J. Math. Tokushima Univ.
Vol. 8. (1974), 9—12

Irreducible Bases of N-semigroups

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(Received May 10, 1974)

1. Introduction and statement of theorem. Throughout this note G is an N-semigroup and the operation is written additively. We use the same terminology and notations as in [1]. $H_+(G)$ denotes the set of all $\boldsymbol{R}_+$-valued homomorphisms on G and $H(G)$ denotes the subspace of the $\boldsymbol{R}$-vector space $\mathrm{Hom}(G, \boldsymbol{R})$ generated by $H_+(G)$, where $\boldsymbol{R}$ is the real number field and $\boldsymbol{R}_+$ is the additive semigroup of all positive real numbers. A base of $H(G)$ contained in $H_+(G)$ is called a base of $H_+(G)$ and we define $\dim G = \dim_{\boldsymbol{R}} H(G)$.

PROPOSITION 0. *Let G be an N-semigroup. The following conditions are equivalent:*

(1) *For any distinct* $x, y \in G$ *there exist* $z \in G$ *and a positive integer* m *such that either* $mx = my + z$ *or* $my = mx + z$.

(2) $\dim G = 1$ *and G is embedded in* $\boldsymbol{R}_+$.

(3) *Every homomorphism on G into any N-semigroup is injective (i. e. G has no N-congruence except the identity congruence).*

The equivalence of (1) and (2) is proved in [1], Theorem 10, and the equivalence of (1) and (3) is proved in [3], Proposition 2. 2. An N-semigroup satisfying the conditions of Proposition 0 is called linear (Tamura calls it irreducible). A homomorphism $f \in H_+(G)$ is called irreducible if the image $f(G)$ is linear. A base (f_α) of $H(G)$ is called irreducible if every f_α is irreducible. It is known that any N-semigroup G has an irreducible homomorphism $f \in H_+(G)$ (see [2] and [3]). The purpose of this note is to prove

THEOREM *A finite dimensional N-semigroup has an irreducible* *base. Therefore a finite dimensional affine N-semigroup*[*] *is isomorphic onto a subdirect sum of a finite number of linear N-semigroups.*

2. Unitary subsemigroups of an N-semigroup. A subsemigroup G' of an N-semigroup G is called unitary iff $x \in G'$ and $x+y \in G'$ for $y \in G$ implies $y \in G'$.

PROPOSITION 1. *Let G' be a unitary subsemigroup of an N-semigroup G and let $i: G' \to G$ be the inclusion map. Then G' is itself an N-semigroup and the induced $\boldsymbol{R}$-homomorphism $i^*: H(G) \to H(G')$ is surjective. Therefore* $\dim G' \leq \dim G$.

PROOF. Since G' is unitary, $nx = y+z$ for $x, y \in G'$ and for $z \in G$ implies $z \in G'$. This shows that G' is archimedean, so it is an N-semigroup. Next, for every $f \in H_+(G')$ the pair (G', f) satisfies the condition (#) in [1], § 6. Therefore every $f \in H_+(G')$ is extensible to G, thus $i^*: H(G) \to H(G')$ is surjective.

Let S be a subset of G. $\langle S \rangle$ denotes the smallest unitary subsemigroup of G containing S. Let G' be a unitary subsemigroup of G and let $x \in G$. It is easy to see that $\dim \langle G', x \rangle \leq \dim G'+1$. Now, let $\sum_r$ be a family of all unitary subsemigroups of dimension r of G. Using Zorn's lemma we have

PROPOSITION 2. *If $\sum_r \neq \phi$, then $\sum_r$ has a maximal element.*

Let G' be a maximal element of $\sum_r$. Then G' has the properties (1) $\dim G' = r$ and (2) $\dim \langle G', x \rangle = r+1$ for any $x \in G \backslash G'$.

COROLLARY. *Let G be an N-semigroup with dimension r and let $r \geq r' \geq 1$. Then there is a unitary subsemigroup of G with dimension r'.*

3. Proof of Theorem. We begin with the case of dimension 2. Let G be an N-semigroup of dimension 2. Let $x, y \in G$ and $x \nsim y$. Then there exist $g_1, g_2 \in H_+(G)$ such that $g_1(x) = g_2(x) = 1$ and $g_1(y) <$

(*) As to affine N-semigroups see [1], § 8.

$g_2(y)$. Let m_1, m_2, n_1, n_2 be positive integers such that $g_1(y)< m_1/n_1 <m_2/n_2<g_2(y)$. In the same way as in the proof of [1], Lemma 4, we have two homomorphisms $f_1, f_2 \in H_+(G)$ such that $m_1 f_1(x) = n_1 f_1(y)$, $m_2 f_2(x) = n_2 f_2(y)$ and $f_1(x) = f_2(x)$. From these inequalities f_1 and f_2 are non-degenerate, hence they are irreducible since dim $G=2$. Thus (f_1, f_2) is an irreducible base of G.

In the general case the proof is proceeded by induction on dim G. In order to do this it would be enough to prove the following two lemmata.

LEMMA 1. *Let G' be a unitary subsemigroup of G and let $f \in H_+(G')$ be irreducible. Then there exists an irreducible homomorphism $\bar{f} \in H_+(G)$ such that $\bar{f}|_{G'} = f$.*

LEMMA 2. *Let G' be a unitary subsemigroup of a finite dimensional N-semigroup G such that* dim G' = dim $G-1$ *and let $f \in H_+(G')$ be irreducible. Then there exist two distinct irreducible homomorphisms $\bar{f}_1$ and $\bar{f}_2$ in $H_+(G)$ such that $\bar{f}_1|_{G'} = \bar{f}_2|_{G'} = f$.*

PROOF OF LEMMA 1. By Proposition 1, f is extensible to a homomorphism $\tilde{f} \in H_+(G)$. Let $g \in H_+(\tilde{f}(G))$ be an irreducible homomorphism. Since $f(G')$ is linear, $g|_{f(G')} = ri$ for some $r \in \boldsymbol{R}_+$, where $i: f(G') \to \boldsymbol{R}_+$ is the inclusion map. Then $\bar{f} = \frac{1}{r}\tilde{f} \circ g \in H_+(G)$ is a desired homomorphism.

PROOF OF LEMMA 2. Extend f to a homomorphism $\tilde{f} \in H_+(G)$. Assume that $\tilde{f}$ is not irreducible. We see dim $\tilde{f}(G)=2$ since dim $\tilde{f}(G')=1$. Therefore there exists an irreducible base (g_1, g_2) on $\tilde{f}(G)$. For the same reason as in the proof of Lemma 1, we have $g_1|_{f(G')} = r_1 i$ and $g_2|_{f(G')} = r_2 i$ for some $r_1, r_2 \in \boldsymbol{R}_+$. The homomorphisms $\bar{f}_1 = \frac{1}{r_1}\tilde{f} \circ g_1$ and $\bar{f}_2 = \frac{1}{r_2}\tilde{f} \circ g_2$ satisfy the conditions of the lemma.

4. PROBLEM. Does an infinite dimensional N-semigroup have an irreducible base?

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References

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