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Homomorphisms on N-semigroups into R_+ and the Structure of N-semigroups

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§1. Introduction

Throughout this paper every semigroup we treat is commutative and the operation is written additively. It is a fundamental fact that any commutative semigroup is decomposed into a semilattice of archimedean semigroups (a semigroup is called archimedean if for every two elements x, y there exist an element z and a positive integer n such that $nx=y+z$). A commutative archimedean cancellative semigroup is an abelian group if it has an identity element. A commutative archimedean cancellative semigroup without identity is called an N-semigroup, which we are going to study in this paper.

An additive semigroup of all positive integers (resp. rational numbers, real numbers) is denoted by $\boldsymbol{Z}_+$ (resp. $\boldsymbol{Q}_+$, $\boldsymbol{R}_+$). A subsemigroup of $\boldsymbol{Z}_+$ (resp. $\boldsymbol{Q}_+$, $\boldsymbol{R}_+$) is called a positive integer (resp. rational, real) semigroup.

Sasaki and Tamura proved in [11] that any power-joined (i.e. for any elements x, y there exist positive integers m, n such that $mx=ny$) N-semigroup is isomorphic onto a subdiriect sum(*) of a positive rational semigroup and an abelian group. It is natural to put the following more general question: Is every N-semigroup isomorphic onto a subdirect sum of a positive real semigroup and an abelian group?

Let G be an N-semigroup and let a be an element of G. We define an equivalence relation $\underset{a}{\sim}$ on G as follows: $x \underset{a}{\sim} y$ iff $x+ma=y+na$ for some positive integers m, n. The quotient set $G_a=G/\underset{a}{\sim}$ is an abelian group by addition induced from G. G_a is called the structure group of G with respect to a.

Now, if G is a subdirect sum of a positive real semigroup and an abelian group, then clearly $\mathrm{Hom}(G, \boldsymbol{R}_+)\neq\phi$. Conversely, if we assume that there

(*) A subdirect sum G of semigroups A and B is a subsemigroup of the direct sum $A\oplus B$ where the projections of G into A and B are surjective.

exists a homomorphism φ on G into $\boldsymbol{R}_+$, then we can define a homomorphism f_a on G into $\boldsymbol{R}_+ \oplus G_a$ by $f_a(x)=(\varphi(x), \zeta(x))$ where $x \in G$ and ζ is the canonical surjection on G into G_a. It is easy to see that f_a is injective, so we see that G is a subdirect sum of a positive real semigroup and its structure group G_a. Thus the problem is reduced to find a homomorphism on G into $\boldsymbol{R}_+$.

Hewitt and Zuckerman already proved that Hom $(G, \boldsymbol{R}_+) \neq \phi$ for any N-semigroup G in their famous paper [4] (Theorem 8. 10), but their proof is not purely algebraic. Tamura also proved the same fact recently when he was engaged in studying N-congruences on N-semigroups ([13], [14], [15]).

Is there a concrete and natural way to construct a homomorphism on G into $\boldsymbol{R}_+$? Sasaki and Tamura defined in [11] a function $\bar{\varphi}$ on a power-joined N-semigroup into $\boldsymbol{Q}_+$ as follows:

$$\bar{\varphi}(x)=\frac{1}{n}\sum_{i=1}^{n} I(i\alpha, \alpha), \quad x \in G,$$

where $\alpha=\zeta(x) \in G_a$, n is the order of α and I is Tamura's $\mathscr{I}$-function introduced in [12]. The function $\bar{\varphi}$ gives us a homomorphism on G. In the case where G is not power-joined, it is inadequate to take a finite sum of $I(i\alpha, \alpha)$ since G_a is not periodic. But if there exists a limit

$$\bar{\varphi}(x)=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} I(i\alpha, \alpha), \quad \alpha=\zeta(x),$$

for every $x \in G$, does $\bar{\varphi}$ give us a way to define a homomorphism on G into $\boldsymbol{R}_+$? This question was our starting point.

In § 2 we construct two functions φ and ψ on G into $\boldsymbol{R}_+$ in a very natural way without using Tamura's representation. The method of constructing these is similar to the one used in the proof of the classical embedding theorem of ordered semigroups into $\boldsymbol{R}_+$. Some important inequalities about these functions are given in §§ 2 and 3. In § 4 we introduce a concept of *almost power-joined N-semigroups* which is a generalization of the concept of power-joined N-semigroups and prove that φ and ψ are homomorphisms on G if and only if G is almost power-joined. Every N-semigroup is decomposed into a disjoint union of almost power-joined N-semigroups. In § 6 we give an extension theorem of homomorphisms on G into $\boldsymbol{R}_+$, from which we see immediately that Hom $(G, \boldsymbol{R}_+) \neq \phi$ for any N-semigroup G. If we define a dimension of G by $\dim G = \dim_{\boldsymbol{R}} H(G)$ where $H(G)=\mathrm{Hom}(G, \boldsymbol{R}_+) \otimes_{\boldsymbol{R}_+} \boldsymbol{R}$, then we can say that an N-semigroup G is almost power-joined if and only if $\dim G=1$. The $\boldsymbol{R}$-vector space $H(G)$ has a deep relation with the structure of G and we study this subject in §§ 7 and 8. In § 8 we give a concept of *affine N-semigroups* and prove an embedding theorem of N-

semigroups into $\prod \boldsymbol{R}_+$, which may be considered to be a generalization of the classical embedding theorem.

I would like to express my thanks to Professor Hiroshi Okuyama for his constant encouragement during the preparation of this paper, and to Professor Takayuki Tamura who informed me of his results recently obtained.

§2. Some basic functions on an N-semigroup into $\boldsymbol{R}_+$.

Throughout this section G is an N-semigroup and an element a of G is fixed which we call a base element.

We define a positive integer $L(x)$ for $x \in G$ by

$$L(x) = \min\{l \mid l > 0,\ la = x + c \text{ for some } c \in G\}.$$

Since G is archimedean, the integer $L(x)$ always exists.

We begin with the following lemma.

Lemma 1. *Let $x, b \in G$ and n be a positive integer. If $x = na + b$, then $n < L(x)$.*

Proof. Let $l = L(x)$ and $la = x + c$ for $c \in G$. Assume $n \geq l$, then we have $x = x + b_0$, where $b_0 = (n-l)a + b + c$. So for any $y \in G$ we have $y + x = y + x + b_0$. Hence $y = y + b_0$ since G is cancellative. This means b_0 is an identity element (contradiction).

We define a non-negative integer $N(x)$ for $x \in G$ by

$$N(x) = \max\{n \mid n \geq 0,\ x = na + b \text{ for some } b \in G\},$$

where $x = na + b$ implies $x = b$ if $n = 0$.

We see $N(x) < L(x)$ from Lemma 1. Moreover we have

Proposition 1. *For any $x, y \in G$ and for any positive integer n, we have*

(1) $N(na) = n - 1$, $L(na) = n + 1$,

(2) $N(x + na) = N(x) + n$, $L(x + na) = L(x) + n$,

(3) $N(x) + N(y) \leq N(x + y) \leq \min\{N(x) + L(y), L(x) + N(y)\}$
$\leq \max\{N(x) + L(y), L(x) + N(y)\} \leq L(x + y) \leq L(x) + L(y)$.

Proof. (1) and (2) are immediate from the definitions. The inequalities $N(x) + N(y) \leq N(x + y)$ and $L(x + y) \leq L(x) + L(y)$ are also immediately obtained from the definitions. Now we shall prove the inequality $N(x + y) \leq N(x) + L(y)$. Let $N(x + y) = m$ and $L(y) = l$. Then $x + y = ma + b$ and

$la = y + c$ for some b, $c \in G$. Hence $x + la = ma + b + c$. Then $N(x + la) = N(x) + l \geq m$, thus we obtain $N(x) + L(y) \geq N(x + y)$. It is similarly proved that $L(x) + N(y) \leq L(x + y)$.

It is natural to consider that $N(nx)/n$ and $L(nx)/n$ approximate to the values expressing the "size" of x as $n \to \infty$. The proof of the existence of the limits in the next theorem is due to Samuel [10].

THEOREM 1. *There exist the limits* $\varphi(x) = \lim_{n\to\infty} N(nx)/n$ *and* $\psi(x) = \lim_{n\to\infty} L(nx)/n$ *for all* $x \in G$ *and we have*

$$N(x) < \varphi(x) \leq \psi(x) < L(x).$$

PROOF. Put $\alpha = \overline{\lim_{n\to\infty}} N(nx)/n$ (resp. $\beta = \underline{\lim_{n\to\infty}} L(nx)/n$). Then for any $\varepsilon > 0$ there exists a positive integer n_0 such that $N(n_0 x) \geq n_0(\alpha - \varepsilon)$ (resp. $L(n_0 x) \leq n_0(\beta + \varepsilon)$). For a given integer n we write $n = n_0 q + r$ (resp. $n = n_0 q - r$) with $0 \leq r < n_0$. Then $N(nx) \geq N(n_0 q x) \geq qN(n_0 x)$ (resp. $L(nx) \leq qL(n_0 x)$), hence $\frac{N(nx)}{n} \geq \frac{n_0 q}{n}(\alpha - \varepsilon)\left(\text{resp. } \frac{L(nx)}{n} \leq \frac{n_0 q}{n}(\beta + \varepsilon)\right)$. Then for n large enough $\frac{N(nx)}{n} \geq \alpha - 2\varepsilon\left(\text{resp. } \frac{L(nx)}{n} \leq \beta + 2\varepsilon\right)$. Thus we see that $\alpha = \lim_{n\to\infty} \frac{N(nx)}{n}\left(\text{resp. } \beta = \lim_{n\to\infty} \frac{L(nx)}{n}\right)$. It is clear that $\varphi(x) \leq \psi(x)$. Now we shall prove that $N(x) < \varphi(x)$. We have $x = N(x)a + b$ for some $b \in G$. Since G is archimedean, there is a positive integer l_0 such that $l_0 b = a + c$ for some $c \in G$. Then for any positive integer n, $nl_0 x = nl_0 N(x)a + na + nc$, hence $N(nl_0 x) \geq nl_0 N(x) + n$. Then we have

$$\varphi(x) = \lim_{n\to\infty} \frac{N(nl_0 x)}{nl_0} \geq N(x) + \frac{1}{l_0} > N(x).$$

The last inequality is similarly proved.

Thus we get two functions φ and ψ on G into $\boldsymbol{R}_+$. From Propositon 1 and the definitions we have

PROPOSITION 2. *For any* x, $y \in G$ *and for any positive integer* n *we have*

(1) $\varphi(na) = \psi(na) = n$,

(2) $\varphi(x + na) = \varphi(x) + n, \quad \psi(x + na) = \psi(x) + n$,

(3) $\varphi(nx) = n\varphi(x), \quad \psi(nx) = n\psi(x)$,

(4) $\varphi(x) + \varphi(y) \leq \varphi(x + y) \leq \min\{\varphi(x) + \psi(y), \ \psi(x) + \varphi(y)\}$
$\leq \max\{\varphi(x) + \psi(y), \ \psi(x) + \varphi(y)\} \leq \psi(x + y) \leq \psi(x) + \psi(y)$.

The method of constructing the functions φ and ψ originally appeared in the classical embedding theorem of ordered semigroups into $\boldsymbol{R}_+$ (see

Hölder [6], Alimof [1], Hion [5], Fuchs [3] and Kist & Leestma [7]). We shall refer to this point in §8 again and try to generalize the classical result.

As to the relation with Tamura's $\mathscr{I}$-function I, we see the following:

$$I(\alpha, \beta) = N(x+y) - N(x) - N(y),$$

$$\varphi(x) - N(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} I(i\alpha, \alpha),$$

where $\alpha=\zeta(x)$, $\beta=\zeta(y)$ and ζ is the canonical surjection on G into the structure group G_a.

§3. The change of base elements.

The functions N, L, φ and ψ which are defined in §2 are of course depend on a base element a. Hereafter, we write them N_a, L_a, φ_a and ψ_a in order to make a base element clear on which they depend. In this section we study the relation between φ_a (resp. ψ_a) and φ_b (resp. ψ_b) for two elements a and b of G.

Proposition 3. *For any a, b, $x \in G$ we have*

$$N_b(x) \geq N_b(a) N_a(x), \quad L_b(x) \leq L_b(a) L_a(x).$$

The proof is easily obtained from the definitions of the functions N and L.

Proposition 4. *For any a, b, $x \in G$ we have*

$$\varphi_b(a)\varphi_a(x) \leq \varphi_b(x) \leq \min\{\varphi_b(a)\psi_a(x), \psi_b(a)\varphi_a(x)\}$$
$$\leq \max\{\varphi_b(a)\psi_a(x), \psi_b(a)\varphi_a(x)\} \leq \psi_b(x) \leq \psi_b(a)\psi_a(x).$$

Proof. For any positive integer n, $nx = N_a(nx)a + c_n$ for some $c_n \in G$. Hence $n\varphi_b(x) \geq N_a(nx)\varphi_b(a)$ and $n\psi_b(x) \geq N_a(nx)\psi_b(a)$. Since $N_a(nx)/n \to \varphi_a(x)$ as $n\to\infty$, we have (1) $\varphi_b(a)\varphi_a(x) \leq \varphi_b(x)$ and (2) $\psi_b(a)\varphi_a(x) \leq \psi_b(x)$. From the equation $L_a(nx)a = nx + d_n$ with $d_n \in G$, similarly we obtain (3) $\psi_b(x) \leq \psi_b(a)\psi_a(x)$ and (4) $\varphi_b(x) \leq \varphi_b(a)\psi_a(x)$. Substituting b for x in (2) and (4), we have $\psi_b(a)\varphi_a(b) \leq 1$ and $\varphi_b(a)\psi_a(b) \geq 1$. But interchanging a and b in these inequalities, we see more precisely that $\varphi_b(a)\psi_a(b) = \psi_b(a)\varphi_a(b) = 1$. Therefore from (1) and (3) it follows that $\varphi_a(x) \leq \psi_a(b)\varphi_b(x)$ and $\varphi_a(b)\psi_b(x) \leq \psi_a(x)$. Interchanging a and b in these inequalities, we obtain (5) $\varphi_b(x) \leq \psi_b(a)\varphi_a(x)$ and (6) $\varphi_b(a)\psi_a(x) \leq \psi_b(x)$. From (1), (2), (3), (4),

(5) and (6) we complete the proof.

The inequality in Proposition 4 will play an important role in this paper as well as the inequality (4) in Proposition 2 in §2.

COROLLARY 1. *For any a, $b \in G$ we have*
(1) $\varphi_a(b)\psi_b(a)=\psi_a(b)\varphi_b(a)=1$,
(2) $\varphi_a(b)\varphi_b(a)\leq 1\leq\psi_a(b)\psi_b(a)$.

COROLLARY 2. *For any a, $x\in G$ and for any positive integer n we have*

$$\varphi_{na}(x)=\frac{\varphi_a(x)}{n},\quad \psi_{na}(x)=\frac{\psi_a(x)}{n}.$$

PROOF. By (1) of Corollary 1 we see

$$\varphi_{na}(x)=\frac{1}{\psi_x(na)}=\frac{1}{n\psi_x(a)}=\frac{\varphi_a(x)}{n}.$$

The rest is similarly obtained.

§4. Almost power-joined N-semigroups.

In this section we give some necessary and sufficient conditions under which φ_a and ψ_a are homomorphisms on G and introduce a concept of almost power-joined N-semigroups.

THEOREM 2. *Let G be an N-semigroup and $a\in G$. Then the following conditions are equivalent:*
(1) $\varphi_a(x)=\psi_a(x)$ *for all* $x\in G$,
(2) φ_a *is a homomorphism on G into* $\boldsymbol{R}_+$,
(3) ψ_a *is a homomorphism on G into* $\boldsymbol{R}_+$.

PROOF. (1)$\rightarrow$(2) and (3): If (1) is valid, both sides of the inequality in Proposition 2 coincide, so (2) and (3) result from this.

(2) or (3) $\rightarrow$ (1): For any $x\in G$, $na=x+c$ for some $c\in G$ and some positive integer n. Hence $\varphi_a(x+c)=\psi_a(x+c)=n$. If (2) is valid, $\varphi_a(x+c)=\varphi_a(x)+\varphi_a(c)$. On the other hand we have $\varphi_a(x)+\varphi_a(c)\leq\psi_a(x)+\varphi_a(c)\leq\psi_a(x+c)$. From these we obtain $\varphi_a(x)=\psi_a(x)$. We can induce (1) from (3) similarly.

COROLLARY. *If φ_a (or ψ_a) is a homomorphism on G for some $a\in G$, then*

φ_b and ψ_b are also homomorphism on G for all $b \in G$ and we have

$$\varphi_b = \psi_b = \varphi_b(a)\varphi_a.$$

PROOF. From Proposition 4 we have

$$\varphi_b(a)\varphi_a \leq \varphi_b \leq \varphi_b(a)\psi_a.$$

If φ_a is a homomorphism, then $\varphi_a = \psi_a$ by Theorem 2. Then both sides of the inequality above coincide, so $\varphi_b = \varphi_b(a)\varphi_a$. Hence φ_b is also a homomorphism and $\varphi_b = \psi_b$.

Let G be an N-semigroup and let $a, x, y \in G$. We say that x and y are almost power-joined if for any positive real number ε there exist elements $c, d \in G$ and positive integers m, n such that

$$mx + c = ny + d, \quad L_a(c) \leq n\varepsilon, \quad L_a(d) \leq m\varepsilon.$$

This definition does not depend on the base element a. In fact, for any element $b \in G$ we see by Proposition 3 that

$$L_b(c) \leq L_b(a)L_a(c) \leq nL_b(a)\varepsilon$$

and

$$L_b(d) \leq L_b(a)L_a(d) \leq mL_b(a)\varepsilon.$$

An N-semigroup whose every two elements are almost power-joined is called almost power-joined.

LEMMA 2. *Let G be an N-semigroup and let $a, x, y \in G$. Then the following conditions are equivalent:*

(1) *x and y are almost power-joined,*

(2) *for any positive real number ε there exist $c \in G$ and positive integers m, n such that*

$$mx = ny + c, \quad L_a(c) \leq m\varepsilon,$$

(3) $\varphi_y(x) = \psi_y(x)$.

PROOF. (1)→(3): Since x and y are almost power-joined, for any positive integer l there exist $c_l, d_l \in G$ and positive integers m_l, n_l such that

$$m_l x + c_l = n_l y + d_l,\ L_y(c_l) \leq n_l/l,\ L_y(d_l) \leq m_l/l.$$

Then by Proposition 1,

$$N_y(lm_l x)+L_y(lc_l)\geq N_y(lm_l x+lc_l)\geq ln_l.$$

Since $L_y(lc_l)\leq n_l$, we see $N_y(lm_l x)\geq n_l(l-1)$. Hence

$$\frac{N_y(lm_l x)}{lm_l}\cdot\frac{l}{l-1}\geq\frac{n_l}{m_l}.$$

On the other hand we see

$$L_y(lm_l x)\leq L_y(ln_l y+ld_l)\leq ln_l+lL_y(d_l)\leq ln_l+m_l.$$

Then

$$\frac{L_y(lm_l x)}{lm_l}-\frac{1}{l}\leq\frac{n_l}{m_l}.$$

Since $\frac{N_y(lm_l x)}{lm_l}\to\varphi_y(x)$, $\frac{L_y(lm_l x)}{lm_l}\to\psi_y(x)$, $\frac{l}{l-1}\to 1$, $\frac{1}{l}\to 0$ as $l\to\infty$ and since $\varphi_y(x)\leq\psi_y(x)$, we have

$$\varphi_y(x)=\psi_y(x)=\lim_{l\to\infty}\frac{n_l}{m_l}.$$

(3)→(2): For any positive integer m we have $mx=N_y(mx)y+c_m$ and $L_y(mx)y=mx+d_m$ for some c_m, $d_m\in G$, hence $c_m+d_m=(L_y(mx)-N_y(mx))y$. Then we have

$$L_y(c_m)\leq m\left(\frac{L_y(mx)}{m}-\frac{N_y(mx)}{m}\right).$$

By the assumption we see

$$\frac{L_y(mx)}{m}-\frac{N_y(mx)}{m}\to\psi_y(x)-\varphi_y(x)=0 \text{ as } m\to\infty.$$

Therefore for any positive real number ε there exists a positive integer m such that $L_y(c_m)\leq\frac{m\varepsilon}{L_a(y)}$, hence

$$L_a(c_m)\leq L_a(y)L_y(c_m)\leq m\varepsilon.$$

Thus for any $\varepsilon>0$ we have positive integers m, $n=N_y(mx)$ and $c=c_m\in G$ such that $mx=ny+c$ and $L_a(c)\leq m\varepsilon$.

(2)→(1): Clear.

From Theorem 2, Lemma 2 above and what we mentioned in Introduction, we obtain the following theorem.

THEOREM 3. *An N-semigroup G is almost power-joined if and only if*

φ_a(or ψ_a) is a homomorphism on G for all (equivalently for some) $a \in G$. Therefore an almost power-joined N-semigroup is isomorphic onto a subdirect sum of a positive real semigroup and an abelian group.

Corollary (Sasaki *and* Tamura). *Let G be a power-joined N-semigroup. Then $\varphi_a(G)=\psi_a(G)\subset \boldsymbol{Q}_+$ for all $a\in G$, therefore G is isomorphic onto a subdirect sum of a positive rational semigroup and an abelian group.*

Proof. For any $x\in G$ there exist positive integers m,n such that $mx=na$, so we obtain

$$\varphi_a(x)=\frac{n}{m}\in \boldsymbol{Q}_+.$$

Remark. An N-semigroup G is almost power-joined iff there exist an element $a\in G$ and a positive number L satisfying the following property.

For every $x, y\in G$ there exist positive integers m, n and $c\in G$ such that

$$mx=ny+c, \quad L_a(c)\leq mL.$$

Theorem 4. *Let a be an element of an N-semigroup G and let f be a homomorphism on G into $\boldsymbol{R}_+$. Then we have*

$$f(a)\varphi_a\leq f\leq f(a)\psi_a.$$

Proof. For a positive integer n and $x\in G$ we have

$$nx=N_a(nx)a+c_n, \quad L_a(nx)a=nx+d_n, \quad c_n,\ d_n\in G.$$

Since f is a homomorphism, we have

$$nf(x)\geq N_a(nx)f(a), \quad L_a(nx)f(a)\geq nf(x).$$

Then

$$f(a)\,\frac{N_a(nx)}{n}\leq f(x)\leq f(a)\,\frac{L_a(nx)}{n}.$$

This gives

$$f(a)\varphi_a(x)\leq f(x)\leq f(a)\psi_a(x).$$

The following immediate consequence of Theorem 4 is a generalization of Theorem 3 in [11].

Corollary. *In the same situation as in Theorem 4, suppose that G is almost power-joined. Then we have*

$$f=f(a)\varphi_a=f(a)\psi_a.$$

Therefore we have an isomorphism

$$\mathrm{Hom}(G, \boldsymbol{R}_+)\simeq\boldsymbol{R}_+.$$

The converse of the corollary above is also true. The proof will be given in §6.

EXAMPLE 1. A subset of $\boldsymbol{R}_+$ in the form $\{x\in\boldsymbol{R}_+ \mid x\geq r\}$ is called a segment. A positive real semigroup containing a segment is an almost power-joined N-semigroup.

EXAMPLE 2. Let $\boldsymbol{C}_1$ be a multiplicative semigroup of complex numbers whose absolute values are greater than 1. Then $\boldsymbol{C}_1$ is an almost power-joined N-semigroup and we have

$$\varphi_a(x)=\psi_a(x)=\log_{|a|}(|x|) \qquad \text{for } a,\ x\in\boldsymbol{C}_1.$$

Moreover there is an isomorphism

$$\boldsymbol{C}_1\simeq\boldsymbol{R}_+\oplus\frac{\boldsymbol{R}}{\boldsymbol{Z}},$$

where $\dfrac{\boldsymbol{R}}{\boldsymbol{Z}}$ is a quotient group of the additive group $\boldsymbol{R}$ modulo $\boldsymbol{Z}$.

§5. The decomposition of an N-semigroup into a disjoint union of almost power-joined N-semigroups.

Let G be an N-semigroup. For x, $y\in G$ we write $x\sim y$ if x and y are almost power-joined. It is easily checked up that this relation $\sim$ is an equivalence relation on G. The equivalence class of x is denoted by $G(x)$ and we call it the almost power-joined component containing x.

LEMMA 3. *Assume* $x\in G(a)$, *then for any* $y\in G$ *we have*

$$\varphi_a(x+y)=\varphi_a(x)+\varphi_a(y),\quad \psi_a(x+y)=\psi_a(x)+\psi_a(y).$$

PROOF. By (4) in Proposition 2 we have

$$\varphi_a(x)+\varphi_a(y)\leq\varphi_a(x+y)\leq\psi_a(x)+\varphi_a(y).$$

Since $x\sim a$, we have $\varphi_a(x)=\psi_a(x)$, so both sides of the inequality above coincide and we get the first equality. The second one is similarly obtained.

Lemma 3 tells us the following.

COROLLARY. *If two of* x, y *and* $x+y$ *are contained in* $G(a)$, *then the other is also contained in* $G(a)$.

THEOREM 5. *Let* G *be an* N*-semigroup and let* $a\in G$, *then* $G(a)$ *is an almost power-joined* N*-semigroup. Therefore any* N*-semigroup is decomposed into a disjoint union of almost power-joined* N*-semigroups.*

PROOF. $G(a)$ is a subsemigroup of G by the preceding corollary. Now let x, $y\in G(a)$. Since G is archimedean, there are $b\in G$ and a positive integer m such that $mx=y+b$. Since we see that $b\in G(a)$ from the same corollary, $G(a)$ is itself archimedean, so it is an N-semigroup.

Next let $x\in G(a)$ and $x=N_a(x)a+c$ for $c\in G$. Again we see that $c\in G(a)$, this implies that $N_{a,G(a)}(x)=N_a(x)$, where $N_{a,G(a)}$ is the function N defined on $G(a)$ (not on G) on the base element a. Thus $N_{a,G(a)}=N_a|_{G(a)}$ and consequently $\varphi_{a,G(a)}=\varphi_a|_{G(a)}$, where $N_a|_{G(a)}$ and $\varphi_a|_{G(a)}$ are the restrictions of N_a and φ_a to $G(a)$ respectively. Similarly $L_{a,G(a)}=L_a|_{G(a)}$ and $\psi_{a,G(a)}=\psi_a|_{G(a)}$. But we know that $\varphi_a=\psi_a$ on $G(a)$ by Lemma 2, hence $\varphi_{a,G(a)}=\psi_{a,G(a)}$. Therefore $G(a)$ is almost power-joined and the the proof is completed.

COROLLARY. *With the same notations as in the proof of Theorem* 5, *we have*

$$N_{a,G(a)}=N_a|_{G(a)},\quad L_{a,G(a)}=L_a|_{G(a)},$$

$$\varphi_{a,G(a)}=\varphi_a|_{G(a)}=\psi_{a,G(a)}=\psi_a|_{G(a)}.$$

THEOREM 6. *Let* a, $b\in G$. *Then we have*

(1) $\varphi_b|_{G(a)}=\varphi_b(a)\varphi_{a,G(a)}$, $\psi_b|_{G(a)}=\psi_b(a)\psi_{a,G(a)}$,

(2) *if* $b\in G(a)$, *then* $\varphi_b=r\varphi_a$, $\psi_b=r\psi_a$, *where* $r=\varphi_b(a)=\psi_b(a)$.

PROOF. From Proposition 4 we have

$$\varphi_b(a)\varphi_a(x)\leq\varphi_b(x)\leq\min\{\varphi_b(a)\psi_a(x),\ \psi_b(a)\varphi_a(x)\}.$$

If $x\in G(a)$, then $\varphi_a(x)=\psi_a(x)$. Hence both sides of the inequality above coincide, so $\varphi_b(a)\varphi_a(x)=\varphi_b(x)$ and the half of (1) is proved. Next, if $b\in G(a)$, then $\varphi_b(a)=\psi_b(a)$. Then similarly we see $\varphi_b(a)\varphi_a(x)=\varphi_b(x)$ for $x\in G$. Thus the half of (2) is proved. The rest of the theorem is similarly proved.

COROLLARY. *Let $a, b \in G$. Then φ_b and ψ_b are homomorphisms on $G(a)$ and φ_b/ψ_b is a constant $\varphi_b(a)/\psi_b(a)$ on $G(a)$.*

By Corollary of Theorem 4 we have

THEOREM 7. *Let f be a homomorphism on G into $\boldsymbol{R}_+$. Then $f=f(a)\varphi_a=f(a)\psi_a$ on $G(a)$ for all $a \in G$.*

REMARK. If an N-semigroup G is not almost power-joined, G has an infinite number of almost power-joined components. In fact, assume that $x, y \in G$ and $x \not\sim y$, then it is proved that $x+my \not\sim x+ny$ for every positive integers m, n $(m \neq n)$.

§6. Existence of homomorphisms on N-semigroups into R_+.

As we mentioned in Introduction, $\mathrm{Hom}(G, \boldsymbol{R}_+)$ is always non-empty for any N-semigroup G and from this it follows that any N-semigroup is isomorphic onto a subdirect sum of a positive real semigroup and an abelian group. The most efficient tool to prove this may be Ross' extention theorem of semicharacters given in [9]. The direct application of Ross' theorem yields us the following theorem. Here we give an outline of a direct proof of the theorem modifying the proof of Ross a little.

Let H be a subsemigroup of an N-semigroup G and let f be a homomorphism on H into $\boldsymbol{R}_+$. For a pair (H, f) we consider the following condition (♯):

(♯) $f(x) \geq f(y)$ for all $x, y \in H$ such that $y \mid x$(*) in G.

THEOREM 8. *Let H_0 be a subsemigroup of an N-semigroup G and let f_0 be a homomorphism on H_0 into $\boldsymbol{R}_+$. Then f_0 is extensible to a homomorphism on G if and only if the pair (H_0, f_0) satisfies the condition (♯).*

PROOF. The necessity of the condition is clear. Now we shall prove the sufficiency. Using Zorn's lemma we see that there exist a maximal subsemigroup H and a homomorphism f on H into $\boldsymbol{R}_+$ such that $f|_{H_0}=f_0$ and (H, f) satisfies (♯). We define a subsemigroup $\bar{H}$ of G by

$$\bar{H}=\{x \in G \mid ((x)+H) \cap H \neq \phi\},$$

where (x) denotes a subsemigroup of G generated by x. Next we define a homomorphism $\bar{f}$ on $\bar{H}$ into $\boldsymbol{R}_+$ by

(*) For two elements $x, y \in G$ we write $y \mid x$ if $x=y+z$ for some $z \in G$. Notice that $x \nmid x$.

$$\bar{f}(x)=\frac{f(nx+h)-f(h)}{n} \quad \text{with } h,\ nx+h\in H.$$

It may be routine to check it up that $\bar{f}$ is well-defined, $\bar{f}|_H=f$ and $(\bar{H},\bar{f})$ satisfies ($\sharp$), so we conclude that $\bar{H}=H$ by the maximality of H. Now we define a function $\tilde{f}$ on G into $\boldsymbol{R}_+$ by

$$\tilde{f}(x)=\inf\left\{\frac{f(nx+z)}{n}: n>0,\ z\in G,\ nx+z\in H\right\}.$$

We can prove that (1) $\tilde{f}|_H=f$, (2) $\tilde{f}(x+y)\geq\tilde{f}(x)$, (3) $\tilde{f}(mx)=m\tilde{f}(x)$ and (4) $\tilde{f}(x)+\tilde{f}(y)\geq\tilde{f}(x+y)$, for any $x,\ y\in G$.

Now assume that $H\neq G$. Let $x\in G-H$ and let $mx+h$ be any element of $(x)+H$. If $n(mx+h)+z\in H$ for $n>0$ and $z\in G$, then $nmx+z\in H$ since $\bar{H}=H$. Hence by the definition of $\tilde{f}$ we see that $\tilde{f}(mx+h)\geq m\tilde{f}(x)+\tilde{f}(h)$. Combining this with (4) above, $\tilde{f}$ is a homomorphism on $((x)+H)\cup H$, this contradicts to the maximality of H, so we must have $H=G$.

Corollary 1. *The homomorphism $\varphi_a|_{G(a)}=\psi_a|_{G(a)}$ on the subsemigroup $G(a)$ satisfies the condition ($\sharp$). Therefore* $\mathrm{Hom}(G, \boldsymbol{R}_+)\neq\phi$ *and G is isomorphic onto a subdirect sum of a positive real N-semigroup and an abelian group.*

The following is the converse of Corollary of Theorem 4 in §4.

Corollary 2. *If G is not almost power-joined, there exist two homomorphisms f and g on G into $\boldsymbol{R}_+$ such that the function f/g is not constant on G.*

Proof. Let $x\in G-G(a)$, then $\varphi_a(x)\neq\psi_a(x)$. By Lemma 3, φ_a and ψ_a are homomorphisms on $G'=(G(a)+(x))\cup G(a)$. Extend $\varphi_a|_{G'}$ and $\psi_a|_{G'}$ to homomorphisms f and g on G respectively. Then $f(a)/g(a)=1$ and $f(x)/g(x)\neq 1$.

§7. The $\boldsymbol{R}$-vector space associated with an N-semigroup.

Let G be an N-semigroup. We introduce the following notations:

$\boldsymbol{R}$: the field of real numbers,

$\boldsymbol{R}_0$: the additive semigroup of all non-negative real numbers,

$H_+(G)=\mathrm{Hom}(G, \boldsymbol{R}_+)$,

$H_0(G)=\mathrm{Hom}(G, \boldsymbol{R}_0)$,

$H(G)$: the $\boldsymbol{R}$-vector subspace of $\mathrm{Hom}(G, \boldsymbol{R})$ generated by $H_+(G)$.

$H(G)$ is called the $\boldsymbol{R}$-vector space associated with G. A base of $H(G)$ contained in $H_+(G)$ is called a base of $H_+(G)$. The dimension of $H(G)$ over $\boldsymbol{R}$ is called the dimension of G, i.e. $\dim G = \dim_{\boldsymbol{R}} H(G)$.

PROPOSITION 5. *If $f \in H_0(G)$ and $f \neq 0$, then $f \in H_+(G)$, that is,*

$$H_0(G) - \{0\} = H_+(G).$$

PROOF. Assume $f(x_0) = 0$ for some $x_0 \in G$. For any $x \in G$ there exists a positive integer n such that $x \mid n x_0$. Therefore $0 = f(n x_0) \geq f(x)$, hence $f(x) = 0$.

We can express the results of Corollary of Theorem 4 and Corollaries of Theorem 8 that $\dim G \geq 1$ *for any N-semigroup G and* $\dim G = 1$ *if and only if G is almost power-joined.*

THEOREM 9. *Let G_1 and G_2 be N-semigroups, then*

$$H_0(G_1 \oplus G_2) \simeq H_0(G_1) \oplus H_0(G_2).$$

Therefore, $\dim G_1 \oplus G_2 = \dim G_1 + \dim G_2$.

PROOF. Let $G = G_1 \oplus G_2$ and $f \in H_0(G)$. Fix $x^0 = (x_1^0, x_2^0) \in G$ and set

$$f_1(x_1, n) = \frac{1}{n} f(x_1^0 + n x_1, x_2^0), \ f_2(x_2, n) = \frac{1}{n} f(x_1^0, x_2^0 + n x_2)$$

for $n > 0$ and $x = (x_1, x_2) \in G$. Then $f_i(x_i, n)$ $(i = 1, 2)$ are monotone decreasing on n and

$$f_1(x_1, n) + f_2(x_2, n) = f(x) + \frac{2}{n} f(x^0).$$

Hence

$$f_1(x_1) + f_2(x_2) = f(x), \text{ where } f_i(x_i) = \lim_{n \to \infty} f_i(x_i, n) \quad (i = 1, 2).$$

It is easy to see that $f_i \in H_0(G_i)$.

Conversely, it is clear that f_1 and f_2 are linearly independent in $H(G)$ for each $f_i \in H_+(G_i)$. Thus we obtain $H_0(G) \simeq H_0(G_1) \oplus H_0(G_2)$.

COROLLARY. *Let $G = G_1 \oplus G_2 \oplus \cdots \oplus G_d$, where G_i is an almost power-joined N-semigroup for each i. Then*

$$H_0(G) \simeq \underbrace{\boldsymbol{R}_0 \oplus \boldsymbol{R}_0 \oplus \cdots \oplus \boldsymbol{R}_0}_{d}.$$

Therefore, $\dim G = d$.

The application of Petrich's theorem proved in [8] would give us another proof of Theorem 9.

The following will be used in the next section.

LEMMA 4. *Let* $x, y \in G$. *If* $x \nsim y$, *then there exist* f, $g \in H_+(G)$ *and positive integers* m, n *such that* $mf(x)=nf(y)$ *and* $g(x) \neq g(y)$. *If* $x \sim y$, *then either* $f(x)=f(y)$ *for all* $f \in H_+(G)$ *or* $f(x) \neq f(y)$ *for all* $f \in H_+(G)$.

PROOF. Assume $x \nsim y$. Then in the same way as in the proof of Corollary 2 of Theorem 8, there exist g_1, $g_2 \in H_+(G)$ such that $g_1(x)=g_2(x)=1$ and $g_1(y)<g_2(y)$. Hence either $g_1(x) \neq g_1(y)$ or $g_2(x) \neq g_2(y)$. Next let m, n be positive integers such that $g_1(y)<m/n<g_2(y)$. Then there are positive real numbers r_1, r_2 such that

$$r_1 g_1(y)+r_2 g_2(y)=\frac{m}{n}, \quad r_1+r_2=1.$$

Put $f=r_1 g_1+r_2 g_2$, then $nf(y)=m$ and $mf(x)=m(r_1+r_2)=m$. Thus we obtain the proof of the half.

If $x \sim y$, then x, $y \in G(a)$ for some $a \in G$. Let $f \in H_+(G)$. Theorem 7 shows that $f=r\varphi_a$ on $G(a)$ for some $r \in \boldsymbol{R}_+$. This proves the latter half of the lemma.

COROLLARY. *If* $f(x)=f(y)$ *for all* $f \in H_+(G)$, *then* $x \sim y$.

§8. Affine N-semigroups.

We define an order $>$ on an N-semigroup G as follows:

$$x > y \text{ iff } ny \mid nx \text{ for some positive integer } n.$$

An N-semigroup which is linearly ordered by this order is called linear.

PROPOSITION 6. *A linear N-semigroup is almost power-joined.*

PROOF. Let G be a linear N-semigroup and assume that there are two elements x, $y \in G$ such that $x \nsim y$. By Lemma 4 there exist $f \in H_+(G)$ and positive integers m, n such that $f(mx)=f(ny)$. This implies $mx=ny$ because G is linear, hence $x \sim y$ (contradiction).

An N-semigroup is called affine if all its almost power-joined components are linear.

PROPOSITION 7. *Let G be an affine N-semigroup and let* x, $y \in G$. *If*

$f(x)=f(y)$ for all $f\in H_+(G)$, then $x=y$.

PROOF. From Corollary of Lemma 4 we see $x\sim y$. Since G is affine, one of the statements $x>y$, $x=y$, $x<y$ holds. But $f(x)=f(y)$ for $f\in H_+(G)$, then we must have $x=y$.

A linear N-semigroup was introduced by Austin in [2] (he calls it a linear semigroup). Tamura's irreducible N-semigroup is also the same concept ([14], [15]). They proved that a linear N-semigroup is embedded in $\boldsymbol{R}_+$, which is a variation of the classical embedding theorem by Hölder and others. We give here more generally an embedding theorem of affine N-semigroups.

To begin with we need the following.

PROPOSITION 8. *Let G be an almost power-joined positive real semigroup and let a, $x\in G$. Then*

$$\varphi_a(x)=\psi_a(x)=x/a.$$

PROOF. Clearly $f_a: x\longmapsto x/a$ is a homomorphism on G into $\boldsymbol{R}_+$. Since G is almost power-joined, $\varphi_a=\psi_a=rf_a$ for some $r\in\boldsymbol{R}_+$. But $r=rf_a(a)=\varphi_a(a)=1$, hence $\varphi_a=\psi_a=f_a$.

THEOREM 10. *An N-semigroup is affine if and only if it is embedded in $\prod\boldsymbol{R}_+$. An N-semigroup with dimension d is affine if and only if it is embedded in $\underbrace{\boldsymbol{R}_+\oplus\boldsymbol{R}_+\oplus\cdots\oplus\boldsymbol{R}_+}_{d}$. An almost power-joined N-semigroup is linear if and only if it is embedded in $\boldsymbol{R}_+$.*

PROOF. Let G be an N-semigroup and let (f_α) be a base of $H_+(G)$. Define a homomorphism $\eta:G\to\prod_\alpha\boldsymbol{R}_+$ by $\eta(x)=(f_\alpha(x))\in\prod_\alpha\boldsymbol{R}_+$. If G is affine, $\eta(x)=\eta(y)$ implies $x=y$ from Proposition 7, hence η is injective. Thus the only part of the theorem is proved.

Now assume $G\subset\prod_\alpha\boldsymbol{R}_+$ and let $a\in G$. It is sufficient to prove that $G(a)$ is linear. Let $p_{\alpha_0}:G(a)\to\boldsymbol{R}_+$ be one of the projections. Since $G(a)$ is almost power-joined, $p_\alpha=r_\alpha p_{\alpha_0}$, $r_\alpha\in\boldsymbol{R}_+$ for all α. Therefore if $p_{\alpha_0}(x)=p_{\alpha_0}(y)$ for x, $y\in G(a)$, then $p_\alpha(x)=p_\alpha(y)$ for all α, so $x=y$, this implies that p_{α_0} is injective. Then we may assume that $G(a)\subset\boldsymbol{R}_+$. Let x, $y\in G(a)$ and $x>y$ (where $>$ denotes the ordinal order in $\boldsymbol{R}_+$). Since $x\sim y$, for any positive number ε we have $m_\varepsilon x=n_\varepsilon y+c_\varepsilon$, $L_a(c_\varepsilon)\leq m_\varepsilon\varepsilon$ for some $c_\varepsilon\in G(a)$ and some positive integers m_ε, n_ε. From Proposition 8 and the proof of Lemma 2 we see

$$\frac{x}{y}=\varphi_y(x)=\lim_{\varepsilon\to 0}\frac{n_\varepsilon}{m_\varepsilon}.$$

Then $n_\varepsilon > m_\varepsilon$ for sufficiently small ε. Hence

$$m_\varepsilon x = m_\varepsilon y + z,$$

where $z=(n_\varepsilon - m_\varepsilon)y+c$. This implies that $x \succ y$ and hence $G(a)$ is linear.

Corollary. *Let G be a positive real N-semigroup. Then the following conditions are equivalent*:

(1) *G is almost power-joined*,

(2) *G is linear*,

(3) *for any $x, y \in G$ such that $x > y$, there exists a positive integer n such that $n(x-y)\in G$.*

Moreover, if these conditions are satisfied, the order $\succ$ on G coincides with the order induced from the ordinal order of $\boldsymbol{R}$.

Let (f_α) and (f'_β) be two bases of $H_+(G)$ and let η (resp. η'): $G\to\prod\boldsymbol{R}_+$ be a homomorphism defined by $\eta(x)=(f_\alpha(x))$ (resp. $\eta'(x)=(f'_\beta(x))$). Then there exists an isomorphism $h: \eta(G)\simeq\eta'(G)$ such that the following diagram commutes:

(D.1)
$$\begin{array}{ccc} G & \xrightarrow{\ \eta\ } & \eta(G) \\ & {}_{\eta'}\searrow & \downarrow\wr\, h \\ & & \eta(G). \end{array}$$

Thus $\eta(G)$ is uniquely determined up to an isomorphism and we call it the affine part of G and it is denoted by $A(G)$. $A(G)$ has the following universal property.

Proposition 9. *Let G and G' be N-semigroups and let $g: G\to G'$ be a homomorphism. Let $\eta: G\to A(G)$ be the surjection defined by a base (f_α) of $H_+(G)$. If G' is affine, there exists a unique homomorphism $\bar{g}: A(G)\to G'$ such that the following diagram commutes*:

(D.2)
$$\begin{array}{ccc} G & \xrightarrow{\ \eta\ } & A(G) \\ & {}_{g}\searrow & \downarrow \bar{g} \\ & & G'. \end{array}$$

Proof. We define a homomorphism $\bar{g}: A(G)\to G'$ by $\bar{g}(\eta(x))=g(x)$ for $x\in G$. We must prove that $\bar{g}$ is well-defined. Assume that $\eta(x)=\eta(y)$ for

$x, y \in G$, that is, $f(x)=f(y)$ for every $f \in H_+(G)$. Then $f'(g(x))=f'(g(y))$ for every $f' \in H_+(G')$. Since G' is affine, we have $g(x)=g(y)$ by Proposition 7, this proves that $\bar{g}$ is well-defined. It is clear that $\bar{g}$ is a homomorphism and uniquely determined.

Let G and G' be N-semigroups and let $g: G \to G'$ be a homomorphism. Let $\eta: G \to A(G)$ and $\eta': G' \to A(G')$ be the surjections. Then by the proposition above there exists a unique homomorphism $\bar{g}: A(G) \to A(G')$ such that the following diagram commutes:

(D.3)
$$\begin{array}{ccc} G & \xrightarrow{g} & G' \\ \eta\downarrow & & \eta'\downarrow \\ A(G) & \xrightarrow{\bar{g}} & A(G'). \end{array}$$

Thus the map $A: G \longmapsto A(G)$ is a covariant functor on the category of N-semigroups to the category of affine N-semigroups.

On the other hand g induces an $\boldsymbol{R}$-homomorphism $g^*: H(G') \to H(G)$ which is defined by $g^*(f')(x)=f'(g(x))$ for $x \in G$ and $f' \in H(G')$. Thus the map $H: G \longmapsto H(G)$ is a contravariant functor on the category of N-semigroups to the category of $\boldsymbol{R}$-vector spaces.

PROPOSITION 10. *In the same situation as above, the homomorphisms* $\eta^*: H(A(G)) \to H(G)$ *and* $\eta'^*: H(A(G')) \to H(G')$ *are isomorphisms and the following diagram commutes:*

(D.4)
$$\begin{array}{ccc} H(A(G')) & \xrightarrow{\bar{g}^*} & H(A(G)) \\ \eta^* \downarrow\wr & & \eta'^* \downarrow\wr \\ H(G') & \xrightarrow{g^*} & H(G). \end{array}$$

Therefore we have an isomorphism of the functors: $H \circ A \simeq H$.

PROOF. From Proposition 9 we see that for any $f \in H_+(G)$ there is a unique $\bar{f} \in H_+(A(G))$ such that $f=\eta \circ \bar{f}=\eta^*(\bar{f})$, this implies that η^* is one-to-one. The commutativity of the diagram (D.4) follows from the commutative diagram (D.3).

The following is simple but important.

PROPOSITION 11. *Let* $g: G \to G'$ *be a surjective homomorphism of N-semigroups. Then* $g^*: H(G') \to H(G)$ *is injective. Therefore* $\dim G \geq \dim G'$.

COROLLARY 1. *On the same assumption as above, if G is almost power-joined, G' is also almost power-joined.*

Corollary 2. *Let $g: G \to G'$ be a homomorphism of N-semigroups and let $x, y \in G$. If $x \sim y$, then $g(x) \sim g(y)$.*

Let $g: G \to G'$ be a homomorphism of N-semigroups. We say g is degenerate if there exist $x, y \in G$ such that $x \nsim y$ and $g(x) \sim g(y)$. Otherwise we say g is non-degenerate.

Proposition 12. *Let $g: G \to G'$ and $g': G' \to G''$ be homomorphisms of N-semigroups. Assume that g is surjective. Then $g' \circ g$ is non-degenerate if and only if g and g' are non-degenerate.*

Theorem 11. *Let $g: G \to G'$ be a surjective homomorphism of N-semigroups. Then the following conditions are equivalent:*

(1) *g is non-degenerate,*

(2) *$\bar{g}: A(G) \to A(G')$ is an isomorphism,*

(3) *$g^*: H(G') \to H(G)$ is an isomorphism.*

Proof. (3)→(1): Assume that g is degenerate, i.e. there exist $x, y \in G$ such that $x \nsim y$ and $g(x) \sim g(y)$. Then from Lemma 4 we can find $f_0 \in H_+(G)$ and positive integers m, n such that $mf_0(x) = nf_0(y)$. Since g^* is an isomorphism, there exists $f'_0 \in H_+(G')$ such that $f_0 = f'_0 \circ g$, hence $f'_0(m\,g(x)) = f'_0(n\,g(y))$. Since $m\,g(x) \sim n\,g(y)$, it follows from Lemma 4 that $f'(m\,g(x)) = f'(n\,g(y))$ for all $f' \in H(G')$. But applying Lemma 4 again, we see that $h_0(mx) \neq h_0(ny)$ for some $h_0 \in H_+(G)$. Then h_0 cannot be induced from an element of $H(G')$, hence g^* is not surjective, this contradicts to (3).

(1)→(2): Let $\eta: G \to A(G)$ and $\eta': G' \to A(G')$ be the surjections. By Proposition 10, η^* and η'^* are isomorphisms, hence η and η' are non-degenerate as we have just proved. Since $\eta' \circ g = \bar{g} \circ \eta$ and g is non-degenerate, it follows from Proposition 12 that $\bar{g}$ is also non-degenerate. Let $x, y \in A(G)$ and assume $\bar{g}(x) = \bar{g}(y)$. Then $x \sim y$ because $\bar{g}$ is non-degenerate. Therefore $f(x) = f(y)$ for all $f \in H_+(A(G))$ by Lemma 4, hence $x = y$ by Proposition 7. Thus $\bar{g}$ is injective. Moreover $\bar{g}$ is surjective since g is so.

(2)→(3): Clear from the isomorphism $H \circ A \cong H$.

Corollary 1. *The surjection $\eta: G \to A(G)$ is non-degenerate.*

Corollary 2. *Let $g: G \to G'$ be a non-degenerate homomorphism. If G is affine, then g is injective.*

COROLLARY 3. *Let $g: G \rightarrow G'$ be a surjective homomorphism and assume G is finite dimensional. Then g is non-degenerate if and only if* $\dim G = \dim G'$.

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