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On Exact Expressions for the Expected Values of Certain Normal Order Statistics

By

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Abstract

The expected values of normal order statistics have hitherto been expressed in terms of elementary functions just for samples of size $n \leqslant 5$, their detailed computation for samples of size $n > 5$ having been accomplished by methods of numerical integration.

In this paper it is shown that by reducing a particular plane double integral to a convergent series, explicit expressions can be set out for the expected values of the order statistics of samples of size $n \leqslant 7$ drawn from a unit normal parent.

1. Introduction and Summary

The moments of normal order statistics have attracted the attention of a number of writers possibly because in addition to their theoretical interest, the laborious nature of the computations involved in their evaluation has provided a continuing challenge to many mathematicians. Ruben [8] has set them into a generalised context by demonstrating that they are (together with the moment generating functions of the squares of normal order statistics) related to the surface contents of certain hyperspherical simplices. But as he remarks on p. 213 of [8] these contents cannot be expressed in terms of elementary functions for dimensions greater than three. After noting the prior existence of expressions involving elementary circular functions for the odd moments of samples of size 2, 3 and 4, he goes on to suggest that his results throw some light on the unavailability of moments relating to samples of size $n > 4$. Curiously, he refers in his introduction to a paper by Godwin [3] which, on p.p. 284, 285, sets out exact expected values of all order statistics of samples of size $n \leqslant 5$ together with certain higher moments (that is, excluding the first) of order statistics of samples of size 6. There is of course no reason why the higher transcendental functions that Ruben had in mind should not reduce to recognizable expressions for certain values of the variables involved, and that is a point of more than routine mathematical interest in itself. For his part, Godwin writes that it is "possible" that elementary functions no longer suffice for results with a higher number of variables than those used in his analysis, a conjecture which covers in particular, the expected values of all normal order statistics of samples of size $n>5$.

Other writers have contributed to the literature on exact moments of normal order statistics. They include Jones [7], (means for $n\leqslant 3$, other moments $n\leqslant 4$), Watanabe et al [11] (means and other moments $n\leqslant 7$), Bose and Gupta [1] (all odd moments for $n\leqslant 5$, all even moments $n\leqslant 6$). In a paragraph reviewing this literature, David [2] p. 34 observes that "the designation of certain functions as elementary is rather arbitrary", and he displays an integral due to Watanabe et al [11] (which is in fact one of four insoluble integrals set out on p.63 of [11]) namely, $\int \sin^{-1}[3/(8-\tan^2\Psi)]^{\frac{1}{2}}\,d\Psi$, as representative of expressions used by them in the derivation of their 'explicit forms' ([11] p.75) for the moments and product moments for $n\leqslant 7$.

In general, the detailed evaluation of the moments of normal order statistics has been achieved, with various refinements, by numerical integration based on the zeros and the weight factors of the Hermite-polynomials, and there are numerous sets of tables available that have been obtained by these and other approximate processes. (See bibliography). The derivation of alternative integral forms, including that attributed to Watanabe et al above, naturally adds further problems in numerical integration to those that are already on hand. One might add that the use of the term explicit as it appears in much of the literature is also rather arbitrary, but we would volunteer that expressions that are cleared of insoluble integrals are more 'explicit' than those that are not.

In this paper we follow in part the development due to Watanabe et al [11], our equations resembling theirs, apart from our use initially of those particular values for various constants that are appropriate to the integrals for the expected values of the largest items only. However where, in their treatment of the moments relating to $n=6$ and $n=7$, they discard the symmetry with respect to the variables of integration of plane double integrals of the type

$$(1.1)\qquad \int_0^{\pi/4}\int_0^{\pi/4}\left[1+\frac{1}{2}\sec^2\theta_1+\frac{1}{2}\sec^2\theta_2\right]^{-\frac{1}{2}}d\theta_1\,d\theta_2$$

by integrating once to form

(1.2) $\int_0^{\pi/4}\left(1+\frac{1}{2}\sec^2\theta_2\right)^{-\frac{1}{2}}\sin^{-1}[(1/\sqrt{2})(2+\sec^2\theta_2)^{\frac{1}{2}}(3+\sec^2\theta_2)]^{-\frac{1}{2}}d\theta_2.$

We retain this symmetry in our expressions until we reach a convergent series of plane double integrals, each one of which is capable of being evaluated as the square of a known integral of a single variable. Thus we derive a series (and its remainder after a given number of terms) with which we can set out explicit expressions for the expected values of the largest items of samples of size six and seven in addition to those of size five and less. The expected values of the remaining normal order statistics of these samples are then expressed as linear forms of terms that appear in the expressions for those of the largest items.

2. Assumptions and Preliminary Results

Let $(X_1, X_2 \dots, X_n)$ be a random sample from a continuous distribution having probability density function (pdf) $f(x)$ and cumulative distribution function (cdf) $F(x)$, both defined on the entire axis of real numbers, and let $X_{1:n} \leqslant X_{2:n} \leqslant \cdots \leqslant X_{p:n} \leqslant \cdots \leqslant X_{n:n}$, be the order statistics of this sample. The expected value of the p-th order statistic will be denoted by $E(X_{p:n})$. Since, for a normal parent distribution, there exist linear forms relating all the $E(X_{p:n})$ for each n, it is convenient to examine only the simplest case, that is, the expected value or mean of the largest item given by

(2.1) $$E(X_{n:n}) = n\int_{-\infty}^{\infty}[F(x)]^{n-1}f(x)x\,dx.$$

If the random sample $(X_1, X_2, \dots, X_n)$ is drawn from a standard normal parent (zero mean, unit variance) then

(2.2) $$f(x)=(1/\sqrt{2\pi})\exp(-x^2/2);\ F(x)=(1/\sqrt{2\pi})\int_{-\infty}^{x}\exp(-w^2/2)\,dw$$

(2.3) $$f'(x)=-f(x)x;\ F'(x)=f(x)$$

(2.4) $$F(x)=\frac{1}{2}+\int_0^x f(w)\,dw, \quad \text{all } x.$$

We will make use of the following,

(2.5) $$0<\int_0^{\infty}\exp(-Ax^2)\,dx=\sqrt{\pi}/(2\sqrt{A}),\ A \text{ is a positive constant.}$$

We define functions $g(x)$ and $G(x)$ as follows,

$$g(x)=2f(x);\ G(x)=\int_0^x g(w)\,dw=2\int_0^x f(w)\,dw. \tag{2.6}$$

The expression (2.1), with the interpretations placed on $f(x)$ and $F(x)$ as set out in (2.2), defines the expected value of the largest item in a sample of size n from a unit normal parent. Referring now to line (2.4), it is clear that $F(x)=\frac{1}{2}+\frac{1}{2}G(x)$ so that writing (2.1) in terms of $g(x)$ and $G(x)$ we obtain,

$$E(X_{n:n})=n2^{-n}\int_{-\infty}^{\infty}[1+G(x)]^{n-1}g(x)x\,dx$$

$$=n2^{-n}\int_0^{\infty}[1+G(x)]^{n-1}g(x)x\,dx+n2^{-n}\int_{-\infty}^{0}[1+G(x)]^{n-1}g(x)x\,dx \tag{2.7}$$

$$=n2^{-n}\int_0^{\infty}\{[1+G(x)]^{n-1}-[1-G(x)]^{n-1}\}g(x)x\,dx$$

$$=n2^{1-n}\sum_{n\geqslant 1}\binom{n-1}{2r-1}\int_0^{\infty}[G(x)]^{2r-1}g(x)x\,dx. \tag{2.8}$$

The essential step in the development above being the nominal transformation from x to $-x$ in the second integral on the right of (2.7), bearing in mind that $g(x)$ and $G(x)$ are even and odd functions respectively.

Equation (2.8) brings to light an aspect of the relationship between the expected values of normal order statistics and those of the chi-distribution (one degree of freedom) or 'folded' normal distribution. (See Govindarajulu [4]). Since $x\geqslant 0$ in each integral of (2.8), $g(x)$ and $G(x)$ become, with the additional definition $g(x)=G(x)=0$ for $x<0$, the pdf and cdf respectively of a random variable χ having the 'folded' normal distribution. Comparing the integral terms of (2.8) with equation (2.1) it is apparent that $E(\chi_{n:n})$ is a linear combination of the $E(\chi_{2r:2r})$ where $2r\leqslant n$.

3. Evaluation of Integrals

$$\text{Let } I_{2r-1}=\int_0^{\infty}[G(x)]^{2r-1}g(x)x\,dx. \tag{3.1}$$

From (2.2), (2.3) and (2.6) we have; $g'(x)=-g(x)x$; $G'(x)=g(x)$; $g(0)=\sqrt{2/\pi}$; $G(0)=0$; $\lim_{x\to\infty} g(x)=0$; $\lim_{x\to\infty} G(x)=1$. Integrating by parts on the right of (3.1) we get,

$$I_{2r-1}=-[G(x)]^{2r-1}g(x)\Big|_0^{\infty}+(2r-1)\int_0^{\infty}[G(x)]^{2r-2}[g(x)]^2\,dx$$

giving

$$I_{2r-1}=(2r-1)\int_0^{\infty}[G(x)]^{2r-2}[g(x)]^2\,dx \tag{3.2}$$

where

$$[g(x)]^2=(2/\pi)\exp(-x^2) \tag{3.3}$$

and

$$\begin{aligned}[G(x)]^{2r-2}&=\left\{\sqrt{2/\pi}\int_0^x \exp(-w^2/2)\,dw\right\}^{2r-2}\\ &=(2/\pi)^{r-1}\left\{\int_0^x\int_0^x \exp\left(-\frac{1}{2}x_1^2-\frac{1}{2}x_2^2\right)dx_1\,dx_2\right\}^{r-1}\\ &=(4/\pi)^{r-1}\left\{\int_0^{\pi/4}\int_0^{x\sec\theta}\exp\left(-\frac{1}{2}R^2\right)R\,dR\,d\theta\right\}^{r-1}\\ &=(4/\pi)^{r-1}\left\{\int_0^{\pi/4}\left[1-\exp\left(-\frac{1}{2}x^2\sec^2\theta\right)\right]d\theta\right\}^{r-1}\end{aligned}$$

and this may be expressed as the $(r-1)$-fold multiple integral of a product given by

$$\begin{aligned}[G(x)]^{2r-2}=(4/\pi)^{r-1}\int_0^{\pi/4}\cdots\int_0^{\pi/4}&\left[1-\exp\left(-\frac{1}{2}x^2\sec^2\theta_1\right)\right]\cdots\\ &\cdots\left[1-\exp\left(-\frac{1}{2}x^2\sec^2\theta_{r-1}\right)\right]d\theta_1\ldots d\theta_{r-1}.\end{aligned} \tag{3.4}$$

There are $\binom{r-1}{j}$ possible products of j exponential functions of the type $-\exp\left(-\frac{1}{2}x^2\sec^2\theta_p\right)$ from $(r-1)$ available. Hence multiplying out the $(r-1)$ factors in the integrand of (3.4) and following through with the integration on each term, we would find that for $r>1$ there is $\binom{r-1}{0}$ or one term of the form

$$(4/\pi)^{r-1}\int_0^{\pi/4}\cdots\int_0^{\pi/4}d\theta_1\ldots d\theta_{r-1}=1, \tag{3.5}$$

there are $\binom{r-1}{1}$ equal terms of the form

$$(4/\pi)^{r-1}\int_0^{\pi/4}\cdots\int_0^{\pi/4}\left[-\exp\left(-\frac{1}{2}x^2\sec^2\theta_p\right)\right]d\theta_1\ldots d\theta_{r-1}$$

$$(3.6) \qquad = (4/\pi)\int_0^{\pi/4}\left[-\exp\left(-\frac{1}{2}x^2\sec^2\theta\right)\right]d\theta,$$

and there are $\binom{r-1}{2}$ equal terms of the form

$$(4/\pi)^{r-1}\int_0^{\pi/4}\cdots\int_0^{\pi/4}\left[\exp\left(-\frac{1}{2}x^2\sec^2\theta_p-\frac{1}{2}x^2\sec^2\theta_q\right)\right]d\theta_1\ldots d\theta_{r-1}$$

$$(3.7) \qquad = (4/\pi)^2\int_0^{\pi/4}\int_0^{\pi/4}\exp\left(-\frac{1}{2}x^2\sec^2\theta_1-\frac{1}{2}x^2\sec^2\theta_2\right)d\theta_1\,d\theta_2.$$

A formal pattern is evident and although for $E(X_{6:6})$ and $E(X_{7:7})$ we have $2r<7$ or $r=1, 2, 3$ so that there are no terms beyond (3.7), we substitute the general term of the progression for $[G(x)]^{2r-2}$ that is suggested by the results (3.5), (3.6) and (3.7) into (3.2), with an additional definition namely, that (3.5) is 1 for $r=1$. Accordingly,

$$(3.8) \quad I_{2r-1}=(2r-1)\left(\frac{2}{\pi}\right)\int_0^{\infty}\left\{\sum_{j=0}^{r-1}\binom{r-1}{j}\left(\frac{-4}{\pi}\right)^j\int_0^{\pi/4}\cdots\right.$$

$$\left.\cdots\int_0^{\pi/4}\exp\left[-x^2\left(1+\frac{1}{2}\sec^2\theta_1+\cdots+\frac{1}{2}\sec^2\theta_j\right)\right]d\theta_1\cdots d\theta_j\right\}dx.$$

Each of the integrals on the right of (3.8) is uniformly convergent. We can therefore interchange the order of integration, performing the integration first with respect to x in each, and employing (2.5). The resulting factor $\sqrt{\pi}/2$ is taken outside the sum so that (3.8) reduces to

$$(3.9) \quad I_{2r-1}=\frac{2r-1}{\sqrt{\pi}}\sum_{j=0}^{r-1}\binom{r-1}{j}\left(\frac{-4}{\pi}\right)^j\int_0^{\pi/4}\cdots\int_0^{\pi/4}\frac{d\theta_1\ldots d\theta_j}{\sqrt{1+\frac{1}{2}\sec^2\theta_1+\cdots+\frac{1}{2}\sec^2\theta_j}}.$$

We wish to evaluate the two integrals for which $j=1$ and 2, but in the general case, the Jacobian of the transformation $\sin\theta_p=(\sqrt{3}/2)\sin u_p$, $p=1$ to j, is a diagonal form which reduces to the product $(\sqrt{3}/2)^j\sec\theta_1\ldots\sec\theta_j \times\cos u_1\ldots\cos u_j$; also, $\left(\frac{1}{2}+\cos^2\theta_p\right)=(3/2)\cos^2 u_p$ all p; the lower limits of integration all transform to zero; the upper limits to $\sin^{-1}(1/\sqrt{3})$ which we will denote by α. Hence for $j=1$,

$$(3.9\text{a}) \quad \int_0^{\pi/4}\frac{d\theta_1}{\sqrt{1+\frac{1}{2}\sec^2\theta_1}}=\int_0^{\pi/4}\frac{\cos\theta_1\,d\theta_1}{\sqrt{\frac{1}{2}+\cos^2\theta_1}}=\int_0^{\pi/4}du_1=\alpha=\sin^{-1}\left(\frac{1}{\sqrt{3}}\right).$$

And for $j=2$ we recognize the integral (1.1)

$$\int_0^{\pi/4}\int_0^{\pi/4}\frac{d\theta_1\,d\theta_2}{\sqrt{1+\frac{1}{2}\sec^2\theta_1+\frac{1}{2}\sec^2\theta_2}}$$

$$=\int_0^{\pi/4}\int_0^{\pi/4}\frac{\cos\theta_1\cos\theta_2\,d\theta_1\,d\theta_2}{\sqrt{\left(\frac{1}{2}+\cos^2\theta_1\right)\left(\frac{1}{2}+\cos^2\theta_2\right)-\frac{1}{4}}}$$

$$=\int_0^{\alpha}\int_0^{\alpha}\frac{du_1\,du_2}{\sqrt{1-(\sec^2u_1\sec^2u_2)/9}}$$

$$=\int_0^{\alpha}\int_0^{\alpha}\sum_{i\geqslant 0}\binom{-\frac{1}{2}}{i}\left(\frac{-1}{9}\right)^i\sec^{2i}u_1\sec^{2i}u_2\,du_1\,du_2$$

$$=\sum_{i\geqslant 0}\binom{-\frac{1}{2}}{i}\left(\frac{-1}{9}\right)^i\left[\int_0^{\alpha}\sec^{2i}u\,du\right]^2. \tag{3.9b}$$

The use of the binomial series for $[1-(\sec^2u_1\sec^2u_2)/9]^{-\frac{1}{2}}$ follows since, given $\sec^2 0=1$ and $\sec^2\alpha=3/2$, we have $0<1/9\leqslant(\sec^2u_1\sec^2u_2)/9\leqslant 1/4<1$ at every point of the region over which the double integral is taken. This is a sufficient condition for the uniform convergence of this series in the region and also for the validity of its integration term by term. The nature of the convergence of the expression (3.9b) is discussed later. We note that (3.9b) provides an alternative means to (1.2) for the evaluation of (1.1)

Employing the standard recurrence relation

$$\int\sec^m u\,du=\frac{1}{m-1}(\sec^{m-2}u\tan u)+\frac{m-2}{m-1}\int\sec^{m-2}u\,du,$$

and substituting $m=2i$, $\sec\alpha=\sqrt{3/2}$ and $\tan\alpha=1/\sqrt{2}$,

$$\int_0^{\alpha}\sec^{2i}u\,du=\alpha=\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)\qquad\text{for } i=0 \tag{3.9c}$$

and

$$\int_0^{\alpha}\sec^{2i}u\,du=\frac{1}{(2i-1)}\cdot\frac{1}{\sqrt{2}}\cdot\left(\frac{3}{2}\right)^{i-1}s_i\qquad\text{for } i>0 \tag{3.9d}$$

where

$$s_i=1+\frac{2i-2}{2i-3}\cdot\frac{2}{3}\cdot s_{i-1};\quad s_1=1. \tag{3.9e}$$

Each s_i is a distinct series of i positive terms, and since $(2i-2)/(2i-3) < 4/3$ for $i>3$, then from (3.9e) we have for any positive integer m, $s_{3+m} < 1+(8/9)+\cdots(8/9)^{m-1}+(8/9)^m s_3$, where $s_3 = 83/27$. But $\lim_{m\to\infty}(8/9)^m s_3 = 0$ so that $\lim_{m\to\infty} s_{3+m} \leqslant [1-(8/9)]^{-1} = 9$, therefore s_i is bounded. Similarly, $1 < (2i-2)/(2i-3) < (2N-2)/(2N-3)$ for all $i>N$. Again by (3.9e) it follows that,

$$1+\frac{2}{3}+\cdots+\left(\frac{2}{3}\right)^{i-2}+\left(\frac{2}{3}\right)^{i-1} s_1 < s_i$$

$$< 1+\frac{2N-2}{2N-3}\cdot\frac{2}{3}+\cdots+\left(\frac{2N-2}{2N-3}\cdot\frac{2}{3}\right)^{m-1}+\left(\frac{2N-2}{2N-3}\cdot\frac{2}{3}\right)^{m} s_N$$

where $s_1 = 1$, $i = m+N$, $N>3$, and $s_N \leqslant 9$. In the limit $m\to\infty$, $i\to\infty$, and the last term on the right tends to zero since $(4N-4)/(6N-9) < 8/9$ for $N>3$, so that

$$\left[1-\frac{2}{3}\right]^{-1} \leqslant \lim_{i\to\infty} s_i \leqslant \left[1-\frac{2N-2}{2N-3}\cdot\frac{2}{3}\right]^{-1},$$

$$3 \leqslant \lim_{i\to\infty} s_i \leqslant 3+\frac{6}{2N-5}, \qquad \text{for all } N>3.$$

It follows that the limiting value for s_i is 3. This result is confirmed by assuming that the limit exists and is equal to s, then taking the limit $i\to\infty$ on both sides of equation (3.9e) to obtain $s = 1+(2/3)s$, the solution of which is clearly $s=3$. In fact, the largest value of s_i is slightly less than 3.7 occurring for $i=6$. From the computational viewpoint, the recurrence relation (3.9e) for s_i is simple to program and has the advantage that truncation errors do not accumulate.

We substitute results (3.9c) and (3.9d) into the right hand side of (3.9b) to complete the evaluation of the double integral. Hence

$$\text{(3.9f)} \qquad \int_0^{\pi/4}\int_0^{\pi/4} \frac{d\theta_1\, d\theta_2}{\sqrt{1+\frac{1}{2}\sec^2\theta_1+\frac{1}{2}\sec^2\theta_2}} = \left[\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)\right]^2 + S$$

where

$$\text{(3.9g)} \qquad S = \sum_{i\geqslant 1} \binom{-\frac{1}{2}}{i} \frac{(-1)^i}{(2i-1)^2}\left(\frac{1}{2}\right)^{2i-1}\left(\frac{s_i}{3}\right)^2.$$

However, since some or all of these results are displayed in papers by Jones [7], Godwin [3], Watanabe et al [11] and Bose and Gupta [1], we go on to find $E(X_{p:6})$ and $E(X_{p:7})$.

It is convenient to denote the angle $[(\pi/4)-\sin^{-1}(1/\sqrt{3})]$ by β so that from equations (3.10), (3.11), and (3.12) we now have

$$(4.3) \qquad I_1=\pi^{-\frac{1}{2}}; \quad I_3=3\pi^{-\frac{1}{2}}(4/\pi)\beta; \quad I_5=5\pi^{-\frac{1}{2}}(4/\pi)^2(\beta^2+S).$$

(We note that from (2.1) and (3.1), $E(X_{2:2})=2I_1$, $E(X_{4:4})=4I_3$ and $E(X_{6:6})=6I_5$. See also Govindarajulu [5] p.1302 for $E(X_{2:2})$ and $E(X_{4:4})$).

To evaluate I_{2r-1} we substitute results (3.9a) and (3.9f) into (3.9).

$$(3.10) \qquad r=1; \quad I_1=\frac{1}{\sqrt{\pi}}$$

$$(3.11) \qquad r=2; \quad I_3=\frac{3}{\sqrt{\pi}}\left[1-\frac{4}{\pi}\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)\right]$$

$$(3.12) \qquad r=3; \quad I_5=\frac{5}{\sqrt{\pi}}\left\{\left[1-\frac{4}{\pi}\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)\right]^2+\left(\frac{4}{\pi}\right)^2 S\right\}.$$

This section is concluded with a note related to the convergence of S (see also (3.9b)). The absolute value of the binomial coefficient $\binom{-\frac{1}{2}}{i}<\frac{1}{2}$ for all $i>1$; $(2i-1)^{-2}<(2N-1)^{-2}$ for all $i>N$; we have seen that $s_i\to 3$. If R_N is the remainder after $(N-1)$ terms

$$R_N=\sum_{i\geqslant N}\binom{-\frac{1}{2}}{i}\frac{(-1)^i}{(2i-1)^2}\left(\frac{1}{2}\right)^{2i-1}\left(\frac{s_i}{3}\right)^2$$

all terms being positive.
Therefore

$R_N<\frac{1}{(2N-1)^2}\sum_{i\geqslant N}\left(\frac{1}{2}\right)^{2i}$ for N sufficiently large. That is

$$(3.13) \qquad R_N<\frac{1}{(2N-1)^2}\left(\frac{1}{2}\right)^{2N}\left(\frac{4}{3}\right).$$

By making adjustments for the exact behaviour of $(s_i/3)^2$ it can be shown that this inequality is true for all $N>1$. Summing the first seven terms of S we have obtained the estimate

$$S=0.03140\ 6940\cdots+R_8;\ R_8<9.1\times10^{-8}$$

which suggests that, for purposes of checking, a workable approximation should be

$$S=0.03140\ 7. \tag{3.14}$$

4. Conclusion

It remains now to express $E(X_{p:n})$ as a linear combination of the I_{2r-1} where $n=6, 7$ and $r=1, 2, 3$. We have

$$E(X_{p:n})=\int_{-\infty}^{\infty} n\binom{n-1}{p-1}[F(x)]^{p-1}[1-F(x)]^{n-p}f(x)x\,dx. \tag{4.1}$$

We observe that when $p=n$ (4.1) reduces to (2.1). Substituting for $f(x)$ and $F(x)$ from (2.6) and following the same development as at line (2.7) and below, (4.1) becomes

$$E(X_{p:n})=n2^{-n}\binom{n-1}{p-1}\int_0^{\infty}\Big\{[1+G(x)]^{p-1}[1-G(x)]^{n-p} - [1-G(x)]^{p-1}[1+G(x)]^{n-p}\Big\}\,g(x)x\,dx. \tag{4.2}$$

If $n-p=q-1$, then $\binom{n-1}{p-1}=\binom{n-1}{n-p}=\binom{n-1}{q-1}$, and since $p-1=n-q$ it follows from (4.2) that $E(X_{p:n})=-E(X_{q:n})=-E(X_{n+1-p:n})$. Further, by substituting $-G(x)$ for $G(x)$ in the expression in braces on the right of (4.2), the complete expression changes sign, suggesting that only odd powers of $G(x)$ are present. By comparison with equation (2.8) and recalling definition (3.1), this implies that $E(X_{p:n})$ is a linear combination of the I_{2r-1}. Accordingly, equation (4.2) (of which (2.8) is a special case), definition (3.1), together with results (3.10), (3.11), (3.12), enable us to set out the expressions for $n\leqslant 7$. Foregoing the elementary algebra the required means are set out below.

$$E(X_{4:6})=\frac{15}{8\sqrt{\pi}}\left[1-6\left(\frac{4}{\pi}\right)\beta+5\left(\frac{4}{\pi}\right)^2(\beta^2+S)\right]=0.20154=-E(X_{3:6}),$$

$$E(X_{5:6})=\frac{15}{61\sqrt{\pi}}\left[3-6\left(\frac{4}{\pi}\right)\beta-5\left(\frac{4}{\pi}\right)^2(\beta^2+S)\right]=0.64175=-E(X_{2:6}),$$

$$E(X_{6:6})=\frac{15}{16\sqrt{\pi}}\left[1+6\left(\frac{4}{\pi}\right)\beta+\left(\frac{4}{\pi}\right)^2(\beta^2+S)\right]=1.26720=-E(X_{1:6}),$$

$$E(X_{4:7})=-E(X_{4:7})=0,$$

$$E(X_{5:7})=\frac{105}{32\sqrt{\pi}}\left[1-6\left(\frac{4}{\pi}\right)\beta+5\left(\frac{4}{\pi}\right)^2(\beta^2+S)\right]=0.35270=-E(X_{3:7}),$$

$$E(X_{6:7})=\frac{21}{8\sqrt{\pi}}\left[1-5\left(\frac{4}{\pi}\right)^2(\beta^2+S)\right]=0.75737=-E(X_{2:7}),$$

$$E(X_{7:7})=\frac{21}{32\sqrt{\pi}}\left[1+10\left(\frac{4}{\pi}\right)\beta+5\left(\frac{4}{\pi}\right)^2(\beta^2+S)\right]=1.35217=-E(X_{1:7}).$$

The estimates to five DP included as a check with the expressions for $E(X_{p:n})$ above, are identical in each case with the corresponding figures in tables by Teichroew [9]. They were calculated on a FACIT manual calculator by first combining the various integral multiples of $(4/\pi)\beta=0.21634\ 6$ and $(4/\pi)^2(\beta^2+S)=0.09772\ 1$, where S is given by (3.14), then forming the product with the appropriate factor terms involving $\pi^{-\frac{1}{2}}$. The powers of π required are set out in Abramowitz and Stegun [13] p.3; $\sin^{-1}(/\sqrt{3})$ was obtained from the series for $\tan^{-1}(1/\sqrt{2})$ so that the estimate for β that was used was 0.16991 8.

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