X_{p+1}(\Omega) \to \cdots$$

and $H_c^p(\Omega, {}^H\mathscr{Q})$ is isomorphic to the p-th cohomology group of the complex

$$\cdots \to {}^H Y_{p-1}(\Omega) \overset{-{}^H\bar\partial}{\to} {}^H Y_p(\Omega) \overset{-{}^H\bar\partial}{\to} {}^H Y_{p+1}(\Omega) \to \cdots .$$

§3. Vanishing theorems and duality theorems of cohomology groups.

In this section, we shall mention some vanishing theorems and duality theorems of cohomology groups with ${}^H\tilde{\mathscr{O}}$ or ${}^H\mathscr{Q}$ as their coefficient sheaves. We shall also mention the relative cohomology groups with support in a compact set K of $\boldsymbol{D}^n \times \sqrt{-1}\boldsymbol{R}^n$.

DEFINITION 3.1 (Kawai). *We call an open set Ω in $\boldsymbol{D}^n \times \sqrt{-1}\boldsymbol{R}^n$ an* $\tilde{\mathscr{O}}$-pseudoconvex domain *if it satisfies the following conditions*:

(i) $\sup_{z \in K \cap C^n} |\operatorname{Im} z| = M < \infty$.

(ii) *There exists a plurisubharmonic function $\theta(z)$ on $\Omega \cap \boldsymbol{C}^n$ which satisfies $\{z; \theta(z) < c\} \subset\subset \Omega$ for any real c and $\sup_{z \in L \cap C^n} \theta(z) \leqq M_L$ for any $L \subset\subset \Omega$, where M_L is some constant.*

THEOREM 3.2. *For any $\tilde{\mathscr{O}}$-pseudoconvex domain Ω in $\boldsymbol{D}^n \times \sqrt{-1}\boldsymbol{R}^n$, we have $H^s(\Omega, {}^H\tilde{\mathscr{O}}) = 0$ $(s \geqq 1)$.*

PROOF. In our case where Ω is paracompact, $H^s(\Omega, {}^H\tilde{\mathscr{O}})$ is isomorphic to the Čech cohomology group. Hence we have only to prove the vanishing of the Čech cohomology group. It is sufficient to prove $\varinjlim_{\{\Omega_\nu\}} H^s(\{\Omega_\nu\}, {}^H\tilde{\mathscr{O}}) = 0$, where $\{\Omega_\nu\}$ satisfies the following conditions:

(i) $\Omega = \cup_{\nu \in N} \Omega_\nu$ (locally finite open covering),

(ii) $\Omega_\nu \cap \boldsymbol{C}^n = V_\nu$ is convex.

The theorem is the special case of the following lemma 3.3.

We define ${}^H C^s(Z^{\rm loc}_{(p,q)}(\{V_\nu\}; \text{infraexponential}))$ to be the set of all H-valued cochains $c = \{c_\nu\}$ which satisfies the following conditions:

(i) ${}^H\bar\partial c_\nu = 0$ in V_ν,

(ii) For any positive ε and any finite subset M of N^{s+1}

$$\sum_{\nu \in M} \int_{V_\nu} \|c_\nu\|_H^2 e^{-\varepsilon\eta(z)} d\lambda < \infty.$$

LEMMA 3.3. *For any $c \in {}^H C^s(Z^{\rm loc}_{(p,q)}(\{V_\nu\}; \text{infraexponential}))$ which satisfies $\delta c = 0$, there exists some $c' \in {}^H C^{s-1}(Z^{\rm loc}_{(p,q)}(\{V_\nu\}; \text{infraexponential}))$ such that $\delta c' = c$. (δ means the coboundary operator.)*

PROOF OF THE LEMMA. Let $\{\chi_\nu\}$ be the partition of unity subordinate to $\{V_\nu\}$ which satisfies the condition $\sup |\nabla \chi_\nu| < \infty$. Put $b_\alpha = \sum_j \chi_j c_{j\alpha}$.

Since $\delta c = 0$, we have $\delta b = c$. Hence $\delta {}^H\bar\partial b = 0$ because ${}^H\bar\partial c = 0$. By Cauchy's inequality,

$$\int_{V_\alpha} \| b_\alpha \|_H^2 e^{-\varepsilon\eta(z)} d\lambda \leqq \sum_j \int_{V_\alpha} \chi_j \| c_{j\alpha} \|_H^2 e^{-\varepsilon\eta(z)} d\lambda$$

for any positive ε. Since ${}^H\bar\partial b_\alpha = \sum_j \bar\partial\chi_j \wedge c_{j\alpha}$ from the condition on c, by the assumption of $\{\chi_\nu\}$, we have

$$\sum_{\alpha\in M}\int_{V_\alpha} \| {}^H\bar\partial b_\alpha \|_H^2 e^{-\varepsilon\eta(z)} d\lambda \leqq C\sum_j\sum_{\alpha\in M}\int_{V_{j\alpha}} \| c_{j\alpha} \|_H^2 e^{-\varepsilon\eta(z)} d\lambda < \infty,$$

for any positive ε and any finite subset M of N^s, where C is some constant.

We consider the case $s = 1$. The fact that $\delta({}^H\bar\partial b) = 0$ implies that ${}^H\bar\partial b$ defines a global section f in this case. Then f satisfies the condition

$$\int_{K\cap C^n} \| f \|_H^2 e^{-\varepsilon\eta(z)} d\lambda < \infty$$

for any positive ε and any $K \subset\subset \Omega$. Hence, by the lemma 1.8 and the existence of $\theta(z)$, we can find a solution u of ${}^H\bar\partial u = f$ such that

$$\int_{K\cap C^n} \| u \|_H^2 e^{-\varepsilon\eta(z)} (1 + |z|^2)^{-2} d\lambda < \infty$$

for any positive ε and any $K \subset\subset \Omega$. We define $c'_\alpha = b_\alpha - u \mid V_\alpha$. Then ${}^H\bar\partial c'_\alpha = 0$ and $c' \in {}^HC^{s-1}(Z^{\mathrm{loc}}_{(p,q)}(\{V_\nu\}; \text{infraexponential}))$ and $\delta c' = \delta b = c$. This proves the lemma in the case $s = 1$.

In the case $s > 1$, we use the induction on s. By the induction hypothesis there exists $b' \in {}^HC^{s-2}(Z^{\mathrm{loc}}_{(p,q+1)}(\{V_\nu\}; \text{infraexponential}))$ such that $\delta b' = {}^H\bar\partial b$. Then, by the lemma 1.8, we can find $b'' \in {}^HL^{2,\mathrm{loc}}_{(p,q)}(V_\alpha)$ such that $b'_\alpha = {}^H\bar\partial b''$ and

$$\sum_{\alpha\in M}\int_{V_\alpha} \| b''_\alpha \|_H^2 e^{-\varepsilon\eta(z)} (1 + |z|^2)^{-2} d\lambda < \infty$$

for any positive ε and any finite subset M of N^{s-1}. If we define $c' = b - \delta b''$, it satisfies all the required conditions. This completes the proof.

THEOREM 3.4. *Let K be a compact set in $\boldsymbol{D}^n$. Then* $H^p(K, {}^H\tilde{\mathcal{O}}) = 0$ $(p \geqq 1)$.

PROOF. Since K is a compact set in $\boldsymbol{D}^n$, K has a fundamental system of neighbourhoods composed of $\tilde{\mathcal{O}}$-pseudoconvex domains Ω_j (see Kawai [7] p. 473, Theorem 2.1.6). Then $H^p(K, {}^H\tilde{\mathcal{O}}) = \varinjlim_j H^p(\Omega_j, {}^H\tilde{\mathcal{O}})$. Hence it is sufficient to prove $\varinjlim_j H^p(\Omega_j, {}^H\tilde{\mathcal{O}}) = 0$ $(p \geqq 1)$. For any cocycle $\{c_\nu\}$ in $H^p(\Omega_j, {}^H\tilde{\mathcal{O}})$ we can assume that $d = \{d_\nu = \cosh \varepsilon z \times c_\nu \mid \Omega_{j+1}\}$ defines a cocycle in $H^p(\Omega_{j+1}, {}^H\tilde{\mathcal{O}})$ for some positive ε. Since $H^p(\Omega_{j+1}, {}^H\tilde{\mathcal{O}}) = 0$, $\{d_\nu\}$ is a coboundary in $H^p(\Omega_{j+1}, {}^H\tilde{\mathcal{O}})$, that is, $d = \delta d'$ where δ denotes the coboundary operator. If we put $c' = \{\cosh(-\varepsilon z) \times d'_\nu\}$, $c \mid \Omega_{j+1} = \delta c'$.

Hence the image of $H^p(\Omega_j, {}^H\mathscr{Q})$ in $H^p(\Omega_{j+1}, {}^H\mathscr{Q})$ is zero. Therefore $\varinjlim_j H^p(\Omega_j, {}^H\mathscr{Q}) = 0$. This completes the proof.

THEOREM 3.5. *Let Ω be an open set in $\boldsymbol{D}^n \times \sqrt{-1}\boldsymbol{R}^n$. Then $H^n(\Omega, {}^H\tilde{\mathscr{O}}) = 0$.*

PROOF. We use the notation in §2. From the last statement of §2, in order to prove the theorem, it is sufficient to prove

$${}^HX_{n-1}(\Omega) \overset{{}^H\bar\partial}{\to} {}^HX_n(\Omega) \to 0 \quad \text{(exact)}.$$

Let $\{K_j\}$ be an increasing sequence of compact sets which are contained in Ω and exhaust Ω, and define

$${}^HX^j_\ell(K_j) = \{u \in {}^HL^{2,\mathrm{loc}}_{(0,\ell)}(\Omega \cap \boldsymbol{C}^n)\ ;\ \int_{K_j \cap \boldsymbol{C}^n} \|u\|^2_H e^{-(1/j)\eta(z)}\, d\lambda < \infty\}.$$

Then ${}^HX_\ell(\Omega) = \varprojlim_j {}^HX^j_\ell(K_j)$ and ${}^HY_{n-\ell}(\Omega) = \varinjlim_j ({}^HX^j_\ell(K_j))'$. We represent $({}^HX^j_\ell(K_j))'$ by

$$\{u \in {}^HL^2_{(0,\ell)}(\boldsymbol{C}^n)\ ;\ supp\ u \subset K_j \cap \boldsymbol{C}^n \text{ and } \int_{\boldsymbol{C}^n} \|u\|^2_H e^{(1/j)\eta(z)}\, d\lambda < \infty\}$$

and $({}^H\bar\partial)' = {}^H\vartheta$. (Here we have used the natural identification of $(0,\ell)$ form with $(0, n-\ell)$ form.) Then, in order to prove the surjectivity of ${}^H\bar\partial$, it is sufficient to prove that ${}^H\vartheta$ is injective and of closed range. However, since ${}^H\vartheta$ becomes an elliptic operator from ${}^HY_n(\Omega)$ to ${}^HY_{n-1}(\Omega)$, it is trivial by the unique continuation property that ${}^H\vartheta$ is injective. Now we will prove that ${}^H\vartheta$ is of closed range. For that purpose, by the usual argument of DFS*-space (see the proof of the lemma 1.8), it is sufficient to show that $f \in Im\,({}^H\vartheta)$ assuming that ${}^H\vartheta u_\nu$ converges to f weakly in $({}^HX^j_{n-1}(K_j))'$.

We denote by $\hat K_{j+1}$ the closure in $\boldsymbol{D}^n \times \sqrt{-1}\boldsymbol{R}^n$ of the union of $K_{j+1} \cap \boldsymbol{C}^n$ and all the connected components of $(\boldsymbol{C}^n - K_{j+1})$ which are relatively compact with respect to the topology of $\boldsymbol{R}^{2n}$. Then, for each ${}^H\vartheta u_\nu$, there ists some $v_\nu \in ({}^HX^j_n(\hat K_{j+1}))'$ such that ${}^H\vartheta u_\nu = {}^H\vartheta v_\nu$. In fact, if ${}^H\vartheta u \in ({}^HX^j_{n-1}(K_j))'$ and $u \in ({}^HX^k_n(K_k))'$ $(j<k)$ then $supp\ u \subset \hat K_{j+1}$ because ${}^H\vartheta$ is elliptic. On the other hand $\sup_{z \in K_{j+1} \cap C^n} |\mathrm{Im}\, z| < \infty$ follows from the definition of the topology of $\boldsymbol{D}^n \times \sqrt{-1}\boldsymbol{R}^n$. Therefore it is seen from Lebesgue's theorem that

$$0 = \int ((\varphi_m u), {}^H\bar\partial g)_H d\lambda = \int ({}^H\vartheta(\varphi_m u), g)_H d\lambda = \int (\varphi_m {}^H\vartheta u, g)_H d\lambda \to \int ({}^H\vartheta u, g)_H d\lambda,$$

where $\varphi_m(z) = \exp(-(1/m)\sum_{j=1}^n \bar z_j{}^2)$, $\Omega \supset\supset L \supset\supset \hat K_{j+1}$ and $g \in {}^HX^j_{n-1}(L)$ satisfying ${}^H\bar\partial g = 0$ and $(\ ,\ )_H$ denotes the inner product in H.
$({}^HX^j_{n-1}(L) = \{g \in {}^HL^{2,\mathrm{loc}}_{(0,n-1)}(\Omega \cap \boldsymbol{C}^n)\ ;\ \int_{L \cap \boldsymbol{C}^n} \|g\|^2_H e^{-(1/j)\eta(z)}\, d\lambda < \infty\}.)$

Hence, by the proposition 1.10, we may consider that ${}^{H}\vartheta u \in {}^{H}\vartheta({}^{H}X^{j}_{n}(L))'$ Then, using the ellipticity of ${}^{H}\vartheta$ again, we can find some $w \in ({}^{H}X^{j}_{n}(\hat{K}_{j+1}))'$. such that ${}^{H}\vartheta u = {}^{H}\vartheta w$. Hence, by taking $K_j = \hat{K}_j$, we may assume that ${}^{H}\vartheta u_\nu \in ({}^{H}X^{j}_{n-1}(K_j))'$ and $u_\nu \in ({}^{H}X^{j}_{n}(K_j))'$ and ${}^{H}\vartheta u_\nu \underset{w}{\rightarrow} f$ in $({}^{H}X^{j}_{n-1}(K_j))'$ from the beginning. Choose L so that $K_j \subset\subset L \subset\subset \Omega$, then

$$0 = \int ({}^{H}\vartheta u_\nu, g)_H d\lambda \rightarrow \int (f, g)_H d\lambda$$

for any $g \in {}^{H}X^{j}_{n-1}(L)$ such that ${}^{H}\bar{\partial} g = 0$. Then we have $f = {}^{H}\vartheta v \in {}^{H}\vartheta({}^{H}X^{j}_{n}(L))'$ by the proposition 1.10, again. Since by the unique continuation theorem $supp\ v \subset \hat{K}_j (= K_j)$, this means that f is in the range of ${}^{H}\vartheta$: $({}^{H}X^{j}_{n}(K_j))' \rightarrow ({}^{H}X^{j}_{n-1}(K_j))'$, and this completes the proof.

THEOREM 3.6. *Let Ω be an open set in $\boldsymbol{D}^n \times \sqrt{-1}\boldsymbol{R}^n$. If* dim $H^p(\Omega, {}^{H}\tilde{\mathcal{O}}) < \infty$ $(p \geq 1)$, *then* $(H^j(\Omega, {}^{H}\tilde{\mathcal{O}}))' \cong H^{n-j}_c(\Omega, {}^{H}\underset{\sim}{\mathcal{O}})$ for $j = 0, 1, \cdots, n$.

PROOF. We consider the following dual complexes which was introduced in the last part of § 2

$$\begin{array}{ccccccccccc} 0 & \rightarrow & {}^{H}X_0(\Omega) & \xrightarrow{{}^{H}\bar{\partial}} & {}^{H}X_1(\Omega) & \xrightarrow{{}^{H}\bar{\partial}} \cdots \xrightarrow{{}^{H}\bar{\partial}} & {}^{H}X_{n-1}(\Omega) & \xrightarrow{{}^{H}\bar{\partial}} & {}^{H}X_n(\Omega) & \rightarrow & 0 \\ & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \\ 0 & \rightarrow & {}^{H}Y_n(\Omega) & \xleftarrow{-{}^{H}\bar{\partial}} & {}^{H}Y_{n-1}(\Omega) & \xleftarrow{-{}^{H}\bar{\partial}} \cdots \xleftarrow{-{}^{H}\bar{\partial}} & {}^{H}Y_1(\Omega) & \xleftarrow{-{}^{H}\bar{\partial}} & {}^{H}Y_0(\Omega) & \leftarrow & 0. \end{array}$$

From the assumption, we may conclude by Schwartz's lemma that each ${}^{H}\bar{\partial}$ in the upper row is of closed range (see Komatsu [9], p. 382, Theorem 20). Hence the theorem follows from the Serre-Komatsu duality theorem for FS*-spaces (see Komatsu [8], p. 381, Theorem 19).

THEOREM 3.7. *Let K be a compact set in $\boldsymbol{D}^n \times \sqrt{-1}\boldsymbol{R}^n$ for which $H^p(K, {}^{H}\underset{\sim}{\mathcal{O}}) = 0$ $(p \geq 1)$ hold. Let V be an open neighbourhood of K. Then we have $H^p_K(V, {}^{H}\tilde{\mathcal{O}}) = 0$ $(p \neq n)$ and $H^n_K(V, {}^{H}\tilde{\mathcal{O}}) \cong [{}^{H}\underset{\sim}{\mathcal{O}}(K)]'$, where $H^p_K(V, {}^{H}\tilde{\mathcal{O}})$ denotes the relative cohomology group with support in K.*

PROOF. By the excision theorem, we may assume $H^p(V, {}^{H}\tilde{\mathcal{O}}) = 0$ $(p \geq 1)$. Consider the following exact sequence

$$\begin{aligned} 0 &\rightarrow H^0_K(V, {}^{H}\tilde{\mathcal{O}}) \rightarrow H^0(V, {}^{H}\tilde{\mathcal{O}}) \rightarrow H^0(V-K, {}^{H}\tilde{\mathcal{O}}) \\ &\rightarrow H^1_K(V, {}^{H}\tilde{\mathcal{O}}) \rightarrow H^1(V, {}^{H}\tilde{\mathcal{O}}) \rightarrow H^1(V-K, {}^{H}\tilde{\mathcal{O}}) \\ &\rightarrow \cdots\cdots\cdots\cdots\cdots\cdots \rightarrow H^{n-1}(V-K, {}^{H}\tilde{\mathcal{O}}) \\ &\rightarrow H^n_K(V, {}^{H}\tilde{\mathcal{O}}) \rightarrow H^n(V, {}^{H}\tilde{\mathcal{O}}) \rightarrow H^n(V-K, {}^{H}\tilde{\mathcal{O}}) \rightarrow 0. \end{aligned}$$

Here $H^p(V, {}^{H}\tilde{\mathcal{O}}) = 0$ $(p \geq 1)$ by the assumption on V and $H^0_K(V, {}^{H}\tilde{\mathcal{O}}) = 0$ by the unique continuation theorem. Hence we have the isomorphisms

$$\begin{cases} H^1_K(V, {}^{H}\tilde{\mathcal{O}}) \cong H^0(V-K, {}^{H}\tilde{\mathcal{O}}) / H^0(V, {}^{H}\tilde{\mathcal{O}}), \\ H^p_K(V, {}^{H}\tilde{\mathcal{O}}) \cong H^{p-1}(V-K, {}^{H}\tilde{\mathcal{O}}) \ (p \geq 2). \end{cases}$$

On the other hand, we have the following exact sequence

$$\begin{aligned} 0 &\to H_c^0(V-K, {}^H\mathscr{O}) \to H_c^0(V, {}^H\mathscr{O}) \to H^0(K, {}^H\mathscr{O}) \\ &\to H_c^1(V-K, {}^H\mathscr{O}) \to H_c^1(V, {}^H\mathscr{O}) \to H^1(K, {}^H\mathscr{O}) \\ &\to \cdots\cdots \\ &\to H_c^p(V-K, {}^H\mathscr{O}) \to H_c^p(V, {}^H\mathscr{O}) \to H^p(K, {}^H\mathscr{O}) \to \cdots\cdots . \end{aligned}$$

Here $H^p(K, {}^H\mathscr{O}) = 0$ ($p \geqq 1$) by the assumption on K. Therefore we obtain the isomorphisms

$$\begin{cases} H^0(K, {}^H\mathscr{O}) \cong H_c^1(V-K, {}^H\mathscr{O}), \\ H_c^p(V-K, {}^H\mathscr{O}) \cong H_c^p(V, {}^H\mathscr{O}) \ (p \geqq 2). \end{cases}$$

By the theorem 3.6, $H_c^p(V, {}^H\mathscr{O}) = 0$ ($p \neq n$). Thus we have the following isomorphisms

$$\begin{cases} H_c^p(V-K, {}^H\mathscr{O}) = 0 \ (p \neq 1, n) \\ H_c^n(V-K, {}^H\mathscr{O}) \cong [{}^H\widetilde{\mathscr{O}}(V)]'. \end{cases}$$

Now we consider the following dual complexes,

$$\begin{array}{l} 0 \to {}^HX_0(V-K) \xrightarrow{{}^H\bar{\partial}_0} {}^HX_1(V-K) \xrightarrow{{}^H\bar{\partial}_1} \cdots \xrightarrow{{}^H\bar{\partial}_{n-2}} {}^HX_{n-1}(V-K) \xrightarrow{{}^H\bar{\partial}_{n-1}} {}^HX_n(V-K) \to 0 \\ \qquad \updownarrow \qquad\qquad\qquad \updownarrow \qquad\qquad\qquad\qquad\qquad \updownarrow \qquad\qquad\qquad \updownarrow \\ 0 \leftarrow {}^HY_n(V-K) \xleftarrow{-{}^H\bar{\partial}_{n-1}} {}^HY_{n-1}(V-K) \xleftarrow{-{}^H\bar{\partial}_{n-2}} \cdots \xrightarrow{-{}^H\bar{\partial}_1} {}^HY_1(V-K) \xleftarrow{-{}^H\bar{\partial}_0} {}^HY_0(V-K) \leftarrow 0. \end{array}$$

Then, since $H_c^p(V-K, {}^H\mathscr{O}) = 0$ ($p \neq 1, n$), the range of $(-{}^H\bar{\partial}_j)$ is closed except for $j=0$, $n-1$. However, ${}^H\bar{\partial}_{n-1}$ is of closed range by the theorem 3.5. Hence, by the closed range theorem, $-{}^H\bar{\partial}_0$ is of closed range (see Komatsu [8], p. 381, Theorem 19).

In order to prove the closed rangeness of $(-{}^H\bar{\partial}_{n-1})$, we consider the following commutative diagram,

$$\begin{array}{ccc} 0 \leftarrow {}^HY_n(V-K) & \xleftarrow{-{}^H\bar{\partial}_{n-1}^{V-K}} & {}^HY_{n-1}(V-K) \\ \downarrow i & & \downarrow \\ 0 \leftarrow {}^HY_n(V) & \xleftarrow{-{}^H\bar{\partial}_{n-1}^{V}} & {}^HY_{n-1}(V) \end{array}$$

(the map i is the natural injection). However, in the dual complexes for V, ${}^H\bar{\partial}_0^V$ is of closed range since $H^1(V, {}^H\widetilde{\mathscr{O}}) = 0$. Thus, by the closed range theorem, $Im(-{}^H\bar{\partial}_{n-1}^{V-K}) = i^{-1}(Im\,{}^H\bar{\partial}_{n-1}^V)$ is closed. Therefore all $-{}^H\bar{\partial}_j^{V-K}$ are of closed range. Hence by the Serre-Komatsu duality theorem, we have the isomorphisms $[H^p(V-K, {}^H\widetilde{\mathscr{O}})]' \cong H_c^{n-p}(V-K, {}^H\mathscr{O})$. Hence we have $[H^0(V-K, {}^H\widetilde{\mathscr{O}})]' \cong H_c^n(V-K, {}^H\mathscr{O}) \cong H_c^n(V, {}^H\mathscr{O}) \cong [H^0(V, {}^H\widetilde{\mathscr{O}})]'$.
Here $H^0(V-K, {}^H\widetilde{\mathscr{O}})$ and $H^0(V, {}^H\widetilde{\mathscr{O}})$ are both FS*-spaces, a posteriori, reflexive. Hence we have the isomorphism $H^0(V, {}^H\widetilde{\mathscr{O}}) \cong H^0(V-K, {}^H\widetilde{\mathscr{O}})$.
Thus $H_K^1(V, {}^H\widetilde{\mathscr{O}}) \cong H^0(V-K, {}^H\widetilde{\mathscr{O}}) / H^0(V, {}^H\widetilde{\mathscr{O}}) = 0$. If $p \geqq 2$, $p \neq n$, we have $0 = H_c^{n-p+1}(V, {}^H\mathscr{O}) \cong H_c^{n-p+1}(V-K, {}^H\mathscr{O}) \cong [H^{p-1}(V-K, {}^H\widetilde{\mathscr{O}})]' \cong [H_K^p(V, {}^H\widetilde{\mathscr{O}})]'$. Hence $H_K^p(V, {}^H\widetilde{\mathscr{O}}) = 0$. In the case $p = n$, we have $[H_K^n(V, {}^H\widetilde{\mathscr{O}})]' \cong [H^{n-1}(V-K, {}^H\widetilde{\mathscr{O}})]' \cong H_c^1(V-K, {}^H\mathscr{O}) \cong H^0(K, {}^H\mathscr{O}) \cong {}^H\mathscr{O}(K)$.
Since ${}^H\mathscr{O}(K)$ is DFS*-space, it follows from the Serre-Komatsu duality

theorem that the above isomorphism is a topological isomorphism. Hence we have the isomorphism $H^n_K(V, {}^H\tilde{\mathcal{O}}) \cong [{}^H\underset{\sim}{\mathcal{Q}}(K)]'$. This completes the proof.

§ 4. Approximation theorem in ${}^H\underset{\sim}{\mathcal{Q}}(K)$.

In this section we prove the analogue of Runge's theorem.

Theorem 4.1. *Let K be a compact set in $\boldsymbol{D}^n$. Then ${}^H\mathcal{P}_* = {}^H\underset{\sim}{\mathcal{Q}}(\boldsymbol{D}^n)$ is dense in ${}^H\underset{\sim}{\mathcal{Q}}(K)$.*

PROOF. We need the following lemma.

LEMMA 4.2 (Kawai). *Let K be a compact set in $\boldsymbol{D}^n$, and put $U_j = \boldsymbol{D}^n \times \sqrt{-1}\{y \mid \sum_{k=1}^n y_k^2 < (1/j)\}$. Then there exists $\{\Omega_j\}$, a fundamental sysem of neighbourhoods of K, which have the following properties:*

(a) *$U_j \supset\supset \Omega_j \supset\supset K$ and Ω_j's tend to K decreasingly.*

(b) *For any j and any $T (\subset\subset \Omega_j)$ there exist an open set V and $\theta(z)$ which is strictly plurisubharmonic in U_j such that they satisfy the following conditions:*

(i) *$T \subset\subset V \subset\subset \Omega_j$,*

(ii) *$\theta(z) < 0$ on $T \cap \boldsymbol{C}^n$,*

(iii) *$\theta(z) > 0$ near $\partial V \cap \boldsymbol{C}^n$,*

(iv) *For any $L \subset\subset \Omega_j$ there exists a constant M_L such that $\sup_{z \in L \cap \boldsymbol{C}^n} \theta(z) \leqq M_L < \infty$.*

PROOF OF THE LEMMA. See Kawai[7], p. 476.

Now we continue the proof of the theorem. For the proof of the theorem, we need some spaces. Let Ω be an open set in $\boldsymbol{D}^n \times \sqrt{-1}\boldsymbol{R}^n$. For a continuous function ψ on $\Omega \cap \boldsymbol{C}^n$, we define ${}^H L^2_{\text{loc}}(\Omega, \psi)$ to be the set of all H-valued measurable functions $f(z)$ on $\Omega \cap \boldsymbol{C}^n$ which satisfy

$$\int_{K \cap \boldsymbol{C}^n} \|f\|_H^2 e^{-\psi(z)} d\lambda < \infty$$

for any $K \subset\subset \Omega$. Then, for a positive ε, we define ${}^H\mathscr{A}^{2,-2\varepsilon}_{\text{loc}}(\Omega) = {}^H L^2_{\text{loc}}(\Omega, -2\varepsilon\eta(z)) \cap {}^H\mathcal{O}(\Omega \cap \boldsymbol{C}^n)$ and ${}^H L^{2,-\varepsilon}_{\text{loc}}(\Omega) = {}^H L^2_{\text{loc}}(\Omega, -\varepsilon\eta(z))$ and ${}^H X^{-\varepsilon}(\Omega)$ to be the closure of ${}^H\mathscr{A}^{2,-2\varepsilon}_{\text{loc}}(\Omega)$ in ${}^H L^{2,-\varepsilon}_{\text{loc}}(\Omega)$.
Then we have the following lemma.

LEMMA 4.3. *If $\varepsilon < \delta\ (<2\varepsilon)$, then ${}^H\mathscr{A}^{2,-\delta}_{\text{loc}}(\Omega)$ is contained in ${}^H X^{-\varepsilon}(\Omega)$.*

PROOF OF THE LEMMA. If we put
$${}^{H}\mathscr{A}^{2,-\delta,-2\log(1+|z|^2)}_{\mathrm{loc}}(\Omega) = {}^{H}L^2_{\mathrm{loc}}(\Omega, -\delta\eta(z) - 2\log(1+|z|^2)) \cap {}^{H}\mathscr{O}(\Omega\cap \boldsymbol{C}^n),$$
and $\delta' < \delta$, we have
$$\begin{aligned}{}^{H}\mathscr{A}^{2,-2\varepsilon,-2\log(1+|z|^2)}_{\mathrm{loc}}(\Omega) &\subset {}^{H}\mathscr{A}^{2,-2\varepsilon}_{\mathrm{loc}}(\Omega) \subset {}^{H}\mathscr{A}^{2,-\delta}_{\mathrm{loc}}(\Omega)\\ &\subset {}^{H}\mathscr{A}^{2,-\delta',-2\log(1+|z|^2)}_{\mathrm{loc}}(\Omega).\end{aligned}$$
Hence, in order to prove the lemma, it is sufficient to show that ${}^{H}\mathscr{A}^{2,-\delta,-2\log(1+|z|^2)}_{\mathrm{loc}}(\Omega)$ is dense in ${}^{H}\mathscr{A}^{2,-\delta',-2\log(1+|z|^2)}_{\mathrm{loc}}(\Omega)$ when $\delta' < \delta$. We also put ${}^{H}L^{2,-\varepsilon,-2\log(1+|z|^2)}_{\mathrm{loc}}(\Omega) = {}^{H}L^2_{\mathrm{loc}}(\Omega, -\varepsilon\eta(z) - 2\log(1+|z|^2))$. We wish to prove that $\mu \in ({}^{H}L^{2,-\delta',-2\log(1+|z|^2)}_{\mathrm{loc}}(\Omega))'$ is orthogonal to ${}^{H}\mathscr{A}^{2,-\delta',-2\log(1+|z|^2)}_{\mathrm{loc}}(\Omega)$ if it is orthogonal to ${}^{H}\mathscr{A}^{2,-\delta,-2\log(1+|z|^2)}_{\mathrm{loc}}(\Omega)$. If we assume that μ is not orthogonal to ${}^{H}\mathscr{A}^{2,-\delta',-2\log(1+|z|^2)}_{\mathrm{loc}}(\Omega)$, by the Hahn-Banach theorem we can find some u whose support is compact in Ω with
$$\int_{\boldsymbol{C}^n} \|u\|_H^2 e^{-\delta'\eta(z)-2\log(1+|z|^2)}\,d\lambda < \infty$$
so that $\langle \mu, v\rangle = \int_{\Omega\cap \boldsymbol{C}^n}(v, u)_H d\lambda$ for any $v \in {}^{H}L^{2,-\delta',-2\log(1+|z|^2)}_{\mathrm{loc}}(\Omega)$. If φ belongs to ${}^{H}\mathscr{A}^{2,-\delta',-2\log(1+|z|^2)}_{\mathrm{loc}}(\Omega)$ and $\langle\mu,\varphi\rangle \neq 0$, then $\varphi(z)\exp(-(1/m)\sum_{j=1}^{n} z_j^2)$ belongs to ${}^{H}\mathscr{A}^{2,-\delta,-2\log(1+|z|^2)}_{\mathrm{loc}}(\Omega)$ under the condition $\sup_{z\in\Omega\cap\boldsymbol{C}^n}|\mathrm{Im} z| < \infty$. Therefore we have
$$\begin{aligned}0 &= \langle \mu, \varphi(z)\exp(-(1/m)\sum_{j=1}^{n} z_j^2)\rangle = \int_{\Omega\cap\boldsymbol{C}^n}(\varphi(z)\exp(-(1/m)\sum_{j=1}^{n} z_j^2), u)_H d\lambda\\ &\to \int_{\Omega\cap\boldsymbol{C}^n}(\varphi, u)_H d\lambda = \langle\mu,\varphi\rangle\end{aligned}$$
by Lebesgue's theorem. This contradicts the assumption on φ. Thus we have proved that ${}^{H}\mathscr{A}^{2,-\delta,-2\log(1+|z|^2)}_{\mathrm{loc}}(\Omega)$ is dense in ${}^{H}\mathscr{A}^{2,-\delta',-2\log(1+|z|^2)}_{\mathrm{loc}}(\Omega)$, so that we have proved that ${}^{H}\mathscr{A}^{2,-\delta}_{\mathrm{loc}}(\Omega)$ is contained in ${}^{H}X^{-\varepsilon}(\Omega)$. This completes the proof of the lemma 4.3.

Now we return to the proof of the theorem. Since ${}^{H}\mathscr{P}_* = \varinjlim_j {}^{H}\mathscr{Q}^j_c(U_j)$ and ${}^{H}\mathscr{Q}(K) = \varinjlim_{\varepsilon, j} {}^{H}X^{-\varepsilon}(\Omega_j)$, it is sufficient to prove the following statement (*).

(*) If an element μ of $({}^{H}X^{-\varepsilon}(\Omega_j))'$ is orthogonal to $B \equiv {}^{H}\mathscr{O}(U_j\cap \boldsymbol{C}^n) \cap {}^{H}L^2_{\mathrm{loc}}(U_j, -\delta\eta(z))$, then μ is zero.

From now on we fix ε and j, so we denote by ${}^{H}X$ the space ${}^{H}X^{-\varepsilon}(\Omega_j)$ and by U the open set U_j and by Ω the open set Ω_j. Now we assume that $\mu \neq 0$. Then, by the Hahn-Banach theorem, there exists some u such that *supp* u is comact in Ω and
$$\int_{\boldsymbol{C}^n}\|u\|_H^2 e^{-\varepsilon\eta(z)}\,d\lambda < \infty \quad\text{and}\quad \langle\mu, v\rangle = \int_{\Omega\cap\boldsymbol{C}^n}(v, u)_H d\lambda \ \text{ for any } v \in {}^{H}X.$$

Take *supp* u as T and fix V and $\theta(z)$ which corespond to T in the lemma 4.2. We define $C=\bigcup_{\lambda=1}^{\infty}\{v\in {}^{H}L^2_{loc}(U;\lambda\theta^{+}-\delta'\eta(z)-2\log(1+|z|^2))\mid {}^{H}\bar{\partial}v=0\}$ where $\theta^{+}(z)=\max\{0,\theta(z)\}$ and $2\varepsilon>\delta'>\delta$. Then, by the condition (iv) on $\theta(z)$ in the lemma 4.2, C is contained in B. Since μ is zero on B and *supp* $u\subset\subset\Omega$,

$$\langle\mu, v\rangle=\int_{\Omega\cap C^n}(v, u)_H d\lambda=\int_{U\cap C^n}(v, u)_H d\lambda=0$$

for any v in C. Moreover, by the condition (ii) on $\theta(z)$ in the lemma 4.2, $u(z)$ is zero where $\theta(z)>0$. Defining $g_{\delta''}(z)=cosh(\delta'' z)$ we have

$$\int_{U\cap C^n}(v, u)_H d\lambda=\int_{U\cap C^n}(vg_{\delta''}(z), u/g_{\delta''}(\bar{z}))_H d\lambda,$$

where $2\varepsilon>\delta''>\delta'$. Here, from the first, we choose ε so small as to secure $g_{\delta''}(\bar{z})\neq 0$ in U. (The assumption has no essential significance.) If we put $\tilde{u}=u/g_{\delta''}(\bar{z})$, then, by the proposition 1.10, we have some F which satisfies the following conditions (i), (ii) and (iii):

(i) $\tilde{u}={}^{H}\vartheta F$,

(ii) $F=0$ near $\partial V\cap C^n$,

(iii) $F\in {}^{H}L^2_{(0,1)}(U;-(\delta''-\delta')\eta(z)+2\log(1+|z|^2))$.

Then we may consider that $f(z)=F(z)g_{\delta''}(\bar{z})$ satisfies the following conditions:

(a) ${}^{H}\vartheta f=u$,

(b) supp $f\subset V\subset\subset U$,

(c) $f\in {}^{H}L^2_{(0,1)}(U;\delta'\eta(z)+2\log(1+|z|^2))$.

Therfore, by an integration by parts, we can prove the following equality for any $v\in {}^{H}\mathscr{A}^{2,-2\varepsilon}_{loc}(\Omega)$ where $2\varepsilon>\delta'''>\delta'$.

$$\begin{aligned}0&=\int_{\Omega\cap C^n}({}^{H}\bar{\partial}v, f)_H d\lambda=\int_{\Omega\cap C^n}({}^{H}\bar{\partial}(vg_{\delta'''}(z)), f/g_{\delta'''}(\bar{z}))_H d\lambda\\&=\int_{\Omega\cap C^n}(vg_{\delta'''}(z), {}^{H}\vartheta(f/g_{\delta'''}(\bar{z})))_H d\lambda=\int_{\Omega\cap C^n}(v, {}^{H}\vartheta f)_H d\lambda\\&=\int_{\Omega\cap C^n}(v, u)_H d\lambda=\langle\mu, v\rangle\end{aligned}$$

Thus we have proved that μ is zero on a dense subset of X, so we may conclude that μ is zero. This contradicts the assumption on μ, so that μ must be zero. This completes the proof.

§5. The sheaf ${}^{H}\mathscr{R}$ of H-valued Fourier hyperfunctions over $\boldsymbol{D}^n$.

In this section, we shall prove the pure-codimensionality of $\boldsymbol{D}^n$ with respect to ${}^{H}\tilde{\mathcal{O}}$. Then we shall define the H-valued Fourier hyperfunctions and the sheaf ${}^{H}\mathscr{R}$ of H-valued Fourier hyperfunctions over $\boldsymbol{D}^n$ and study some of their properties.

First we shall prove the pure-codimensionality of $\boldsymbol{D}^n$ with respect to ${}^H\tilde{\mathcal{O}}$.

THEOREM 5.1. *Let Ω be an open set in $\boldsymbol{D}^n$. Let V be an open set in $\boldsymbol{D}^n \times \sqrt{-1}\,\boldsymbol{R}^n$ which contains Ω as its closed subset. Then $H^p_\Omega(V, {}^H\tilde{\mathcal{O}}) = 0$ $(p \neq n)$. Here $H^p_\Omega(V, {}^H\tilde{\mathcal{O}})$ denotes the p-th relative cohomology group with respect to the pair $(V, V-\Omega)$ whose coefficient sheaf is ${}^H\tilde{\mathcal{O}}$.*

PROOF. By the excision theorem, we may assume that $V = \boldsymbol{D}^n \times \sqrt{-1}\,I^n - \partial_{\boldsymbol{D}^n}\Omega$ where $I^n = (-1, 1) \times \cdots \times (-1, 1)$ and the symbol $\partial_{\boldsymbol{D}^n}\Omega$ means the boundary of Ω in $\boldsymbol{D}^n$, which we denote by $\partial\Omega$ in short in the following. We denote $U = \boldsymbol{D}^n \times \sqrt{-1}\,I^n$. Then, since $U - \Omega^a \subset V = U - \partial\Omega \subset U$ where Ω^a means the closure of Ω in $\boldsymbol{D}^n$, we have the following exact sequence

$$\begin{aligned} 0 &\to H^0_{\partial\Omega}(U, {}^H\tilde{\mathcal{O}}) \to H^0_{\Omega^a}(U, {}^H\tilde{\mathcal{O}}) \to H^0_\Omega(V, {}^H\tilde{\mathcal{O}}) \\ &\to H^1_{\partial\Omega}(U, {}^H\tilde{\mathcal{O}}) \to \cdots\cdots \\ &\cdots\cdots \to H^{n-1}_\Omega(V, {}^H\tilde{\mathcal{O}}) \\ &\to H^n_{\partial\Omega}(U, {}^H\tilde{\mathcal{O}}) \to H^n_{\Omega^a}(U, {}^H\tilde{\mathcal{O}}) \to H^n_\Omega(V, {}^H\tilde{\mathcal{O}}) \to H^{n+1}_{\partial\Omega}(U, {}^H\tilde{\mathcal{O}}) \cdots\cdots. \end{aligned}$$

By theorems 3.4 and 3.7 we may conclude that $H^p_{\partial\Omega}(U, {}^H\tilde{\mathcal{O}}) = 0$ and $H^p_{\Omega^a}(U, {}^H\tilde{\mathcal{O}}) = 0$ for $p \geqq n+1$, so $H^p_\Omega(V, {}^H\tilde{\mathcal{O}}) = 0$ when $p \geqq n+1$. In the same way, it follows from theorems 3.4 and 3.7 that $H^p_\Omega(V, {}^H\tilde{\mathcal{O}}) = 0$ $(0 \leqq p \leqq n-2)$. On the other hand, the theorem 4.1 shows that $j : ({}^H\mathcal{Q}(\partial\Omega))' \to ({}^H\mathcal{Q}(\Omega^a))'$ is injective. By theorems 3.4 and 3.7 again, we have the exact sequence

$0 \to H^{n-1}_\Omega(V, {}^H\tilde{\mathcal{O}}) \to ({}^H\mathcal{Q}(\partial\Omega))' \xrightarrow{j} ({}^H\mathcal{Q}(\Omega^a))'$. Since j is injective, we have $H^{n-1}_\Omega(V, {}^H\tilde{\mathcal{O}}) = 0$. This completes the proof.

DEFINITION 5.2. *Let Ω and V be as in the theorem 5.1. Then we define ${}^H\mathcal{R}(\Omega)$, the space of H-valued Fourier hyperfunctions on Ω, by $H^n_\Omega(V, {}^H\tilde{\mathcal{O}})$.* (By the excision theorem, the space ${}^H\mathcal{R}(\Omega)$ is independent of the choice of V.)

THEOREM 5.3. *The presheaf $\{{}^H\mathcal{R}(\Omega)\}$ constitutes a flabby sheaf over $\boldsymbol{D}^n$, whose restriction to $\boldsymbol{R}^n$ coincides with the sheaf of H-valued hyperfunctions over $\boldsymbol{R}^n$.*

PROOF. From the theorem 3.5, we may conclude that flabby dim ${}^H\tilde{\mathcal{O}} \leqq n$. Thus, by the theorem 5.1 and by the theorem 1.8 of Komatsu [10], $\{{}^H\mathcal{R}(\Omega)\}$ is a flabby sheaf. The last statement of the theorem follows from the remark under the definition 2.2. This completes the proof.

DEFINITION 5.4. *We denote by* ${}^{H}\mathscr{R}$ *the sheaf* $\{{}^{H}\mathscr{R}(\Omega)\}$ *over* $\boldsymbol{D}^n$ *and call it the sheaf of* H-valued Fourier hyperfunctions *over* $\boldsymbol{D}^n$.

THEOREM 5.5. *Let* K *be a compact set in* $\boldsymbol{D}^n$. *Then* $H_K^n(V, {}^{H}\tilde{\mathcal{O}}) \cong ({}^{H}\underset{\sim}{\mathcal{Q}}(K))'$, *especially* ${}^{H}\mathscr{R}(\boldsymbol{D}^n) \cong ({}^{H}\mathscr{P}_*)'$.

PROOF. This is a direct consequence of theorems 3.4 and 3.7.

Now we will study the vector valued Fourier hyperfunctions as classes of vector valued slowly increasing holomorphic functions. Let Ω be an open set in $\boldsymbol{D}^n$. Then there exists an $\tilde{\mathcal{O}}$-pseudoconvex neighbourhood V of Ω such that $V \cap \boldsymbol{D}^n = \Omega$ (see Kawai [7], p. 473, Theorem 2.1.6). We define $V_j (j = 0, 1, \cdots, n)$ by $V_0 = V$, $V_j = \{z \in V; \operatorname{Im} z_j \neq 0\}$, $j = 1, 2, \cdots, n$. Then $\mathscr{V} = \{V_0, V_1, \cdots, V_n\}$ and $\mathscr{V}' = \{V_1, \cdots V_n\}$ cover V and $V - \Omega$, respectively. Since V_j and their intersections are also $\tilde{\mathcal{O}}$-pseudconvex domains, the covering $(\mathscr{V}, \mathscr{V}')$ satisfies the conditions of Leray's theorem (see Komatsu [9], [10]). Therefore, by Leray's theorem, we obtain the isomorphism $H_\Omega^n(V, {}^{H}\tilde{\mathcal{O}}) \cong H^n(\mathscr{V}, \mathscr{V}', {}^{H}\tilde{\mathcal{O}})$. Since the covering $\mathscr{V}$ is composed of only $n+1$ open sets $V_j (j = 0, 1, \cdots, n)$, we easily obtain the isomorphisms

$$Z^n(\mathscr{V}, \mathscr{V}', {}^{H}\tilde{\mathcal{O}}) \cong {}^{H}\tilde{\mathcal{O}}(\cap_j V_j), \quad C^{n-1}(\mathscr{V}, \mathscr{V}', {}^{H}\tilde{\mathcal{O}}) \cong \bigoplus_{j=1}^{n} {}^{H}\tilde{\mathcal{O}}(\cap_{i \neq j} V_i)$$

Hence we have

$$\delta C^{n-1}(\mathscr{V}, \mathscr{V}', {}^{H}\tilde{\mathcal{O}}) \cong \sum_{j=1}^{n} {}^{H}\tilde{\mathcal{O}}(\cap_{i \neq j} V_i) \mid V_1 \cap \cdots \cap V_n.$$

Therefore we have the isomorphisms

$$H_\Omega^n(V, {}^{H}\tilde{\mathcal{O}}) \cong H^n(\mathscr{V}, \mathscr{V}', {}^{H}\tilde{\mathcal{O}}) \cong Z^n(\mathscr{V}, \mathscr{V}', {}^{H}\tilde{\mathcal{O}}) / \delta C^{n-1}(\mathscr{V}, \mathscr{V}', {}^{H}\tilde{\mathcal{O}})$$
$$\cong {}^{H}\tilde{\mathcal{O}}(\cap_j V_j) / \sum_{j=1}^{n} {}^{H}\tilde{\mathcal{O}}(\cap_{i \neq j} V_i).$$

Thus we get

THEOREM 5.6. *Let notations be as above. Then we have*
$$H_\Omega^n(V, {}^{H}\tilde{\mathcal{O}}) \cong H^n(\mathscr{V}, \mathscr{V}', {}^{H}\tilde{\mathcal{O}}) \cong {}^{H}\tilde{\mathcal{O}}(\cap_j V_j) / \sum_{j=1}^{n} {}^{H}\tilde{\mathcal{O}}(\cap_{i \neq j} V_i).$$

§6. *H*-valued Fourier hyperfunctions as continuous linear operators from $\underset{\sim}{\mathcal{Q}}(K)$ to H.

In this section, we will show that an H-valued Fourier hyperfunction with support in a compact set K in $\boldsymbol{D}^n$ can be realized as a continuous linear mapping from $\underset{\sim}{\mathcal{Q}}(K)$ $(\cong {}^{C}\underset{\sim}{\mathcal{Q}}(K))$ to H. In the following, we identify $\boldsymbol{D}^n \times \sqrt{-1}\,\boldsymbol{R}^n \cong \boldsymbol{D}^n \times \boldsymbol{R}^n$ and $\boldsymbol{C}^n \cong \boldsymbol{R}^n \times \sqrt{-1}\,\boldsymbol{R}^n \cong \boldsymbol{R}^n \times \boldsymbol{R}^n$.

DEFINITION 6.1. *We define* $S_{1,m}(\boldsymbol{C}^n)$ *as a space of type* S *of* I. M. Gel'fand.

That is,
$S_{1,m}(\boldsymbol{C}^n) = \{f(z) \in C^\infty(\boldsymbol{C}^n)\,;\, \sup_{|\alpha|\leq p, z \in \boldsymbol{C}^n} M_p(z)\,|f^{(\alpha)}(z)| < \infty \quad (p = 2, 3, \cdots)\}$, *where* $M_p(z) = e^{(1/m)(1-1/p)|z|}$ $(p = 2, 3, \cdots)$, and $m = 1, 2, \cdots$.

PROPOSITION 6.2. *The space* $S_{1,m}(\boldsymbol{C}^n)$ *is a nuclear space with seminorms*
$\|f\|_{m,p} = \sup_{|\alpha|\leq p, z \in \boldsymbol{C}^n} M_p(z)\,|f^{(\alpha)}(z)|$ $(p = 2, 3, \cdots)$.

PROOF. See Mityagin [11].

PROPOSITION 6.3. *Let* K *be a compact subset of* $\boldsymbol{D}^n \times \sqrt{-1}\,\boldsymbol{R}^n$. *Let* $S_{1,m}(K)$ *be the set of all* $f(z) \in S_{1,m}(\boldsymbol{C}^n)$ *such that supp* $f \subset K \cap \boldsymbol{C}^n$. *Then* $S_{1,m}(K)$ *is a nuclear space.*

PROOF. $S_{1,m}(K)$ is a linear subspace of $S_{1,m}(\boldsymbol{C}^n)$. Hence $S_{1,m}(K)$ is nuclear because $S_{1,m}(\boldsymbol{C}^n)$ is nuclear. This completes the proof.

DEFINITION 6.4. *Let* Ω *be an open set in* $\boldsymbol{D}^n \times \sqrt{-1}\,\boldsymbol{R}^n$. *We define* $S_{1,m}(\Omega)$ *to be the set of all functions* $f(z) \in C^\infty(\Omega \cap \boldsymbol{C}^n)$ *which satisfy the following conditions: for any compact set* K *in* Ω,
$\sup_{|\alpha|\leq p, z \in K \cap \boldsymbol{C}^n} M_p(z)\,|f^{(\alpha)}(z)| < \infty$ $(p = 2, 3, \cdots)$,
where $M_p(z)$ *is as in the definition* 6.1.

PROPOSITION 6.5. *Let* Ω *be an open set in* $\boldsymbol{D}^n \times \sqrt{-1}\,\boldsymbol{R}^n$. *Then* $S_{1,m}(\Omega)$ *is a Fréchet space with seminorms*
$\|f\|_{K,m,p} = \sup_{|\alpha|\leq p, z \in K \cap \boldsymbol{C}^n} M_p(z)\,|f^{(\alpha)}(z)|$
for compact subsets K *of* Ω *and* $p = 2, 3, \cdots$.

PROOF. *Let* $\{K_j\}$ be a sequence of increasing compact subsets of Ω which exhaust Ω. Then the family of continuous seminorms $\{\|f\|_{K_p,m,p}\}_{p=2}^\infty$ becomes a nondecreasing countable basis of continuous seminorms. Hence $S_{1,m}(\Omega)$ is a locally convex metrizable topological vector space, in particular, it is Hausdorff. However, it is easy to see that $S_{1,m}(\Omega)$ is complete. Therefore $S_{1,m}(\Omega)$ is a Fréchet space. This completes the proof.

PROPOSITION 6.6. *Let* Ω *be an open set in* $\boldsymbol{D}^n \times \sqrt{-1}\,\boldsymbol{R}^n$ *and let* $\{K_j\}$ *be a sequence of increasing compact sets in* Ω *which satisfy* $K_j \subset\subset K_{j+1}$ *and exhaust* Ω. *Then* $S_{1,m}(\Omega)$ *is the projective limit of a sequence of spaces* $S_{1,m}(K_j)$. *Hence* $S_{1,m}(\Omega)$ *is a Fréchet nuclear space.*

PROOF. It is easy to see that $S_{1,m}(\Omega) \cong \varprojlim_j S_{1,m}(K_j)$. Since $S_{1,m}(K_j)$ is a Hausdorff space, $S_{1,m}(\Omega)$ is a nuclear space by the proposition 50.1 of Treves [14]. This completes the proof.

DEFINITION 6.7. *Let Ω be an open set in $\boldsymbol{D}^n\times\sqrt{-1}\boldsymbol{R}^n$. We define $\mathcal{O}^m(\Omega)$ to be the set of all holomorphic functions on $\Omega\cap\boldsymbol{C}^n$ which satisfy the condition: for any compact set K in Ω and for $p=2,3,\cdots$,* $\sup_{z\in K\cap\boldsymbol{C}^n}M_p(z)\,|f(z)|<\infty$ *where $M_p(z)$ is as in the definition* 6.1.

PROPOSITION 6.8. *Let $\mathcal{O}^m(\Omega)$ be as in the definition* 6.7. *Then $\mathcal{O}^m(\Omega)$ is a Fréchet nuclear space with seminorms*
$\|f\|_{K,m,p}=\sup_{z\in K\cap\boldsymbol{C}^n}M_p(z)\,|f(z)|$
for any compact set K in Ω and $p=2,3,\cdots$.

PROOF. It is obvious that $\mathcal{O}^m(\Omega)$ is a closed subspace of $S_{1,m}(\Omega)$. Hence $\mathcal{O}^m(\Omega)$ is a Fréchet nuclear space. This completes the proof.

PROPOSITION 6.9. *Let K be a compact subset of $\boldsymbol{D}^n$. Let $\{U_m\}$ be a fundamental system of neighbourhoods of K in $\boldsymbol{D}^n\times\sqrt{-1}\boldsymbol{R}^n$ which satisfy $U_m\supset\supset U_{m+1}$ and each of whose components contains at least one point of K. Then $\underset{\sim}{\mathcal{O}}(K)\cong\varinjlim_m\mathcal{O}^m(U_m)$.*

PROOF. Since $\underset{\sim}{\mathcal{O}}(K)=\varinjlim_m\mathcal{O}_c^m(U_m)$ by the definition 1.1.4 of Kawai [7], it is sufficient to prove $\mathcal{O}_c^m(U_m)\subset\mathcal{O}^m(U_m)\subset\mathcal{O}_c^{m+1}(U_{m+1})$. However, it is trivial. It is also trivial that $\{\mathcal{O}^m(U_m)\}$ becomes a compact injective sequence of locally convex spaces. This completes the proof.

COROLLARY 6.10. *Let K be a compact subset of $\boldsymbol{D}^n$. Then $\underset{\sim}{\mathcal{O}}(K)$ and $(\underset{\sim}{\mathcal{O}}(K))'\cong H_K^n(V,\tilde{\mathcal{O}})$ are nuclear.*

PROOF. Since, by the proposition 6.9, $\underset{\sim}{\mathcal{O}}(K)$ is a injective limit of a compact injective sequence of nuclear spaces and it is a DFS-space, the corollary follows from propositions 50.1 and 50.6 of Treves [14], p. 514 and p. 523.

For the case of vector valued functions, we give

DEFINITION 6.11. *Let Ω be an open set in $\boldsymbol{D}^n\times\sqrt{-1}\boldsymbol{R}^n$. We define ${}^H\mathcal{O}^m(\Omega)$ to be the set of all H-valued holomorphic functions on $\Omega\cap\boldsymbol{C}^n$ which satisfy the condition: for any compact set K in Ω and for $p=2,3,\cdots$, the seminorms*
$|||f|||_{K,m,p}=\sup_{z\in K\cap\boldsymbol{C}^n}M_p(z)\,\|f(z)\|_H<\infty$,
where $M_p(z)$ is as in the definition 6.1.

PROPOSITION 6.12. *Let K and $\{U_m\}$ be as in the proposition* 6.9. *Then ${}^H\underset{\sim}{\mathcal{O}}(K)\cong\varinjlim_m{}^H\mathcal{O}^m(U_m)$.*

PROOF. In this case we have the inclusion relation ${}^H\mathcal{O}_c^m(U_m)\subset{}^H\mathcal{O}^m(U_m)$ $\subset{}^H\mathcal{O}_c^{m+1}(U_{m+1})$ and the natural injections are continuous. Therefore, since $\{{}^H\mathcal{O}_c^m(U_m), u_{m+1,m}\}$ is a weakly compact injective sequence, $\{{}^H\mathcal{O}^m(U_m),$ $u_{m+2,m}\}$ becomes a weakly compact injective sequence of locally convex spaces. Hence ${}^H\underset{\sim}{\mathcal{O}}(K)=\varinjlim_m {}^H\mathcal{O}_c^m(U_m)\cong\varinjlim_m {}^H\mathcal{O}^m(U_m)$.
This completes the proof.

PROPOSITION 6.13. *Let Ω be an open set in $\boldsymbol{D}^n\times\sqrt{-1}\boldsymbol{R}^n$. Then $\mathcal{O}^m(\Omega)\otimes H$ is dense in ${}^H\mathcal{O}^m(\Omega)$.*

PROOF. Let $f(z)$ be in ${}^H\mathcal{O}^m(\Omega)$ and let K be a compact subset of Ω and let ε be a positive number. Since $\lim_{|z|\to\infty}M_p(z)\|f(z)\|_H=0$ for any $p(=2,3,\cdots)$, $M_p(z)f(z)$ is uniformly continuous in $K\cap\boldsymbol{C}^n$. Thus we can extend $M_p(z)f(z)$ to K continuously. Therefore the extension $(M_p(z)f(z))^{\sim}$ of $M_p(z)f(z)$ to K is continuous on K, so that it is uniformly continuous on K. Hence we can find a finite open covering $V_1,\cdots,V_r$ of K such that for each $j=1,2,\cdots,r$ and each pair $z, z'\in V_j$

$$\|M_p(z)f(z)-M_p(z')f(z')\|_H<\varepsilon.$$

In each set V_j we pick up a point z_j. Let E be a closed subspace of H spanned by $\{f(z_1),\cdots,f(z_r)\}$ and let P_E be a orthogonal projection on E. Then $\mathrm{P}_Ef(z)$ belongs to $\mathcal{O}^m(\Omega)\otimes H$ and for $z\in V_j$ we have

$$\|M_p(z)f(z)-M_p(z)\mathrm{P}_Ef(z)\|_H$$
$$\leqq\|M_p(z)f(z)-M_p(z_j)f(z_j)\|_H+\|M_p(z_j)\mathrm{P}_Ef(z_j)-M_p(z)\mathrm{P}_Ef(z)\|_H$$
$$<2\varepsilon.$$

Hence, since $\{V_j\}$ covers K, we have

$$\sup_{z\in K\cap\boldsymbol{C}^n}M_p(z)\|f(z)-\mathrm{P}_Ef(z)\|_H<2\varepsilon.$$

This completes the proof.

PROPOSITION 6.14. *Let Ω be an open set in $\boldsymbol{D}^n\times\sqrt{-1}\boldsymbol{R}^n$. Then ${}^H\mathcal{O}^m(\Omega)\cong\mathcal{O}^m(\Omega)\hat{\otimes}H$.*

PROOF. Since $\mathcal{O}^m(\Omega)$ is nuclear, $\mathcal{O}^m(\Omega)\hat{\otimes}H=\mathcal{O}^m(\Omega)\hat{\otimes}_\varepsilon H$. Hence, because of the completeness of ${}^H\mathcal{O}^m(\Omega)$ and in virtue of the proposition 6.13, it is sufficient to show that ${}^H\mathcal{O}^m(\Omega)$ induces on $\mathcal{O}^m(\Omega)\otimes H$ the topology ε. We observe, first, that ${}^H\mathcal{O}^m(\Omega)$ can be canonically injected in $L(H'_\tau;\mathcal{O}^m(\Omega))$. Indeed, let $f(z)\in{}^H\mathcal{O}^m(\Omega)$ and consider the complex valued function, defined on Ω, $z\mapsto\langle e',f(z)\rangle$ where e' is an arbitrary element of H'. It is easy to see that the function $\langle e',f(z)\rangle$ belongs to $\mathcal{O}^m(\Omega)$. Now let K be a compact subset of Ω. Then $(M_pf)^{\sim}(K)$ is a compact subset of H, where $(M_pf)^{\sim}$ is the continuous extension of M_pf to K, and then the closed convex balanced hull Γ of $(M_pf)^{\sim}(K)$ is also compact since H is complete. Thus it is, a fortiori, weakly compact. If e' belongs to the polar of $(1/\varepsilon)\Gamma$, which is a neighbourhood of zero in H'_τ, we have

$\sup_{z\in K\cap C^n} M_p(z)\ |\langle e', f(z)\rangle| \leqq \varepsilon.$

This shows that the mapping $e' \mapsto (z \mapsto \langle e', f(z)\rangle)$ is continuous from H'_τ into $\mathcal{O}^m(\Omega)$. Now let $U=\{h;\ \|h\|\leqq\varepsilon\}$ and let U° be its polar and let K be a compact subset of Ω. Then it is equivalent to say that $M_p(z) f(z) \in U$ for all $z\in K\cap\Omega$ or to say that $|M_p(z)\langle e', f(z)\rangle| \leqq 1$ for all $z\in K\cap\Omega$ and for all $e'\in U^\circ$. This shows that the topology of ${}^H\mathcal{O}^m(\Omega)$ is equal to the topology induced by $L_\varepsilon(H'_\tau;\ \mathcal{O}^m(\Omega))$. Hence in virtue of the proposition 6.13, we have

$${}^H\mathcal{O}^m(\Omega) \cong \mathcal{O}^m(\Omega)\hat{\otimes}_\varepsilon H$$

and this completes the proof.

PROPOSITION 6.15. *Let Ω be an open set in $\boldsymbol{D}^n\times\sqrt{-1}\boldsymbol{R}^n$. Then $({}^H\mathcal{O}^m(\Omega))' \cong L_b(\mathcal{O}^m(\Omega);H)$.*

PROOF. By propositions 50.5 and 50.7 of Treves [14], p.522 and p.524, and by the proposition 6.14, we have

$$({}^H\mathcal{O}^m(\Omega))' \cong (\mathcal{O}^m(\Omega)\hat{\otimes} H)' \cong L_b(\mathcal{O}^m(\Omega);H).$$

This completes the proof.

THEOREM 6.16. *Let K be a compact subset of $\boldsymbol{D}^n$. Then $H^n_K(V,{}^H\tilde{\mathcal{O}}) \cong L_b(\mathcal{Q}(K);H)$.*

PROOF. Since, by the theorem 5.5, $H^n_K(V,{}^H\tilde{\mathcal{O}}) \cong ({}^H\mathcal{Q}(K))'$, we have only to show that $({}^H\mathcal{Q}(K))' \cong L_b(\mathcal{Q}(K);H)$. Let $\{U_m\}$ be as in propositions 6.9, and 6.12. Then, since ${}^H\mathcal{Q}(K)=\varinjlim_m {}^H\mathcal{O}^m(U_m)$, we have

$$({}^H\mathcal{Q}(K))' \cong \varprojlim_m ({}^H\mathcal{O}^m(U_m))' \cong \varprojlim_m L_b(\mathcal{O}^m(U_m);H).$$

However, since $\mathcal{Q}(K)=\varinjlim_m \mathcal{O}^m(U_m)$, we have

$$L_b(\mathcal{Q}(K);H) \cong \varprojlim_m L_b(\mathcal{O}^m(U_m);H).$$

Therefore $({}^H\mathcal{Q}(K))' \cong L_b(\mathcal{Q}(K);H)$.

This completes the proof.

THEOREM 6.17. *Let K be a compact set in $\boldsymbol{D}^n$. Then $H^n_K(V,{}^H\tilde{\mathcal{O}}) \cong H^n_K(V,\tilde{\mathcal{O}})\hat{\otimes} H$.*

PROOF. By the corollary 6.10, and by the proposition 50.5 of Treves [14], p.522, we have $L_b(\mathcal{Q}(K);H) \cong (\mathcal{Q}(K))'\hat{\otimes} H$. Since $(\mathcal{Q}(K))' \cong H^n_K(V,\tilde{\mathcal{O}})$ and $L_b(\mathcal{Q}(K);H) \cong H^n_K(V,{}^H\tilde{\mathcal{O}})$, we obtain the isomorphism $H^n_K(V,{}^H\tilde{\mathcal{O}}) \cong H^n_K(V,\tilde{\mathcal{O}})\hat{\otimes} H$. This completes the proof.

COROLLARY 6.18. *Let K be a compact subset of $\boldsymbol{D}^n$. Then* ${}^H\mathscr{Q}(K) \cong \mathscr{Q}(K) \hat{\otimes} H$.

PROOF. We notice that the isomorphism $({}^H\mathscr{Q}(K))' \cong (\mathscr{Q}(K))' \hat{\otimes} H$ follows from the theorem 6.17. Since ${}^H\mathscr{Q}(K)$ and $\mathscr{Q}(K)$ are DFS*- and DFS-spaces, respectively, and $\mathscr{Q}(K)$ as well as $(\mathscr{Q}(K))'$ is nuclear, we have ${}^H\mathscr{Q}(K) \cong ((\mathscr{Q}(K))' \hat{\otimes} H)' \cong \mathscr{Q}(K) \hat{\otimes} H$ by the proposition 50.7 of Treves [14], p. 524. This completes the proof.

§ 7. Fourier transformation and the Paley-Wiener theorem

In this section we introduce the notion of the Fourier transformation of the elements of ${}^H\mathscr{R}(\boldsymbol{D}^n)$, by which we give the explicit formula of operation of ${}^H\mathscr{R}(\boldsymbol{D}^n)$ on $\mathscr{P}_*$. Next we give an analogue of the Paley-Wiener theorem.

PROPOSITION 7.1. *If we define* $\mathscr{F}\varphi$ by $\int e^{i<x,\xi>}\varphi(x)\,dx$ *for* $\varphi \in \mathscr{P}_* = \mathscr{Q}(\boldsymbol{D}^n)$, *then* $\mathscr{F}$ *gives a topological isomorphism from* $\mathscr{P}_*$ *to* $\mathscr{P}_*$.

PROOF. See Kawai [7], p. 483, Proposition 3.2.4.

DEFINITION 7.2. *Let μ be an element of $L_b(\mathscr{P}_*; H)$, then we define $\mathscr{F}^*\mu$ by the formula $\mathscr{F}^*\mu(\varphi) = \mu(\mathscr{F}\varphi)$ ($\forall\, \varphi \in \mathscr{P}_*$). We also define* $\overline{\mathscr{F}}^*\mu(\varphi) = \mu(\overline{\mathscr{F}}\varphi)$, *where* $\overline{\mathscr{F}}\varphi = 1/(2\pi)^n \int e^{-i<x,\xi>}\varphi(x)\,dx$.

Denoting the closure of j-th quadrant in $\boldsymbol{D}^n$ by K_j, we obtain the following theorem.

THEOREM 7.3. *Every element $\mu \in L_b(\mathscr{P}_*; H)$ can be decomposed as* $\mu = \sum_{j=1}^{2^n}\mu_j$ *where* $\mu_j \in L_b(\mathscr{Q}(K_j); H)$.

PROOF. This is a direct consequence of theorems 5.3 and 6.16.

If we define $V_0 = \boldsymbol{D}^n \times \sqrt{-1}I^n$, $V_j = \boldsymbol{D}^n \times \sqrt{-1}\{y \in I^n; y_j \neq 0\}$ (where $I = \{-1 < y < 1\}$), $\mathscr{V} = \{V_j\}_{j=0}^n$ and $\mathscr{V}' = \{V_j\}_{j=1}^n$, we obtain the isomorphism $H^n_{\boldsymbol{D}^n}(\boldsymbol{D}^n \times \sqrt{-1}I^n, {}^H\widetilde{\mathscr{O}}) \cong H^n(\mathscr{V}, \mathscr{V}', {}^H\widetilde{\mathscr{O}})$ by Leray's theorem (see Komatsu [9], [10]). Thus we can represent any element μ of $H^n_{\boldsymbol{D}^n}(\boldsymbol{D}^n \times \sqrt{-1}I^n, {}^H\widetilde{\mathscr{O}})$ by some element in ${}^H\widetilde{\mathscr{O}}(V_1 \cap \cdots \cap V_n)$, which we write by $\{\varphi_1, \cdots, \varphi_{2^n}\} = [\varphi]$.

Using this isomorphism, the operation of ${}^H\mathscr{R}(\boldsymbol{D}^n)$ on $\mathscr{P}_*$ is given by

$$[\varphi](f)=\sum_{j=1}^{2^n}(-1)\prod_{k=1}^{n} sign\ \varepsilon_k\int\overset{\infty}{\underset{-\infty}{\cdots}}\int\varphi_j(x_1+i\varepsilon_1,\cdots,x_n+i\varepsilon_n)$$
$$f(x_1+i\varepsilon_1,\cdots,\ x_n+i\varepsilon_n)\,dx_1\cdots\ dx_n\quad(*)$$

where $|\varepsilon_j|$ is sufficiently small but not zero and $sign\ \varepsilon_j=\varepsilon_j/|\varepsilon_j|$ and the integrals are the Bochner integrals. In fact, it is clear that any $[\varphi]$ defines an element of $L_b(\mathscr{P}_*;H)$ by the well defined integration. We denote by ι this map from ${}^H\mathscr{R}(\boldsymbol{D}^n)$ to $L_b(\mathscr{P}_*;H)$. We want to construct the inverse map $\varkappa$ of ι.

DEFINITION 7.4. *Using the decomposition of* μ *of the theorem* 7,3, *we define* $\mathscr{F}_S\mu=\{F_j(\zeta)\}$, *which is an element of* $H^n(\mathscr{V},\mathscr{V}',{}^H\tilde{\mathscr{O}})$. *Here* $F_j(\zeta)=(-1)^{j+1}\mu_j(e^{i<z,\zeta>})$. (Im ζ *belongs to the j-th open quadrant.*)

Then $\mathscr{F}_S\mu$ is well-defined. For $\{F_j(\zeta)\}$ is independent of the decomposition of μ in the theorem 7.3. In fact, since $\mathscr{P}_*=\underset{\sim}{\mathscr{Q}}(\boldsymbol{D}^n)$ is dense in $\underset{\sim}{\mathscr{Q}}(K_j)$ by the theorem 4.1, we can consider that $L_b(\underset{\sim}{\mathscr{Q}}(K_j);H)\subset L_b(\underset{\sim}{\mathscr{Q}}(\boldsymbol{D}^n);H)$. Hence the ambiguity of the decomposition μ in the theorem 7.3 comes from an element ν belonging to $L_b(\underset{\sim}{\mathscr{Q}}(K_j);H)\cap L_b(\underset{\sim}{\mathscr{Q}}(K_k);H)$. Here we consider the exact sequence

$$0\to\underset{\sim}{\mathscr{Q}}(K_j\cup K_k)\to\underset{\sim}{\mathscr{Q}}(K_j)\oplus\underset{\sim}{\mathscr{Q}}(K_k)\to\underset{\sim}{\mathscr{Q}}(K_j\cap K_k)\to H^1(K_j\cup K_k,\underset{\sim}{\mathscr{Q}})\to\cdots.$$
$$\begin{array}{ccc}\cup & & \cup\\ (f,g) & \longrightarrow & f-g\end{array}$$

Then, since $H^1(K_j\cup K_k,\underset{\sim}{\mathscr{Q}})=0$ (see Kawai [7], p. 481), such an element ν can be considered as an element of $L_b(\underset{\sim}{\mathscr{Q}}(K_j\cap K_k);H)$. Hence, $\nu(e^{i<z,\zeta>})$, considered as an element of $H^n_{D^n}(\boldsymbol{D}^n\times\sqrt{-1}I^n,{}^H\tilde{\mathscr{O}})$, gives a null-element. Therefore the above definition 7.4 of $\mathscr{F}_S\mu$ makes sense.

Now we define the map $\varkappa$.

DEFINITION 7.5. *Let* μ *belong to* $L_b(\mathscr{P}_*;H)$, *then we define* $\varkappa(\mu)=\mathscr{F}_S(\overline{\mathscr{F}}^*\mu)$.

THEOREM 7.6. *The composed map* $\iota\circ\varkappa: L_b(\mathscr{P}_*;H)\to L_b(\mathscr{P}_*;H)$ is the *identity map and* ι *is injective, so* ι *and* $\varkappa$ *are bijective.*

PROOF. At first, we prove that $\iota\circ\varkappa=id$. In fact, we have the following identity for any f of $\mathscr{P}_*$.

$$\iota\circ\varkappa(\mu)(f)=\Sigma_j\int\overset{\infty}{\underset{-\infty}{\cdots}}\int\ \nu_j(e^{i<z,\zeta>})f(\zeta)\ d\xi_1\cdots d\xi_n$$
$$=\Sigma_j\nu_j(\int\overset{\infty}{\underset{-\infty}{\cdots}}\int e^{i<z,\zeta>}f(\zeta)\,d\xi)$$
$$=\nu(\mathscr{F}f)=\overline{\mathscr{F}}^*\mu(\mathscr{F}f)=\mu(f).$$

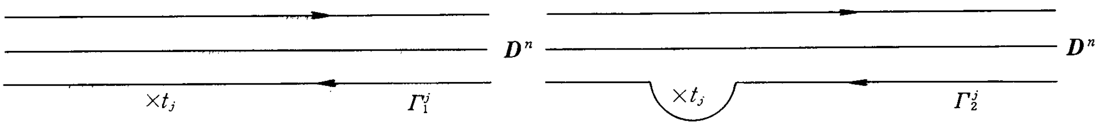
where we put $\overline{\mathscr{F}}^*\mu=\nu$. This proves that $\iota\circ\varkappa=id$.

Next we prove the injectivity of ι. Define

$$f_t(z) = \exp\left(-\sum_{j=1}^{n}(t_j - z_j)^2\right) / (2\pi\sqrt{-1})^n \prod_{j=1}^{n}(t_j - z_j)$$

and consider the following path of integration Γ_1^j and Γ_2^j on the j-th coordinate plane.

Figure



If we assume $\iota[\varphi] = 0$, then

$$\int_{\Gamma_1^1 \times \cdots \times \Gamma_1^n} \varphi(z) f_t(z)\, dz = 0$$

Therefore we have, denoting by $(i_1, \cdots, i_n)$ the n-tuple of 1 or 2,

$$\varphi(t) = \sum_{(i_1, \cdots, i_n) \neq (1, \cdots, 1)} (-1)^{i_1 + \cdots + i_n} \int_{\Gamma_{i_1}^1 \times \cdots \times \Gamma_{i_n}^n} \varphi(z) f_t(z)\, dz$$

by Cauchy's formula. The right hand side, however, belongs to $\sum_j {}^H\mathscr{O}(\bigcap_{i \neq j} V_i)$. Thus we have only to estimate $\varphi(t)$. Now we have the estimate $\|\varphi(z)\|_H \leq A_{\varepsilon, \delta, \delta'} e^{\varepsilon|z|}$ on $\{\delta < |\operatorname{Im} z_j| < \delta' < 1;\ j = 1, \cdots, n\}$ *for* every ε, δ, δ', by the definition of $H^n(\mathscr{V}, \mathscr{V}', {}^H\widetilde{\mathscr{O}})$. This means $\|\varphi(t)\|_H \leq B_\varepsilon e^{\varepsilon|t|}$ as far as $|\operatorname{Im} t|$ is sufficiently small. This proves that $[\varphi]$ is zero as a cohomology class, that is, ι is injective. This completes the proof.

This theorem proves that the operation of ${}^H\mathscr{R}(\boldsymbol{D}^n)$ on $\mathscr{S}_*$ is given by the integral $(*)$. In the definition 7.4, we have defined the Fourier transform of the element of ${}^H\mathscr{R}(\boldsymbol{D}^n)$ via "boundary values" of H-valued holomorphic functions in tublar domains. We call this Fourier-Carleman-Leray-Sato transformation, and denote by $\mathscr{F}_s$. Then

THEOREM 7.7. $\mathscr{F}_s = \mathscr{F}^*$.

PROOF. For an arbitrary $f \in \mathscr{S}_*$

$$\mathscr{F}_s\mu(f) = \sum_j \int \mu_j(e^{i<z,\zeta>}) f(\zeta)\, d\zeta = \mu\left(\int e^{i<z,\zeta>} f(\zeta)\, d\zeta\right)$$
$$= \mu(\mathscr{F}f) = \mathscr{F}^*\mu(f).$$

Therefore we may conclude that $\mathscr{F}_s$ is the same as $\mathscr{F}^*$ in the definition 7.2.
This completes the proof.

REMARK. We denote by $\mathscr{F}$ these two Fourier transformations $\mathscr{F}_s = \mathscr{F}^*$ symbolically, considering as an operator of ${}^H\mathscr{R}(\boldsymbol{D}^n) = H^n_{\boldsymbol{D}^n}(V, {}^H\tilde{\mathscr{O}})$.

PROPOSITION 7.8. *Let F be an H-valued Fourier hyperfunction over $\boldsymbol{D}^n$. Let $[\varphi]$ be its realization as a boundary value of H-valued slowly increasing holomorphic functions and let μ be its realization as a continuous linear operator from $\mathscr{P}_*$ to H. Then $\varkappa(\mu) = [\varphi]$. And then, for any $e \in H$, $([\varphi], e)_H = [(\varphi, e)_H] \in H^n(\mathscr{V}, \mathscr{V}', \tilde{\mathscr{O}})$ and $\nu_{\{\mu, e\}} \in (\mathscr{P}_*)'$ are both scalar valued Fourier hyperfunctions and $k(\nu_{\{\mu, e\}}) = ([\varphi], e)_H$, where $\nu_{\{\mu, e\}}$ is defined by the formula $\langle \nu_{\{\mu, e\}}, f\rangle = (\mu(f), e)_H$ for all $f \in \mathscr{P}_*$, and k is the map from $(\mathscr{P}_*)'$ to $H^n(\mathscr{V}, \mathscr{V}', \tilde{\mathscr{O}})$ defined by* Kawai [7], (p. 484, Definition 3.2.8). *We denote by $(F, e)_H$ this scalar valued Fourier hyperfunction.*

PROOF. The fact that $\varkappa(\mu) = [\varphi]$ follows directly from the definition of $\varkappa$. Using the formula $(*)$, we have, for $f \in \mathscr{P}_*$,

$$\langle \nu_{\{\mu, e\}}, f\rangle = (\mu(f), e)_H$$

$$= \Big(\sum_{j=1}^{2^n} (-1) \prod_{k=1}^{n} sign\, \varepsilon_k \int_{-\infty}^{\infty} \varphi_j(x_1 + i\varepsilon_1, \cdots, x_n + i\varepsilon_n) f(x_1 + i\varepsilon_1, \cdots, x_n + i\varepsilon_n)\, dx_1 \cdots dx_n, e\Big)_H$$

$$= \sum_{j=1}^{2^n} (-1) \prod_{k=1}^{n} sign\, \varepsilon_k \int_{-\infty}^{\infty} (\varphi_j(x_1 + i\varepsilon_1, \cdots, x_n + i\varepsilon_n), e)_H f(x_1 + i\varepsilon_1, \cdots, x_n + i\varepsilon_n)\, dx_1 \cdots dx_n$$

$$= \langle [(\varphi, e)_H], f\rangle.$$

Hence $k(\nu_{\{\mu, e\}}) = [(\varphi, e)_H]$. This completes the proof.

PROPOSITION 7.9. *Let F be an H-valued Fourier hyperfunction over $\boldsymbol{D}^n$. Then for any $e \in H$, $(\mathscr{F}F, e)_H = \mathscr{F}(F, e)_H$.*

PROOF. Let μ be a realization of F as a continuous linear operator from $\mathscr{P}_*$ to H. Then $\nu_{\{\mu, e\}} = (F, e)_H$, $\nu_{\{\mathscr{F}^*\mu, e\}} = (\mathscr{F}F, e)_H$. Therefore for any $f \in \mathscr{P}_*$ we have $\nu_{\{\mathscr{F}^*\mu, e\}}(f) = (\mathscr{F}^*\mu(f), e)_H = (\mu(\mathscr{F}f), e)_H = \nu_{\{\mu, e\}}(\mathscr{F}f) = (\mathscr{F}_d \nu_{\{\mu, e\}})(f)$. Hence $(\mathscr{F}F, e)_H = \mathscr{F}(F, e)_H$. This completes the proof.

Next we study the Fourier transformation from the view point of H-valued holomorphic functions in tublar domains.

THEOREM 7.10. *Let Γ be a closed and strictly convex cone in $\boldsymbol{R}^n$, that is, a closed convex cone which does not contain any line in the whole, and let K be its closure in $\boldsymbol{D}^n$. For the sake of simplicity we assume that the vertex of the cone Γ be at the origin and $\Gamma\subset\subset\{x_1\geqq-\varepsilon\}$. (If A and B are cones, $A\subset\subset B$ means that the closure of A has a compact neighbourhood in the closure of B with respect to the topology of $\boldsymbol{D}^n$.) Then every μ in $L_b(\mathscr{Q}(K);H)$ has the following properties; $\mu(e^{i<z,\zeta>})$ is holomorphic in $\boldsymbol{R}^n\times\sqrt{-1}(\Gamma^\circ)^i$ and satisfies the following estimate (†).*

(†) *For every $\Gamma'\subset\subset\Gamma^\circ$ and $\varepsilon>0$ we have*

$$\|\mu(e^{i<z,\zeta>})\|_H\leqq C_\varepsilon\ \exp(\varepsilon|\mathrm{Re}\ \zeta|+\chi_{\Gamma,\varepsilon}(\mathrm{Im}\ \zeta)),\ \zeta\in\boldsymbol{R}^n\times\sqrt{-1}\Gamma',$$

where $\chi_{\Gamma,\varepsilon}(\eta)=\sup_{x\in\Gamma-\varepsilon(1,0,\cdots,0)}(-\langle x,\eta\rangle+\varepsilon|x|)$. (In the above notation Γ° means the polar set of Γ, that is, $\{\xi;\ \langle x,\xi\rangle\geqq 0,\ \forall x\in\Gamma\}$.)

PROOF. In view of the topology of $\mathscr{Q}(K)$ the proof is immediate.

The inverse of the above theorem is also true. Let $F(\zeta)$ be an H-valued holomorphic function defined in $\boldsymbol{R}^n\times\sqrt{-1}(\Gamma^\circ)^i$ for some closed and strictly convex cone Γ and it satisfies the growth condition (†) given in the preceding theorem. Then we can consider that $F(\zeta)$ defines some cohomology class μ in $H^n_{\boldsymbol{D}^n}(\boldsymbol{D}^n\times\sqrt{-1}I^n,{}^H\tilde{\mathscr{O}})$ in a natural way as "boundary values". Then μ can be considered as an element of $L_b(\mathscr{P}_*;H)$ and we can find some ν uniquely such that $\mathscr{F}^*\nu=\mu$ by the proposition 7.1 and the definition 7.2.

Then we have the following theorem.

THEOREM 7.11. *The element ν can be extended to a continuous linear operator from $\mathscr{Q}(K)$ to H where K is the closure of Γ in $\boldsymbol{D}^n$, that is, ν can be regarded as an element of $L_b(\mathscr{Q}(K);H)$.*

PROOF. The convexity of Γ reduces the situation to the case $n=1$. At first, we give the proof of the theorem when $n=1$. By Kawai's approximation theorem (see Kawai [7], p. 474, Theorem 2.2.1) and the definition of the topology of $\mathscr{Q}(K)$, it is sufficient to prove the following estimate: let $f(\zeta)$ belong to $\mathscr{O}_c{}^m(U_m)$ where $U_m=\boldsymbol{D}\times\sqrt{-1}\{y;\ |y|<1/m\}$, that is, let $\sup_{|\mathrm{Im}\,\zeta|<1/m}|f(\zeta)e^{(1/m)|\zeta|}|<\infty$ be satisfied by $f(\zeta)$, then, for any positive ε, there exists C_ε such that

$$\|F(f)\|_H=\|\int_{-\infty+i\delta}^{\infty+i\delta}F(\zeta)f(\zeta)\,d\zeta\|_H$$

$$\leqq C_\varepsilon\sup_{z\in\Gamma_\varepsilon}|(\mathscr{F}f)(z)e^{\varepsilon|z|}|\qquad(0<\delta<<1)$$

where $\Gamma_\varepsilon=\{x+iy;x\geqq-\varepsilon,\ |y|<\varepsilon\}$ assuming $\Gamma=\{x\geqq 0\}$. Then ν can be extended to a continuous linear operator from $\mathscr{Q}(K)$ to H.

In order to prove the estimate, since $\mathscr{F}:\mathscr{P}_*\to\mathscr{P}_*$ is an isomorphism, it is sufficient to prove

$$\| \int F(\zeta)\,(\int e^{-i\zeta z} g(z)\,dz)\,d\zeta\,\|_H \leqq C_\varepsilon \sup_{z\in\Gamma_\varepsilon} |g(z)\,e^{\varepsilon|z|}|,$$

where $g(z)=(\mathscr{F}f)(z)$.

In order to prove this inequality we denote the integral in the left hand side by J. Moreover we define

$$J_+=\int_{\xi+i\delta,\ \xi\geqq 0} F(\zeta)\,(\int_{x-i\delta} e^{-i\zeta z} g(z)\,dz)\,d\zeta,$$

$$J_-=\int_{\xi+i\delta\ \ \xi\leqq 0} F(\zeta)\,(\int_{x+i\delta} e^{-i\zeta z} g(z)\,dz)\,d\zeta,$$

and

$$J_{++}=\int_{\xi+i\delta,\ \xi\geqq 0} F(\zeta)\,(\int_{x-i\delta,\ x\geqq-\delta'} e^{-i\zeta z} g(z)\,dz)\,d\zeta,$$

$$J_{+-}=\int_{\xi+i\delta,\ \xi\geqq 0} F(\zeta)\,(\int_{x-i\delta,\ x\leqq-\delta'} e^{-i\zeta z} g(z)\,dz)\,d\zeta,$$

$$J_{-+}=\int_{\xi+i\delta,\ \xi\leqq 0} F(\zeta)\,(\int_{x+i\delta,\ x\geqq-\delta'} e^{-i\zeta z} g(z)\,dz)\,d\zeta,$$

$$J_{--}=\int_{\xi+i\delta,\ \xi\leqq 0} F(\zeta)\,(\int_{x+i\delta,\ x\leqq-\delta'} e^{-i\zeta z} g(z)\,dz)\,d\zeta,$$

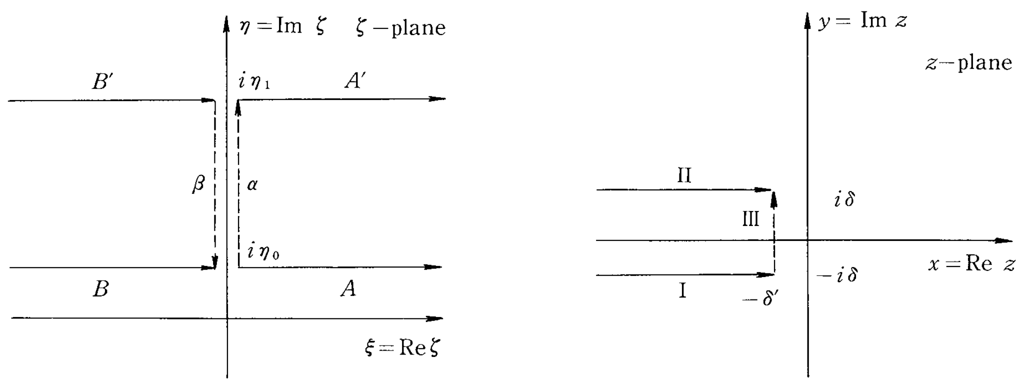
where $0<\delta'<<1$.

Trivially we have $J=J_++J_-$ and $J_+=J_{++}+J_{+-}$, $J_-=J_{-+}+J_{--}$.

Since the values $\|J_{++}\|_H$ and $\|J_{-+}\|_H$ are smaller than the right hand side of the required inequality by their definitions, it is sufficient to prove the following statement in order to obtain the desired inequality: for every $\theta>0$ and $\delta'>0$, $\|J_{+-}+J_{--}\|_H<\theta$ if δ is sufficiently small.

We denote J_{+-} by $J(\mathrm{I},A)$ and J_{--} by $J(\mathrm{II},B)$, where the paths of integration are as below. Just in the same way we denote by $J(\mathrm{I},A')$ etc., the integral over the path of integration $\mathrm{I}\times A'$ etc..

Figure



By the condition (†) we can easily conclude that $J(\mathrm{I},A')$ and

$J(\mathrm{II}, B')$ tend to zero as η_1 tends to infinity. While we have $(J_{+-}+J_{--})-(J(\mathrm{I}, A')+J(\mathrm{II}, B'))=J(\mathrm{I}, \alpha)+J(\mathrm{II}, \beta)=J((\mathrm{I}-\mathrm{II}), \alpha)=-J(\mathrm{III}, \alpha)$ since $J(\mathrm{I}+\mathrm{III}-\mathrm{II}, \alpha)=0$ by Cauchy's integral theorem. Hence it is sufficient to prove $\|J(\mathrm{III}, \alpha)\|_H$ tends to zero as η_1 tends to infinity.

Using the condition (†) again, we have the following estimate for every $\delta'>0$:

$$\left\|\int_{\eta_0}^{\infty} F(i\tau)\left(\int_{-\delta}^{\delta} exp(-iz\zeta)g(z)\,dz\right)d\tau\right\|_H$$

$$\leqq C_{\delta'}\int_{\eta_0}^{\infty} exp(\delta'\tau/2)\left(\int_{-\delta}^{\delta} exp(-\delta'\tau)\,dy\right)d\tau$$

$$=2\delta C_{\delta'}\int_{\eta_0}^{\infty} exp(-\delta'\tau/2)\,d\tau=K_{\delta'}\cdot\delta.$$

Thus we have the desired result .

When the case $n\geqq 2$, we consider as follows. Since Γ is a closed convex cone, we can represent $\Gamma=\bigcap_\xi H_\xi$, where $H_\xi=\{x;\langle x, \xi\rangle\geqq 0\}$. Then we can prove the following estimate just as in the case of $n=1$. After some affine transformation if necessary,

$$\left\|\int F(\zeta_1, \cdots, \zeta_n)\left(\int e^{-i\langle\zeta, z\rangle}g(z_1, \cdots, z_n)\,dz_1\cdots dz_n\right)d\zeta_1\cdots d\zeta_n\right\|_H$$

$$\leqq C_\varepsilon \sup_{x_1\geqq-\varepsilon, |y|\leqq\varepsilon}|g(z)e^{\varepsilon|z|}|$$

for every small $\varepsilon>0$ when $g(z)$ satisfies the estimate $\sup_{|\mathrm{Im}\, z|<1/m}|g(z)e^{(1/m)|z|}|<\infty$. This concludes that ν can be regarded as an element of $L_b(\mathscr{Q}(H_\xi^a); H)$. On the other hand $L_b(\mathscr{Q}(H_\xi^a); H)$ is isomorphic to $H^n_{H_\xi^a}(\boldsymbol{D}^n\times\sqrt{-1}I^n, {}^H\tilde{\mathscr{O}})$ by the theorem 6.16. Thus ν can be considered to belong to $\bigcap_\xi H^n_{H_\xi^a}(\boldsymbol{D}^n\times\sqrt{-1}I^n, {}^H\tilde{\mathscr{O}})=H^n_{\bigcap_\xi H_\xi^a}(\boldsymbol{D}^n\times\sqrt{-1}I^n, {}^H\tilde{\mathscr{O}})$ since $\{{}^H\mathscr{R}(\Omega)\}$ constitutes a sheaf on $\boldsymbol{D}^n$. This proves that ν belongs to $H^n_K(\boldsymbol{D}^n\times\sqrt{-1}I^n, {}^H\tilde{\mathscr{O}})\cong L_b(\mathscr{Q}(K); H)$. This completes the proof.

REMARK. Theorems 7.10 and 7.11 can be regarded as an analogue of the Paley-Wiener theorem.

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