J. Math. Tokushima Univ.
Vol. 6. (1972), 31-35

Direct Limits of Finitary Relation Spaces

By

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(Received May 25, 1972)

In [1] the direct limit of a sequence of semigroups is defined as follows: Consider a sequence $\{D_i: i=1, 2, \cdots\}$ of semigroups with isomorphisms φ_{ji} of D_i into D_j, $i\leqq j$, such that for $i\leqq j\leqq k$, $\varphi_{ki}(x)=\varphi_{kj}\cdot\varphi_{ji}(x)$ and $\varphi_{ii}(x)=x$. The semigroup D of the set union $\bigcup_{i=1}^{\infty} D_i$ which, for every i, j and x such that $i\leqq i$ and $x\in D_i$, identifies x with $\varphi_{ji}(x)$ is called the direct limit of $\{D_i: i=1, 2, \cdots\}$ with respect to the isomorphism family $F=\{\varphi_{ji}: i=1, 2, \cdots, j=1, 2, \cdots; i\leqq j\}$ and is denoted by $D=\varinjlim (D_i; F)$.

Further in [1] the problem is asked to describe the isomorphism condition for S and S' in terms of S_i, S_i', φ_{ji}, φ'_{ji}, if S_i and S_i' are positive integer semigroups and $S=\varinjlim (S_i; \varphi_{ji})$, $S'=\varinjlim (S_i'; \varphi'_{ji})$.

In this paper this concrete problem is not solved, but some conditions are given for a more general case.

The definition of the direct limit of the sequence of semigroups given in [1] can be generalized to the sequence of finitary relation spaces, if the word "semigroup" in it is substituted by the words "finitary relation space". The relation space is a set on which some relations are given. If all of those relations are finitary, this relation space is called finitary.

We may adapt this definition by such a way that S is defined as such a set that for any i an isomorphism φ_i of S_i into S exists and any element of S is an image of some element of S_i in φ_i for some i and $\varphi_j\varphi_{ji}=\varphi_i$ for $j\geqq i$.

Now we shall prove a lemma.

LEMMA. *Let M and M' be two finitary relation spaces. Let α_i for each positive integer i be an isomorphism from M into M'. For $i\leqq j$ let the definition domain of α_i be included in the definition domain of α_j. Let $\alpha_i(x)=\alpha_j(x)$ for any x from the intersection of definition domains of α_i and α_j. Let any element of M be in the definition domain of some α_i. Then there exists an isomorphism of M into M'.*

PROOF. Let us define the mapping α so that $\alpha(x)=\alpha_i(x)$ for such an i that x is in the definition domain of α_i. The element $\alpha(x)$ is determined uniquely, because $\alpha_i(x)=\alpha_j(x)$ for x from the intersection of definition domains of α_i and α_j. Let some elements $x_1, \cdots, x_n$ be in an n-ary relation on the relation space M. Let x_i $(i=1, \cdots, n)$ be in the definition domain of $\alpha_{k(i)}$. Let $N=\max\{k(i): i=1, \cdots, n\}$. Then x_i is in the definition domain of α_N for $i=1, \cdots, n$ and the elements $\alpha_N(x_1), \cdots, \alpha_N(x_n)$ are in the corresponding relation, because α_N is an isomorphism. But $\alpha(x_i)=\alpha_N(x_i)$ for $i=1, \cdots, n$ and so $\alpha(x_1), \cdots, \alpha(x_n)$ are in the corresponding relation. This can be made for any n-tuple which is in some relation on M and so we have proved that α is an isomorphism.

The assumption that M is a finitary relation space was made, because in the contrary case N should not always have to exist.

All groups, semigroups, lattices, rings, fields, graphs etc. are finitary relation spaces.

Now we shall prove a theorem.

THEOREM 1. *Let* $\{S_i : i=1, 2, \cdots\}$, $\{S'_i : i=1, 2, \cdots\}$ *be two infinite sequences of finitary relation spaces, let* $S=\underrightarrow{\lim}\,(S_i\,;\varphi_{ji})$, $S'=\underrightarrow{\lim}\,(S'_i; \varphi'_{ji})$, *where* φ_{ji}, φ'_{ji} *are corresponding isomorphisms (see the definition of the direct limit). The relation spaces* S *and* S' *are isomorphic to each other, if and only if there exists an infinite sequence* $\{T_i : i=1, 2, \cdots\}$, *whose terms are finitary retation spaces or empty sets, and the isomorphisms* ψ_i, ψ'_i, τ_{ji} *for* $i\leqq j$ *so that* ψ_i *is an isomorphism of* T_i *into* S_i, ψ'_i *is an isomorphism of* T_i *into* S'_i *and* τ_{ji} *is an isomorphism of* T_i *into* T_j *so that the following conditions are satisfied*:

(A)
$$\tau_{ki}(x)=\tau_{kj}\cdot\tau_{ji}(x), \quad \tau_{ii}(x)=x \quad \text{for} \quad i\leqq j\leqq k\,,$$
$$\psi_j\tau_{ji}(x)=\varphi_{ji}\psi_i(x)$$
$$\psi_j'\tau_{ji}(x)=\varphi'_{ji}\psi_i'(x)$$
for any i.

(B) *To each positive integer* k *and to each element* $x\in S_k$ *there exists such a positive integer* N *that for each integer* $n>N$ *we have* $\varphi_{nk}(x)\in\psi_n(T_n)$.

(C) *To each positive integer* k *and to each element* $x'\in S_k'$ *there exists such a positive integer* N' *that for each integer* $n>N'$ *we have* $\varphi'_{nk}(x)\in\psi_n'(T_n)$.

REMARK. The conjunction of the conditions (B) and (C) is equivalent to the following condition:

$$\bigcap_{i=1}^{\infty}(S_i-\psi_i(T_i))=\bigcap_{i=1}^{\infty}(S_i'-\psi_i'(T_i))=\emptyset\,.$$

PROOF. Let there exist the sequence $\{T_i: i=1, 2, \cdots\}$ and the isomorphisms ψ_i, ψ_i' and τ_{ji} with the above described properties. For each i consider the isomorphism $\eta_i=\varphi_i\psi_i$ of T_i into S and the isomorphism $\eta_i'=\varphi_i'\psi_i'$ of T_i into S'. Let $y\in S$. We have $y=\varphi_k(x)$ for some positive integer k and some

$x\in S_k$. According to (B) there exists such a positive integer N that for $n>N$ we have $\varphi_{nk}(x)\in\psi_n(T_n)$. But $\varphi_{nk}(x)=\varphi_n^{-1}(y)$. Thus $\varphi_n^{-1}(y)\in\psi_n(T_n)$ and this means that there exists $z\in T_n$ such that $\psi_n(z)=\varphi_n^{-1}(y)$, which implies $z=\psi_n^{-1}\varphi_n^{-1}(y)=\eta_n^{-1}(y)$. We have proved that for any $y\in S$ there exists a positive integer N such that for $n>N$ the element y is in the definition domain of η_n^{-1}. Analogously we can prove that for any $y'\in S'$ there exists a positive integer N' such that for $n>N'$ the element y' is in the definition domain of $\eta_n'^{-1}$. Now let us consider the mappings $\omega_i=\eta_i'\eta_i^{-1}=\varphi_i'\psi'_i\psi_i^{-1}\varphi_i^{-1}$ for positive integers i; they are isomorphisms from S into S' and they can be eventually empty, i.e. defined for no element (if $T_i=\emptyset$). Let us study interrelations between ω_m, ω_n for $m<n$. We have $\varphi_m(x)=\varphi_n\varphi_{nm}(x)$, $\varphi_m'(x)=\varphi_n'\varphi_{nm}'(x)$ for any x for which it is defined, therefore

$$\omega_m=\varphi_m'\psi_m'\psi_m^{-1}\varphi_m^{-1}=\varphi_n'\varphi_{nm}'\psi_m'\psi_m^{-1}\varphi_{nm}^{-1}\varphi_n^{-1}\,.$$

Further $\varphi_{nm}\psi_m=\psi_n\tau_{nm}$, according to (A), and thus $\psi_m^{-1}\varphi_{nm}^{-1}=\tau_{nm}^{-1}\varphi_n^{-1}$. According to (A) also $\varphi_{nm}'\psi_m'=\psi_n'\tau_{nm}$. Thus we have $\omega_m(y)=\varphi_n'\psi_n'\tau_{nm}\tau_{nm}^{-1}\psi_n^{-1}\varphi_n^{-1}(y)$ $=\varphi_n'\psi_n'\psi_n^{-1}\varphi_n^{-1}(y)=\omega_n(y)$ for all y for which $\omega_m(y)$ is defined; therefore ω_n is an extension of ω_m for $n>m$. To each $y\in S$ there exists a positive integer N such that for $n>N$ the mapping η_n^{-1} is defined in y; therefore also $\omega_n=\eta_n'\eta_n^{-1}$ is defined in y, because $\eta_n^{-1}(y)\in T_n$ and η_n' is the mapping of T_n into S'. Therefore let us define the mapping ω of S into S' so that $\omega(y)=\omega_n(y)$ for such a positive integer n that ω_n is defined in y; according to the above proved $\omega(y)$ is determined uniquely. The mapping ω is an extension of ω_n for any positive integer n. According to Lemma ω is an isomorphism of S into S'. Now we shall prove that ω is even an isomorphism of S onto S'. We have $\omega_n^{-1}=\eta_n\eta_n'^{-1}$ for any n. The mapping ω_n^{-1} is defined in all elements of S' in which η'^{-1}_n is defined. We have proved that for each $y'\in S'$ there exists a positive integer N such that for $n>N$ the element y' is in the definition domain of $\eta_n'^{-1}$. We define the mapping ω' of S' into S so that $\omega'(y')=\omega_n^{-1}(y')$ for such a positive integer n that ω_n^{-1} is defined in y'; the mapping ω' is determined uniquely. Now let $y=\omega'(y')$, $y\in S$, $y'\in S'$. Therefore there exists a positive integer m such that $y=\omega_m^{-1}(y')$; thus $y'=\omega_m(y)$ and $y'=\omega(y)$. So $\omega'=\omega^{-1}$; as ω' is defined in all elements of S', ω must be an isomorphism onto S'. Therefore S and S' are isomorphic.

Now suppose that S and S' are isomorphic. Let η be an isomorphism of S onto S'. Put $T_n=\eta\varphi_n(S_n)\cap\varphi_n'(S_n')$ for each positive integer n. Further put $\psi_n=\varphi_n^{-1}\eta^{-1}$, $\psi_n'=\varphi_n'^{-1}$ for every positive integer n, $\tau_{nm}(x)=x$ for $x\in T_m$, m and n positive integers, $m<n$. The condition (A) is fulfilled, because $\psi_j\tau_{ji}(x)$ $=\varphi_j^{-1}\eta^{-1}(x)$ for $x\in T_i$, $\varphi_{ji}\psi_i(x)=\varphi_{ji}\varphi_i^{-1}\eta^{-1}(x)=\varphi_j^{-1}\eta^{-1}(x)$ for $x\in T_i$, therefore $\psi_j\tau_{ji}=\varphi_{ji}\psi_i$; further $\psi_j'\tau_{ji}(x)=\varphi_j^{-1}(x)$ for $x\in T_i$, $\varphi_{ji}'\psi_i'(x)=\varphi_{ji}'\varphi_i'^{-1}(x)=\varphi_j'^{-1}(x)$

for $x \in T_i$, therefore $\psi_j' \tau_{ji} = \varphi_{ji}' \psi_i'$. We shall verify the condition (B). Let $x \in S_k$ and consider the element $\varphi_k(x) \in S$. As η is an isomorphism of S onto S', there exists an element $z = \eta\varphi_k(x) \in S'$. This element is equal to $\varphi_m'(y)$ for some m and some $y \in S_m'$. Let $N = \max(k, m)$. For $n > N$ we have $z = \eta\varphi_n\varphi_{nk}(x) = \varphi_n'\varphi_{nm}'(y)$, therefore $z \in \eta\varphi_n(S_n) \cap \varphi_n'(S_n') = T_n$. Then $x = \varphi_k^{-1}\eta^{-1}(z)$ and $\varphi_{nk}(x) = \varphi_{nk}\varphi_k^{-1}\eta^{-1}(z) = \varphi_n^{-1}\eta^{-1}(z) = \psi_n(z) \in \psi_n(T_n)$. Analogously we verify the condition (C). The proof is ready.

Now if we put $\xi_i = \psi_i'\psi_i^{-1}$, the mapping ξ_i is an isomorphism from S_i into S_i'. Further $\xi_j\varphi_{ji}(x) = \psi_j'\psi_j^{-1}\varphi_{ji}(x) = \psi_j'\tau_{ji}\psi_i'^{-1}(x) = \varphi_{ji}'\psi_i'\psi_i^{-1}(x) = \varphi_{ji}'\xi_i(x)$ for any x for which both $\xi_j\varphi_{ji}$ and $\varphi_{ji}'\,\xi_i$ are defined; this follows from the conditions (A). From the conditions (B) and (C) we can obtain the result that for any positive integer k and $x \in S_k$ there exists a positive integer N such that for $n > N$ the element $\varphi_{nk}(x)$ is in the definition domain of ξ_n and for any positive integer k and $x' \in S_k'$ there exists a positive integer N' such that for $n > N'$ the element $\varphi_{nk}'(x')$ is in the set of values of ξ_n.

On the other hand, if such mappings ξ_i are defined, we may obtain the sets T_i and the mappings ψ_i, ψ_i', τ_{ji}. For each i we define T_i as the subset of S_i consisting of all elements x for which $\xi_i(x)$ is defined. Then $\psi_i(x) = x$, $\psi_i'(x) = \xi_i(x)$, $\tau_{ji}(x) = \varphi_{ji}(x)$ for $x \in T_i$, $j > i$. We can easily verify all the conditions for these sets and mappings.

Therefore we can express a theorem which is a simplification of Theorem 1; we do not need the sets T_i in it.

THEOREM 2. *Let* $\{S_i : i = 1, 2, \cdots\}$, $\{S_i' : i = 1, 2, \cdots\}$ *be two infinite sequences of finitary relation spaces, let* $S = \varinjlim (S_i; \varphi_{ji})$, $S' = \varinjlim (S_i'; \varphi_{ji}')$, *where* φ_{ji}, φ_{ji}' *are corresponding isomorphisms. The relation spaces* S *and* S' *are isomorphic to each other, if and only if there exists a family* $\Xi = \{\xi_i : i = 1, 2, \cdots\}$ *of isomorphisms such that* ξ_i *is an isomorphism from* S_i *into* S_i' *(some of these isomorphisms may be empty) so that the following conditions are satisfied:*

$$\text{(A)} \qquad \xi_j\varphi_{ji}(x) = \varphi_{ji}'(x)\xi_i(x)$$

for any x *for which both* $\xi_j\varphi_{ji}$ *and* $\varphi_{ji}'\xi_i$ *are defined.*
(B) *To each positive integer* k *and to each element* $x \in S_k$ *there exists such a positive integer* N *that for each integer* $n > N$ *the element* $\varphi_{nk}(x)$ *is contained in the definition domain of* ξ_n.
(C) *To each positive integer* k *and to each element* $x' \in S_k'$ *there exists such a positive integer* N' *that for each integer* $n > N'$ *the element* $\varphi_{nk}'(x')$ *is contained in the set of values of* ξ_n.

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Reference

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