J. Math. Tokushima Univ.
Vol. 6. (1972), 17-29

An Explicit Construction of Irreducible Representations for $\mathfrak{so}(6)$

By

Tokuichi KAYAMA

(Received May 25, 1972)

Irreducible representation spaces of $\mathfrak{so}(6)$ are constructed by the solid harmonics for the unitary group. The matrix representation is explicitly obtained by using of the raising-lowering operators of the special function.

§ 1. Introduction

A few years ago, the author defined two kinds of special functions and investigated some properties of them [1]. These functions were introduced as a necesary consequence from a series of studies concerning solid harmonics for the unitary group, which had been developed by Ikeda partially in collaboration with the author [2], [3]. It was an aim to provide mathematical tools for applications of the group theory to physics of many-particle systems. The present paper is also devoted to this purpose.

In the present study an essential role is played by the twelve recurrence operators of $P_{n\gamma}^{\alpha\beta}(x)$, the above mentioned special function. It is well-known that the associated Legendre functions of the first kind are used for the construction of matrix representations for $\mathfrak{so}(3)$, which is a Lie algebra of rank 1. The author has never seen such uses of special functions for a simple Lie algebra of rank 2 or highers. This is the first time that the recurrence formulas have ever employed for $\mathfrak{so}(6)$ of rank 3. The method in the paper may be likely applied to higher dimensional cases.

It is a conclusion of the paper to propose the matrix elements for the irreducible representation of $\mathfrak{so}(6)$ (see [7.1]). The basis of the representation space is fixed according to the chains $\mathfrak{so}(6)\supset\mathfrak{su}(3)\supset\mathfrak{su}(2)$ and $\mathfrak{so}(6)\supset\mathfrak{so}(4)\simeq\mathfrak{su}(2)\oplus\mathfrak{su}(2)$. In other words, the matrix representation of the restriction to one of the subalgebras is the block diagonal. The restriction to $\mathfrak{su}(2)$ agrees with the Condon and Shortley's formula [4]. The matrix elements for $\mathfrak{su}(3)$ are identical with that of several authors [5]. These facts show that the result of the paper is considerably fit for practical uses.

The content of the paper is as follows. §2 deals with the properties of $P_{n\gamma}^{\alpha\beta}(x)$ which are required in the subsequent sections. They have been already obtained in the papers [1]. New notations are also introduced for the recurrence operators of the function. In §3 the structure of $\mathfrak{so}(6)$ is studied. In §4, particular solusions of the Hermite-Laplace equation are written by making use of $P_{n\gamma}^{\alpha\beta}(x)$ and discussed in connection with operators of $\mathfrak{so}(6)$. In §5 the operators are represented by means of the raising-lowering operators. In §6 irreducible representation spaces are constructed. The final result is given in §7.

§2. Some properties of $P_{n\gamma}^{\alpha\beta}(x)$

The function $P_{n\gamma}^{\alpha\beta}(x)$ is a special solution of the equation

$$(2.1)\quad (1-x^2)\frac{d^2y}{dx^2}+\{n-(n+2)x\}\frac{dy}{dx} + \left\{\gamma(\gamma+n+1)-\frac{\alpha^2}{2(1-x)}-\frac{\beta^2-n^2}{2(1+x)}\right\}y=0 \qquad (-1<x<1),$$

and defined by

$$(2.2)\quad P_{n\gamma}^{\alpha\beta}(x)=\{\Gamma(1-\alpha)\}^{-1}(1-x)^{-\alpha/2}(1+x)^{(\beta-n)/2} \times F\{-\gamma-(\alpha-\beta+n)/2,\ \gamma+(-\alpha+\beta+n+2)/2;\ 1-\alpha;\ (1-x)/2\}.$$

Here, n is a non-negative integer, and α, β and γ are complex. (2.2) is significant for each value of the parameters. For, if x_0 is fixed in $-1<x_0<1$, then $P_{n\gamma}^{\alpha\beta}(x_0)$ is an entire function of the complex variables α, β and γ.

(2.2) has the following symmetry relations.

$$(2.3)\qquad P_{n\gamma}^{\alpha\beta}(x)=P_{n,-\gamma-n-1}^{\alpha\beta}(x)=2^{\beta}P_{n\gamma}^{\alpha,-\beta}(x).$$

In the case of a non-negative integer α, we have

$$(2.4)\qquad P_{n\gamma}^{\alpha\beta}(x)=(-2)^{-\alpha}A_n(\alpha,\beta,\gamma)P_{n\gamma}^{-\alpha,\beta}(x),$$

$$(2.5)\qquad A_n(\alpha,\beta,\gamma)=\frac{\Gamma\{\gamma+(\alpha-\beta+n+2)/2\}\Gamma\{\gamma+(\alpha+\beta+n+2)/2\}}{\Gamma\{\gamma-(\alpha-\beta-n-2)/2\}\Gamma\{\gamma-(\alpha+\beta-n-2)/2\}}.$$

Now, we introduce the following operators.

$$(2.6)\quad \Delta_{n,0}^{\alpha\pm,\beta\pm}(x)=\sqrt{1-x^2}\frac{d}{dx}+\begin{Bmatrix}\alpha\\-\alpha\end{Bmatrix}\sqrt{\frac{1+x}{4(1-x)}}+\begin{Bmatrix}-\beta+n\\\beta+n\end{Bmatrix}\sqrt{\frac{1-x}{4(1+x)}},$$

$$(2.7)\quad \Delta_{n,\gamma\pm}^{0,\beta\pm}(x)=(1-x)\sqrt{1+x}\frac{d}{dx}+\begin{Bmatrix}-\gamma-n-1\\\gamma\end{Bmatrix}\sqrt{1+x}+\begin{Bmatrix}-\beta+n\\\beta+n\end{Bmatrix}\frac{1}{\sqrt{1+x}},$$

$$(2.8)\quad \Delta_{n,\gamma\pm}^{\alpha\pm,0}(x)=(1+x)\sqrt{1-x}\frac{d}{dx}+\begin{Bmatrix}\gamma+n+1\\-\gamma\end{Bmatrix}\sqrt{1-x}+\begin{Bmatrix}\alpha\\-\alpha\end{Bmatrix}\frac{1}{\sqrt{1-x}},$$

Here, the upper or lower argument in braces on the right hand side is chosen in accordance with $+$ or $-$ of the double sign behind each suffix of Δ on the left hand side. We often omit α, β and γ so far as no confusion arises. For example $\Delta_{n-}^{+0}(x)$ means $\Delta_{n,\gamma-}^{\alpha+,0}(x)$.

By making use of the above notation we may write twelve recurrence formulas for $P_{n\gamma}^{\alpha\beta}(x)$ as follows.

(2. 9) $\Delta_{n\,0}^{-+}(x)P_{n\gamma}^{\alpha\beta}(x)=2^{-1}\{\gamma+(\alpha-\beta+n)/2\}\{\gamma-(\alpha-\beta-n-2)/2\}P_{n\,\gamma}^{\alpha-1,\beta+1}(x)$,

(2. 10) $\Delta_{n\,0}^{+-}(x)P_{n\gamma}^{\alpha\beta}(x)=-2P_{n\,\gamma}^{\alpha+1,\beta-1}(x)$,

(2. 11) $\Delta_{n\,0}^{--}(x)P_{n\gamma}^{\alpha\beta}(x)=\{\gamma-(\alpha+\beta-n-2)/2\}\{\gamma+(\alpha+\beta+n)/2\}P_{n\gamma}^{\alpha-1,\beta-1}(x)$,

(2. 12) $\Delta_{n\,0}^{++}(x)P_{n\gamma}^{\alpha\beta}(x)=-P_{n\,\gamma}^{\alpha+1,\beta+1}(x)$,

(2. 13) $\Delta_{n-}^{0+}(x)P_{n\gamma}^{\alpha\beta}(x)=\{\gamma+(\alpha-\beta+n)/2\}P_{n,\gamma-1/2}^{\alpha,\beta+1}(x)$,

(2. 14) $\Delta_{n+}^{0-}(x)P_{n\gamma}^{\alpha\beta}(x)=-2\{\gamma-(\alpha+\beta-n-2)/2\}P_{n,\gamma+1/2}^{\alpha,\beta-1}(x)$,

(2. 15) $\Delta_{n+}^{0+}(x)P_{n\gamma}^{\alpha\beta}(x)=-\{\gamma-(\alpha-\beta-n-2)/2\}P_{n,\gamma+1/2}^{\alpha,\beta+1}(x)$,

(2. 16) $\Delta_{n-}^{0-}(x)P_{n\gamma}^{\alpha\beta}(x)=2\{\gamma+(\alpha+\beta+n)/2\}P_{n,\gamma-1/2}^{\alpha,\beta-1}(x)$,

(2. 17) $\Delta_{n+}^{-0}(x)P_{n\gamma}^{\alpha\beta}(x)=\{\gamma-(\alpha-\beta-n-2)/2\}\{\gamma-(\alpha+\beta-n-2)/2\}P_{n,\gamma+1/2}^{\alpha-1,\beta}(x)$,

(2. 18) $\Delta_{n-}^{+0}(x)P_{n\gamma}^{\alpha\beta}(x)=-2P_{n,\gamma-1/2}^{\alpha+1,\beta}(x)$,

(2. 19) $\Delta_{n+}^{+0}(x)P_{n\gamma}^{\alpha\beta}(x)=-2P_{n,\gamma+1/2}^{\alpha+1,\beta}(x)$,

(2. 20) $\Delta_{n-}^{-0}(x)P_{n\gamma}^{\alpha\beta}(x)=\{\gamma+(\alpha-\beta+n)/2\}\{\gamma+(\alpha+\beta+n)/2\}P_{n,\gamma-1/2}^{\alpha-1,\beta}(x)$.

Finally we give an orthogonality relation: *When α is fixed in the region* Re $\alpha<1$ *or a positive integer and β an arbitrary complex, assume γ to take such values that one and only one of $\gamma+(\pm\alpha\pm\beta+n)/2$ and $-\gamma+(\pm\alpha\pm\beta-n-2)/2$ is a non-negative integer. Here, $+$ or $-$ of the double sign in front of α corresponds to* Re $\alpha<1$ *or* $\alpha=1, 2, \cdots$ *respectively, and also $+$ or $-$ in front of β corresponds to* Re $\beta<1$ *or* Re $\beta>-1$. *Then the system of functions $\{P_{n\gamma}^{\alpha\beta}(x)\}$ satisfies the following formula.*

$$(2.21)\qquad \int_{-1}^{1} P_{n\gamma}^{\alpha\beta}(x)P_{n\gamma'}^{\alpha\beta}(x)(1+x)^n dx = \begin{cases} 0 & (\gamma\neq\gamma') \\ \dfrac{2^{-\alpha+\beta+1}}{2\gamma+n+1}A_n(\alpha,\beta,\gamma) & (\gamma=\gamma'), \end{cases}$$

where $A_n(\alpha, \beta, \gamma)$ is given by (2.5).

§ 3. The structure of $\mathfrak{so}(6)$

We consider the three dimensional unitary space. Let z^μ ($\mu=1, 2, 3$) be complex coordinate and $\bar{z}^\mu$ be its conjugate. We introduce fifteen differential operators

$$(3.1)\quad \begin{cases} X^\mu_\nu = -z^\mu \dfrac{\partial}{\partial z^\nu} + \bar{z}^\nu \dfrac{\partial}{\partial \bar{z}^\mu}, & N^\mu_\nu = -\bar{z}^\nu \dfrac{\partial}{\partial z^\mu} + \bar{z}^\mu \dfrac{\partial}{\partial z^\nu}, \\ \bar{N}^\mu_\nu = -z^\nu \dfrac{\partial}{\partial \bar{z}^\mu} + z^\mu \dfrac{\partial}{\partial \bar{z}^\nu} & (N^\mu_\nu + N^\nu_\mu = \bar{N}^\mu_\nu + \bar{N}^\nu_\mu = 0). \end{cases}$$

Here, the differentiation is defined by $\partial/\partial z^\mu = (\partial/\partial x^\mu - i\partial/\partial y^\mu)/2$ and $\partial/\partial \bar{z}^\mu = (\partial/\partial x^\mu + i\partial/\partial y^\mu)/2$, if we put $z^\mu = x^\mu + iy^\mu$ (x^μ, y^μ: real, i: imaginary unit). Commutation relations among (3.1) are as follows.

$$(3.2)\quad \begin{cases} [X^\mu_\nu, X^\rho_\sigma] = \delta_{\mu\sigma} X^\rho_\nu - \delta_{\nu\rho} X^\mu_\sigma, & [X^\mu_\nu, N^\rho_\sigma] = \delta_{\nu\sigma} N^\mu_\rho - \delta_{\nu\rho} N^\mu_\sigma, \\ [X^\mu_\nu, \bar{N}^\rho_\sigma] = \delta_{\mu\sigma} \bar{N}^\rho_\nu - \delta_{\mu\rho} \bar{N}^\sigma_\nu, & [N^\mu_\nu, N^\rho_\sigma] = [\bar{N}^\mu_\nu, \bar{N}^\rho_\sigma] = 0, \\ [N^\mu_\nu, \bar{N}^\rho_\sigma] = \delta_{\mu\rho} X^\nu_\sigma - \delta_{\mu\sigma} X^\nu_\rho + \delta_{\nu\sigma} X^\mu_\rho - \delta_{\nu\rho} X^\mu_\sigma. \end{cases}$$

As is readily shown, the totality of (3.1) over the complex number field forms the complexification of the Lie algebra of the six dimensional rotation group. We refer to it as $\mathfrak{so}(6)$*. Its maximal commutative subalgebra is spanned by X^μ_μ ($\mu=1, 2, 3$). The first commutator in (3.2) is that of $\mathfrak{u}(3)$. X^μ_ν ($\mu \neq \nu$) and $\sum_{\mu=1}^3 a_\mu X^\mu_\mu$ subject to $\sum_{\mu=1}^3 a_\mu = 0$ are linear harls of $\mathfrak{su}(3)$. X^μ_ν (μ, $\nu=1, 2$), N^1_2 and $\bar{N}^2_1$ form $\mathfrak{so}(4)$**. We therefore provide the followings as a canonical basis of $\mathfrak{so}(6)$.

$$(3.3)\quad \begin{cases} M = (X^1_1 - X^2_2)/2, \quad N = (X^1_1 + X^2_2)/2, & M_+ = X^2_1, \quad M_- = X^1_2, \\ Y = (X^1_1 + X^2_2 - 2X^3_3)/3, & N_+ = \bar{N}^2_1, \quad N_- = N^1_2, \\ X^3_1, \quad X^1_3, \quad X^3_2, \quad X^2_3, \quad \bar{N}^3_1, \quad N^1_3, \quad \bar{N}^3_2, \quad N^2_3. \end{cases}$$

Commutators of (3.3) are easily obtained from (3.2). They are given in an appendix.

[3. 1] *If the operators $\boldsymbol{L}_1^2$ and $\boldsymbol{L}_2^2$ are defined by*

$$(3.4)\qquad \boldsymbol{L}_1^2 = \sum_{\mu,\nu=1}^{2} X^\mu_\nu X^\nu_\mu/2 - \Big(\sum_{\mu=1}^{2} X^\mu_\mu\Big)^2/4,$$

$$(3.5)\qquad \boldsymbol{L}_2^2 = \sum_{\mu,\nu=1}^{3} X^\mu_\nu X^\nu_\mu/2 - \Big(\sum_{\mu=1}^{3} X^\mu_\mu\Big)^2/4,$$

then $\boldsymbol{L}_2^2$ is permutable with each element of $\mathfrak{so}(6)$, and $\boldsymbol{L}_1^2$ is commutable with M, N, $M_\pm$ and $N_\pm$.

The commutability is easily proved for X^μ_ν by making use of the first relation in (3.2). On the other hand, for N^μ_ν and $\bar{N}^\mu_\nu$ we should not only use

---

* By Cartan's notation, it is D_3 or A_3. The latter is $\mathfrak{su}(4)$.

** $-N^1_2$ and $-\bar{N}^2_1$ are respectively equal to N_- and N_+ introduced by Ikeda (II in [2]).

the commutators but also the definition (3.1).

§ 4. Particular solutions of $\Delta f = 0$

Let us consider the Hermite-Laplace equation

(4. 1) $$\Delta f = 0, \quad \Delta = \partial^2/\partial z^1 \partial \bar{z}^1 + \partial^2/\partial z^2 \partial \bar{z}^2 + \partial^2/\partial z^3 \partial \bar{z}^3 .$$

In order to solve (4.1), we use the generalized polar coordinates

(4. 2) $$\begin{cases} \rho = z^1\bar{z}^1 + z^2\bar{z}^2 + z^3\bar{z}^3, \quad x_1 = (z^1\bar{z}^1 - z^2\bar{z}^2)/(z^1\bar{z}^1 + z^2\bar{z}^2), \\ x_2 = (z^1\bar{z}^1 + z^2\bar{z}^2 - z^3\bar{z}^3)/(z^1\bar{z}^1 + z^2\bar{z}^2 + z^3\bar{z}^3), \\ \varphi_\mu = (1/2i)\log(z^\mu/\bar{z}^\mu) \qquad (\mu = 1, 2, 3), \end{cases}$$

or

(4. 3) $$\begin{cases} z^1 = \sqrt{\rho(1+x_1)(1+x_2)/4}\ e^{i\varphi_1} \\ z^2 = \sqrt{\rho(1-x_1)(1+x_2)/4}\ e^{i\varphi_2}, \quad z^3 = \sqrt{\rho(1-x_2)/2}\ e^{i\varphi_3} . \end{cases}$$

If we employ the method of separation of variables, we have a particular solution of (4.1) [1], [2],

(4. 4) $$f_{l_1 l_2}^{m_1 m_2 m_3} = \rho^{l_2} P_{0 l_1}^{m_2 m_1}(x_1) P_{1 l_2}^{m_3, 2l_1+1}(x_2) e^{i(m_1\varphi_1 + m_2\varphi_2 + m_3\varphi_3)} ,$$

where l_1, l_2 and m_μ ($\mu=1, 2, 3$) are constants for the separation.

Now, we investigate some properties of (4.4).

[4. 1] *$f_{l_1 l_2}^{m_1 m_2 m_3}$ is a homogeneous function of degree $l_2 - \sum_{\mu=1}^3 m_\mu/2$ in $\bar{z}^\mu$ and of degree $l_2 + \sum_{\mu=1}^3 m_\mu/2$ in z^μ.*

Proof. The operators $\mathscr{P} = \sum_{\mu=1}^3 \bar{z}^\mu \partial/\partial \bar{z}^\mu$ and $\mathscr{Q} = \sum_{\mu=1}^3 z^\mu \partial/\partial z^\mu$ are written in terms of (4.2) as

$$\mathscr{P} = \rho\frac{\partial}{\partial \rho} - \frac{1}{2i}\sum_{\mu=1}^3 \frac{\partial}{\partial \varphi_\mu}, \quad \mathscr{Q} = \rho\frac{\partial}{\partial \rho} + \frac{1}{2i}\sum_{\mu=1}^3 \frac{\partial}{\partial \varphi_\mu} .$$

Then we have

$$\mathscr{P} f_{l_1 l_2}^{m_1 m_2 m_3} = (l_2 - \textstyle\sum_{\mu=1}^3 m_\mu/2) f_{l_1 l_2}^{m_1 m_2 m_3}, \quad \mathscr{Q} f_{l_1 l_2}^{m_1 m_2 m_3} = (l_2 + \textstyle\sum_{\mu=1}^3 m_\mu/2) f_{l_1 l_2}^{m_1 m_2 m_3} .$$

Thus the proof is completed.

If a solution of (4.1) is a homogeneous polynomial of degree p in $\bar{z}^\mu$ and of degree q in z^μ, we call it a *solid harmonic of type* (p, q)*.

* By the terminology in the papers [2], [3], it is called "*a solid harmonic for* $U(3)$ *of type* (p, q)." Here, U(3) is the three dimensional unitary group.

[4.2] *If f is a solid harmonic of type (p, q), then $X^{\mu}_{\nu}f$, $N^{\mu}_{\nu}f$ and $\bar{N}^{\mu}_{\nu}f$ are solid harmonics of type (p, q), $(p-1, q+1)$ and $(p+1, q-1)$ respectively.*

Proof. From the definition (3.2) we easily obtain the following commutators

$$[\Delta, X^{\mu}_{\nu}] = [\Delta, N^{\mu}_{\nu}] = [\Delta, \bar{N}^{\mu}_{\nu}] = 0, \quad [\mathcal{P}, X^{\mu}_{\nu}] = [\mathcal{Q}, X^{\mu}_{\nu}] = 0\,,$$
$$[\mathcal{P}, N^{\mu}_{\nu}] = -N^{\mu}_{\nu}, \quad [\mathcal{Q}, N^{\mu}_{\nu}] = N^{\mu}_{\nu}, \quad [\mathcal{P}, \bar{N}^{\mu}_{\nu}] = \bar{N}^{\mu}_{\nu}, \quad [\mathcal{Q}, \bar{N}^{\mu}_{\nu}] = -\bar{N}^{\mu}_{\nu}\,.$$

Thus the lemma is proved.

[4. 3] *$f^{m_1m_2m_3}_{l_1l_2}$ is a simultaneous eigen function of $\boldsymbol{L}^2_2$, $\boldsymbol{L}^2_1$, M, N and Y with respective eigenvalues $l_2(l_2+2)$, $l_1(l_1+1)$, $(-m_1+m_2)/2$, $(-m_1-m_2)/2$ and $(-m_1-m_2+2m_3)/3$.*

It may be proved by writing the operators in terms of (4.2). In fact we are led to the followings (c.f. (2.1)).

$$M = (-\partial/\partial\varphi_1+\partial/\partial\varphi_2)/2i, \quad N = (-\partial/\partial\varphi_1-\partial/\partial\varphi_2)/2i\,,$$
$$Y = (-\partial/\partial\varphi_1-\partial/\partial\varphi_2+2\partial/\partial\varphi_3)/3i\,,$$
$$\boldsymbol{L}^2_2 = -(1-x_2^2)\frac{\partial^2}{\partial x_2^2}-(1-3x_2)\frac{\partial}{\partial x_2}-\frac{1}{2(1-x_2)}\frac{\partial^2}{\partial\varphi_3^2}+\frac{2}{1+x_2}\boldsymbol{L}^2_1\,,$$
$$\boldsymbol{L}^2_1 = -(1-x_1^2)\frac{\partial^2}{\partial x_1^2}+2x_1\frac{\partial}{\partial x_1}-\frac{1}{2(1-x_1)}\frac{\partial^2}{\partial\varphi_2^2}-\frac{1}{2(1+x_1)}\frac{\partial^2}{\partial\varphi_1^2}\,.$$

§ 5. Representations by means of the raising-lowering operators

We first take up X^1_3 as an example. It is written in terms of (4.2) as follows*.

$$X^1_3 = e^{i(\varphi_1-\varphi_3)}\left\{\sqrt{\frac{(1+x_1)(1-x_2^2)}{2}}\frac{\partial}{\partial x_2}+\sqrt{\frac{(1+x_1)(1-x_2)}{2(1+x_2)}}(1-x_1)\frac{\partial}{\partial x_1}\right.$$
$$\left.-\frac{1}{i}\left(\sqrt{\frac{1-x_2}{2(1+x_1)(1+x_2)}}\frac{\partial}{\partial\varphi_1}+\sqrt{\frac{(1+x_1)(1+x_2)}{8(1-x_2)}}\frac{\partial}{\partial\varphi_3}\right)\right\}.$$

If it is operated on (4.4), then we have

$$X^1_3 f^{m_1m_2m_3}_{l_1l_2} = \rho^{l_2}\left\{\sqrt{\frac{(1+x_1)(1-x_2^2)}{2}}\frac{\partial}{\partial x_2}+\sqrt{\frac{(1+x_1)(1-x_2)}{2(1+x_2)}}(1-x_1)\frac{\partial}{\partial x_1}\right.$$
$$\left.-m_1\sqrt{\frac{1-x_2}{2(1+x_1)(1+x_2)}}-m_3\sqrt{\frac{(1+x_1)(1+x_2)}{8(1-x_2)}}\right\}$$
$$\times P^{m_2m_1}_{0l_1}(x_1)P^{m_3,2l_1+1}_{1l_2}(x_2)e^{i\{(m_1+1)\varphi_1+m_2\varphi_2+(m_3-1)\varphi_3\}}\,.$$

* Expressions for the other operators are given in an appendix.

Now, we intend to show that the right hand side is reducible to a linear combination of (4.4). This is fulfilled, if the factor concerned with x_1 and x_2 is a linear combination of $P_{0l_1'}^{m_2,m_1+1}(x_1)P_{1l_2}^{m_3-1,2l_1'+1}(x_2)$, varying l_1'.

For the purpose we consider

$$\Delta_{0l_1-}^{0m_1+}(x_1)\Delta_{10}^{m_3-,(2l_1+1)-}(x_2)-\Delta_{0l_1+}^{0m_1+}(x_1)\Delta_{10}^{m_3-,(2l_1+1)+}(x_2)\,.$$

From (2.6) and (2.7) it is calculated as follows.

$$\left\{(1-x_1)\sqrt{1+x_1}\frac{\partial}{\partial x_1}+l_1\sqrt{1+x_1}-\frac{m_1}{\sqrt{1+x_1}}\right\}$$

$$\times\left\{\sqrt{1-x_2^2}\frac{\partial}{\partial x_2}-\frac{m_3}{2}\sqrt{\frac{1+x_2}{1-x_2}}+(l_1+1)\sqrt{\frac{1-x_2}{1+x_2}}\right\}$$

$$-\left\{(1-x_1)\sqrt{1+x_1}\frac{\partial}{\partial x_1}-(l_1+1)\sqrt{1+x_1}-\frac{m_1}{\sqrt{1+x_1}}\right\}$$

$$\times\left\{\sqrt{1-x_2^2}\frac{\partial}{\partial x_2}-\frac{m_3}{2}\sqrt{\frac{1+x_2}{1-x_2}}-l_1\sqrt{\frac{1-x_2}{1+x_2}}\right\}$$

$$=(2l_1+1)\left\{\sqrt{(1+x_1)(1-x_2^2)}\frac{\partial}{\partial x_2}+\sqrt{\frac{(1+x_1)(1-x_2)}{1+x_2}}\frac{\partial}{\partial x_1}\right.$$

$$\left.-m_1\sqrt{\frac{1-x_2}{(1+x_1)(1+x_2)}}-m_3\sqrt{\frac{(1+x_1)(1+x_2)}{4(1-x_2)}}\right\}.$$

Thus we obtain

(5. 1) $$\sqrt{2}(2l_1+1)X_3^1 f_{l_1l_2}^{m_1m_2m_3}$$
$$=\{\Delta_{0-}^{0+}(x_1)\Delta_{10}^{--}(x_2)-\Delta_{0+}^{0+}(x_1)\Delta_{10}^{-+}(x_2)\}e^{i(\varphi_1-\varphi_3)}f_{l_1l_2}^{m_1m_2m_3}\,.$$

The first and second terms on the right hand side agree with $f_{l_1-1/2,l_2}^{m_1+1,m_2,m_3-1}$ and $f_{l_1+1/2,l_2}^{m_1+1,m_2,m_3-1}$ respectively within the constant factors.

For the other operators, we can samely obtain the followings.

(5. 2) $$\sqrt{2}\,(2l_1+1)X_1^3=\{-\Delta_{0-}^{0-}(x_1)\Delta_{10}^{+-}(x_2)+\Delta_{0+}^{0-}(x_1)\Delta_{10}^{++}(x_2)\}e^{i(\varphi_3-\varphi_1)}\,,$$

(5. 3) $$\sqrt{2}\,(2l_1+1)X_3^2=\{-\Delta_{0-}^{+0}(x_1)\Delta_{10}^{--}(x_2)+\Delta_{0+}^{+0}(x_1)\Delta_{10}^{-+}(x_2)\}e^{i(\varphi_2-\varphi_3)}\,,$$

(5. 4) $$\sqrt{2}\,(2l_1+1)X_2^3=\{\Delta_{0-}^{-0}(x_1)\Delta_{10}^{+-}(x_2)-\Delta_{0+}^{-0}(x_1)\Delta_{10}^{++}(x_2)\}e^{i(\varphi_3-\varphi_2)}\,,$$

(5. 5) $$\sqrt{2}\,(2l_1+1)N_3^1=\{-\Delta_{0-}^{0+}(x_1)\Delta_{10}^{+-}(x_2)+\Delta_{0+}^{0+}(x_1)\Delta_{10}^{++}(x_2)\}e^{i(\varphi_1+\varphi_3)}\,,$$

(5. 6) $$\sqrt{2}\,(2l_1+1)\bar{N}_1^3=\{\Delta_{0-}^{0-}(x_1)\Delta_{10}^{--}(x_2)-\Delta_{0+}^{0-}(x_1)\Delta_{10}^{-+}(x_2)\}e^{i(-\varphi_1-\varphi_3)}\,,$$

(5. 7) $$\sqrt{2}\,(2l_1+1)N_3^2=\{\Delta_{0-}^{+0}(x_1)\Delta_{10}^{+-}(x_2)-\Delta_{0+}^{+0}(x_1)\Delta_{10}^{++}(x_2)\}e^{i(\varphi_2+\varphi_3)}\,,$$

(5. 8) $$\sqrt{2}\,(2l_1+1)\bar{N}_2^3=\{-\Delta_{0-}^{-0}(x_1)\Delta_{10}^{--}(x_2)+\Delta_{0+}^{-0}(x_1)\Delta_{10}^{-+}(x_2)\}e^{i(-\varphi_2-\varphi_3)}\,,$$

(5. 9) $$M_+=-\Delta_{00}^{+-}(x_1)e^{i(\varphi_2-\varphi_1)},\quad M_-=\Delta_{00}^{-+}(x_1)e^{i(\varphi_1-\varphi_2)}\,,$$

(5. 10) $N_{+}=\Delta_{00}^{--}(x_1)e^{-i(\varphi_1+\varphi_2)},\quad N_{-}=-\Delta_{00}^{++}(x_1)e^{i(\varphi_1+\varphi_2)}.$

Here, (5.2)~(5.10) are to be operated on $f_{l_1l_2}^{m_1m_2m_3}$.

We therefore arrive at the lemma:

[5. 1] *The totality of* $f_{l_1l_2}^{m_1m_2m_3}$ *with a fixed* l_2 *forms a linear representation space of* $\mathfrak{so}(6)$.

§ 6. Irreducible representations

The representation of $\mathfrak{so}(6)$ stated in [5.1] is probably reducible. Since its martix elements are known through (5.1)~(5.10) and (2.9)~(2.20), we could seek out irreducible subspaces. But this process is too complicated. On the other hand, we have already known the theory concerning solid harmonics and representations of the unitary groups [2]. In this section, we employ a number of facts from the theory:

If p *and* q *are fixed non-negative integers, all the solid harmonics of type* (p, q) *form an irreducible representation space of* $\mathfrak{su}(3)$. Let us denote it by $V(p, q)$.

A basis of $V(p, q)$ *is assigned by a triplet of non-negative integers* (r, s, t) *subject to*

(6. 1) $r=0, 1, \cdots, q,\quad s=0, 1, \cdots, p,\quad t=0, 1, \cdots, 2l\qquad (2l=p+r-s).$

Let us refer to it as v_{pq}^{rst}.

v_{pq}^{rst} *is characterized by a simultaneous eigenvector of* $\boldsymbol{L}_1^2$, M, N *and* Y, *whose eigenvalues are* $l(l+1)$, $l-t$, $l-r$ *and* $(p+2q-3r-3s)/3$ *respectively.*

Let us consider $f_{l_2l_2}^{-2l_2,0,0}$, $2l_2$ being a non-negative integer. From (4.4) it is

(6. 2) $$f_{l_2l_2}^{-2l_2,0,0}=\rho^{l_2}P_{0l_2}^{0,-2l_2}(x_1)P_{1l_2}^{0,2l_2+1}(x_2)e^{-i2l_2\varphi_1}.$$

By making use of (2.2), (2.3) and (4.3), we are led to

$$\begin{aligned}f_{l_2l_2}^{-2l_2,0,0}&=\rho^{l_2}2^{-2l_2}(1+x_1)^{l_2}(1+x_2)^{l_2}e^{-i2l_2\varphi_1}\\&=\{\sqrt{\rho(1+x_1)(1+x_2)/4}e^{-i\varphi_1}\}^{2l_2}=(\bar{z}^1)^{2l_2}.\end{aligned}$$

This is a solid harmonic of type $(2l_2, 0)$.

[6.1] *For a fixed non-negative integer* $2l_2$, *all the functions obtained by succesive operations of* (3.3) *on* (6.2) *form an irreducible representation space of* $\mathfrak{so}(6)$. *It agrees with*

(6. 3) $$V(2l_2, 0)\oplus V(2l_2-1, 1)\oplus\cdots\oplus V(0, 2l_2).$$

Proof. Let us take up $N_3^1=-z^3\partial/\partial\bar{z}^1+z^1\partial/\partial\bar{z}^3$, one of (3.3). Then we have

$$(N^1_3)^n f^{-2l_2,0,0}_{l_2 l_2} = (N^1_3)^n(\bar{z}^1)^{2l_2} = (-1)^n(2l_2)(2l_2-1)\cdots(2l_2-n+1)$$
$$\times(\bar{z}^1)^{2l_2-n}(z^3)^n \qquad (n=0,\ 1,\ \cdots,\ 2l_2)\,.$$

It is a harmonic in $V(2l_2-n,\ n)$, which is irreducible with respect to X^μ_ν ($\mu,\ \nu=1,\ 2,\ 3$). Therefore, from [4.2] the totality of the functions stated in the lemma is nothing but (6.3).

In the next place, let us consider $\bar{N}^3_1=-\bar{z}^1\partial/\partial z^3+\bar{z}^3\partial/\partial z^1$. Starting from $(z^3)^{2l_2}$, which is in $V(0,\ 2l_2)$, we are led to

$$(\bar{N}^3_1)^n(z^3)^{2l_2} = (-1)^n(2l_2)(2l_2-1)\cdots(2l_2-n+1)$$
$$\times(\bar{z}^1)^n(z^3)^{2l_2-n} \qquad (n=0,\ 1,\ \cdots,\ 2l_2)\,.$$

This is a non-vanishing harmonic in $V(n,\ 2l_2-n)$. Thus the irreducibility is also proved.

We may adopt v^{rst}_{pq} as a basis of (6.3), where p and q are varied under the condition $p+q=2l_2=$constant.

[6. 2] *If* $f^{m_1m_2m_3}_{l_1l_2}$ *is a solid harmonic in* (6.3), *then it is proportional to* v^{rst}_{pq}, *where*

$$\text{(6. 4)} \qquad \begin{cases} l_2=(p+q)/2, & l_1=l=(p+r-s)/2\,, \\ m_1=-p+s+t, & m_2=r-t, \quad m_3=q-r-s\,. \end{cases}$$

Proof. From [4.1] and [4.3] $f^{m_1m_2m_3}_{l_1l_2}$ is an eigenfunction of $\mathcal{P}$, $\mathcal{Q}$, $\boldsymbol{L}^2_1$, M, N and Y. Therefore it is nothing but v^{rst}_{pq}, unless it is vanishing. By comparing the corresponding eigenvalues, we have

$$l_2-(m_1+m_2+m_3)/2=p, \quad l_2+(m_1+m_2+m_3)/2=q\,,$$
$$l_1(l_1+1)=l(l+1), \quad (-m_1+m_2)/2=l-t, \quad (-m_1-m_2)/2=l-r\,,$$
$$(-m_1-m_2+2m_3)/3=(p+2q-3r-3s)/3 \qquad (2l=p+r-s)\,.$$

If we solve these equations in l_2, l_1, m_1, m_2 and m_3, we are led to (6.4). Here, it should be remarked that two cases of $l_1=l$ and $l_1=-l-1$ give the same harmonic (c.f. (2.3)). We have taken up the former case.

By making use of (2.2)$\sim$(2.5), it is easily shown that $f^{m_1m_2m_3}_{l_1l_2}$ is not vanishing for all values of p, q, r, s, t subject to (6.1). Thus the proof is completed.

§ 7. Matrix elements

Let us normalize the basis according to the orthogonality relation (2.18). We arive at the final result:

[7. 1] *If the basis of* (6.3) *is defined by*

$$u_{pq}^{rst} = \left\{ \frac{2^{-2l_1-m_1+m_2+m_3-3}(2l_1+1)(2l_2+2)}{A_0(m_2, m_1, l_1)A_1(m_3, 2l_1+1, l_2)} \right\}^{1/2} f_{l_1 l_2}^{m_1 m_2 m_3}, \tag{7.1}$$

where

$$\begin{cases} l_2 = (p+q)/2, \quad l_1 = l = (p+r-s)/2, \\ m_1 = -p+s+t, \quad m_2 = r-t, \quad m_3 = q-r-s \\ (r = 0, 1, \cdots, q, \quad s = 0, 1, \cdots, p, \quad t = 0, 1, \cdots, 2l), \end{cases} \tag{7.2}$$

then the operators of $\mathfrak{so}(6)$ *are represented as follows.*

$$\begin{aligned} X_3^1 u_{pq}^{rst} = {} & \{(2l-t)(p-s)(s+1)(p+q-s+1)/(2l)(2l+1)\}^{1/2} u_{pq}^{r\,s+1\,t} \\ & + \{(t+1)(r+1)(q-r)(p+r+2)/(2l+1)(2l+2)\}^{1/2} u_{pq}^{r+1\,s\,t+1}, \end{aligned} \tag{7.3}$$

$$\begin{aligned} X_1^3 u_{pq}^{rst} = {} & \{tr(q-r+1)(p+r+1)/(2l)(2l+1)\}^{1/2} u_{pq}^{r-1\,s\,t-1} \\ & + \{(2l-t+1)(p-s+1)s(p+q-s+2)/(2l+1)(2l+2)\}^{1/2} u_{pq}^{r\,s-1\,t}, \end{aligned} \tag{7.4}$$

$$\begin{aligned} X_3^2 u_{pq}^{rst} = {} & \{t(p-s)(s+1)(p+q-s+1)/(2l)(2l+1)\}^{1/2} u_{pq}^{r\,s+1\,t-1} \\ & - \{(2l-t+1)(r+1)(q-r)(p+r+2)/(2l+1)(2l+2)\}^{1/2} u_{pq}^{r+1\,s\,t}, \end{aligned} \tag{7.5}$$

$$\begin{aligned} X_2^3 u_{pq}^{rst} = {} & -\{(2l-t)r(q-r+1)(p+r+1)/(2l)(2l+1)\}^{1/2} u_{pq}^{r-1\,s\,t} \\ & + \{(t+1)(p-s+1)s(p+q-s+2)/(2l+1)(2l+2)\}^{1/2} u_{pq}^{r\,s-1\,t+1}, \end{aligned} \tag{7.6}$$

$$\begin{aligned} N_3^1 u_{pq}^{rst} = {} & \{(2l-t)(p-s)(p+r+1)(q-r+1)/(2l)(2l+1)\}^{1/2} u_{p-1\,q+1}^{r\,s\,t} \\ & + \{(t+1)(r+1)s(p+q-s+2)/(2l+1)(2l+2)\}^{1/2} u_{p-1\,q+1}^{r+1\,s-1\,t+1}, \end{aligned} \tag{7.7}$$

$$\begin{aligned} \bar{N}_1^3 u_{pq}^{rst} = {} & \{tr(s+1)(p+q-s+1)/(2l)(2l+1)\}^{1/2} u_{p+1\,q-1}^{r-1\,s+1\,t-1} \\ & + \{(2l-t+1)(p-s+1)(q-r)(p+r+2)/(2l+1)(2l+2)\}^{1/2} u_{p+1\,q-1}^{r\,s\,t} \end{aligned} \tag{7.8}$$

$$\begin{aligned} N_3^2 u_{pq}^{rst} = {} & \{t(p-s)(q-r+1)(p+r+1)/(2l)(2l+1)\}^{1/2} u_{p-1\,q+1}^{r\,s\,t-1} \\ & - \{(2l-t+1)(r+1)s(p+q-s+2)/(2l+1)(2l+2)\}^{1/2} u_{p-1\,q+1}^{r+1\,s-1\,t}, \end{aligned} \tag{7.9}$$

$$\begin{aligned} \bar{N}_2^3 u_{pq}^{rst} = {} & -\{(2l-t)r(s+1)(p+q-s+1)/(2l)(2l+1)\}^{1/2} u_{p+1\,q-1}^{r-1\,s+1\,t} \\ & + \{(t+1)(p-s+1)(q-r)(p+r+2)/(2l+1)(2l+2)\}^{1/2} u_{p+1\,q-1}^{r\,s\,t+1}, \end{aligned} \tag{7.10}$$

$$M_+ u_{pq}^{rst} = \{t(2l-t+1)\}^{1/2} u_{pq}^{r\,s\,t-1}, \quad M_- u_{pq}^{rst} = \{(t+1)(2l-t)\}^{1/2} u_{pq}^{r\,s\,t+1}, \tag{7.11}$$

$$N_+ u_{pq}^{rst} = \{r(p-s+1)\}^{1/2} u_{p+1\,q-1}^{r-1\,s\,t}, \quad N_- u_{pq}^{rst} = \{(r+1)(p-s)\}^{1/2} u_{p-1\,q+1}^{r+1\,s\,t}. \tag{7.12}$$

Appendix

The commutators of (3.3) are as follows.

$$\begin{array}{lll}
[M, M_+] = M_+, & [Y, M_+] = 0, & [N, M_+] = 0, \\
[M, M_-] = -M_-, & [Y, M_-] = 0, & [N, M_-] = 0, \\
[M, X^1_3] = -\frac{1}{2}X^1_3, & [Y, X^1_3] = -X^1_3, & [N, X^1_3] = -\frac{1}{2}X^1_3, \\
[M, X^3_1] = \frac{1}{2}X^3_1, & [Y, X^3_1] = X^3_1, & [N, X^3_1] = \frac{1}{2}X^3_1, \\
[M, X^2_3] = \frac{1}{2}X^2_3, & [Y, X^2_3] = -X^2_3, & [N, X^2_3] = -\frac{1}{2}X^2_3, \\
[M, X^3_2] = -\frac{1}{2}X^3_2, & [Y, X^3_2] = X^3_2, & [N, X^3_2] = \frac{1}{2}X^3_2,
\end{array}$$

$$\begin{array}{lll}
[M, N_+] = 0, & [Y, N_+] = \frac{2}{3}N_+, & [N, N_+] = N_+, \\
[M, N_-] = 0, & [Y, N_-] = -\frac{2}{3}N_-, & [N, N_-] = -N_-, \\
[M, N^1_3] = -\frac{1}{2}N^1_3, & [Y, N^1_3] = \frac{1}{3}N^1_3, & [N, N^1_3] = -\frac{1}{2}N^1_3, \\
[M, \bar{N}^3_1] = \frac{1}{2}\bar{N}^3_1, & [Y, \bar{N}^3_1] = -\frac{1}{3}\bar{N}^3_1, & [N, \bar{N}^3_1] = \frac{1}{2}\bar{N}^3_1, \\
[M, N^2_3] = \frac{1}{2}N^2_3, & [Y, N^2_3] = \frac{1}{3}N^2_3, & [N, N^2_3] = -\frac{1}{2}N^2_3, \\
[M, \bar{N}^3_2] = -\frac{1}{2}\bar{N}^3_2, & [Y, \bar{N}^3_2] = -\frac{1}{3}\bar{N}^3_2, & [N, \bar{N}^3_2] = \frac{1}{2}\bar{N}^3_2,
\end{array}$$

$$\begin{array}{ll}
[M_+, M_-] = 2M, & [N_+, N_-] = 2N, \\
[X^1_3, X^3_1] = -M-\frac{3}{2}Y, & [X^2_3, X^3_2] = M-\frac{3}{2}Y, \\
[N^1_3, \bar{N}^3_1] = -M-2N+\frac{3}{2}Y, & [N^2_3, \bar{N}^3_2] = M-2N+\frac{3}{2}Y,
\end{array}$$

$$\begin{array}{lll}
[M_+, X^1_3] = -X^2_3, & [M_+, X^3_2] = X^3_1, & [M_-, X^3_1] = X^3_2, \\
[M_-, X^2_3] = -X^1_3, & [X^3_1, X^2_3] = M_+, & [X^3_2, X^1_3] = M_-,
\end{array}$$

$$\begin{array}{lll}
[N_+, X^1_3] = \bar{N}^3_2, & [N_+, X^2_3] = -\bar{N}^3_1, & [M_+, N^1_3] = -N^2_3, \\
[M_+, \bar{N}^3_2] = \bar{N}^3_1, & [N^1_3, X^2_3] = N_-, & [\bar{N}^3_1, X^3_2] = -N_+, \\
[N_-, X^3_1] = -N^2_3, & [N_-, X^3_2] = N^1_3, & [M_-, \bar{N}^3_1] = \bar{N}^3_2, \\
[M_-, N^2_3] = -N^1_3, & [N^2_3, X^1_3] = -N_-, & [\bar{N}^3_2, X^3_1] = N_+,
\end{array}$$

$$\begin{array}{lll}
[N_+, N^1_3] = X^3_2, & [N_+, N^2_3] = -X^3_1, & [N^1_3, \bar{N}^3_2] = -M_-, \\
[\bar{N}^3_1, N^2_3] = M_+, & [N_-, \bar{N}^3_1] = -X^2_3, & [N_-, \bar{N}^3_2] = X^1_3,
\end{array}$$

other commutators being zero.

X^μ_ν, $\bar{N}^\mu_\nu$ and N^μ_ν are expressed in terms of (4.2) as follows.

$$X_1^3 = e^{i(\varphi_3-\varphi_1)}\Big\{-\sqrt{\frac{(1+x_1)(1-x_2^2)}{2}}\frac{\partial}{\partial x_2}-\sqrt{\frac{(1+x_1)(1-x_2)}{2(1+x_2)}}(1-x_1)\frac{\partial}{\partial x_1}$$
$$-\frac{1}{i}\Big(\sqrt{\frac{1-x_2}{2(1+x_1)(1+x_2)}}\frac{\partial}{\partial\varphi_1}+\sqrt{\frac{(1+x_1)(1+x_2)}{8(1-x_2)}}\frac{\partial}{\partial\varphi_3}\Big)\Big\},$$

$$X_3^2 = e^{i(\varphi_2-\varphi_3)}\Big\{\sqrt{\frac{(1-x_1)(1-x_2^2)}{2}}\frac{\partial}{\partial x_2}-\sqrt{\frac{(1-x_1)(1-x_2)}{2(1+x_2)}}(1+x_1)\frac{\partial}{\partial x_1}$$
$$-\frac{1}{i}\Big(\sqrt{\frac{1-x_2}{2(1-x_1)(1+x_2)}}\frac{\partial}{\partial\varphi_2}+\sqrt{\frac{(1-x_1)(1+x_2)}{8(1-x_2)}}\frac{\partial}{\partial\varphi_3}\Big)\Big\},$$

$$X_2^3 = e^{i(\varphi_3-\varphi_2)}\Big\{-\sqrt{\frac{(1-x_1)(1-x_2^2)}{2}}\frac{\partial}{\partial x_2}+\sqrt{\frac{(1-x_1)(1-x_2)}{2(1+x_2)}}(1+x_1)\frac{\partial}{\partial x_1}$$
$$-\frac{1}{i}\Big(\sqrt{\frac{1-x_2}{2(1-x_1)(1+x_2)}}\frac{\partial}{\partial\varphi_2}+\sqrt{\frac{(1-x_1)(1+x_2)}{8(1-x_2)}}\frac{\partial}{\partial\varphi_3}\Big)\Big\},$$

$$N_3^1 = e^{i(\varphi_1+\varphi_3)}\Big\{-\sqrt{\frac{(1+x_1)(1-x_2)}{2(1+x_2)}}(1-x_1)\frac{\partial}{\partial x_1}-\sqrt{\frac{(1+x_1)(1-x_2^2)}{2}}\frac{\partial}{\partial x_2}$$
$$+\frac{1}{i}\Big(\sqrt{\frac{1-x_2}{2(1+x_1)(1+x_2)}}\frac{\partial}{\partial\varphi_1}-\sqrt{\frac{(1+x_1)(1+x_2)}{8(1-x_2)}}\frac{\partial}{\partial\varphi_3}\Big)\Big\},$$

$$\bar{N}_1^3 = e^{-i(\varphi_1+\varphi_3)}\Big\{\sqrt{\frac{(1+x_1)(1-x_2)}{2(1+x_2)}}(1-x_1)\frac{\partial}{\partial x_1}+\sqrt{\frac{(1+x_1)(1-x_2^2)}{2}}\frac{\partial}{\partial x_2}$$
$$+\frac{1}{i}\Big(\sqrt{\frac{1-x_2}{2(1+x_1)(1+x_2)}}\frac{\partial}{\partial\varphi_1}-\sqrt{\frac{(1+x_1)(1+x_2)}{8(1-x_2)}}\frac{\partial}{\partial\varphi_3}\Big)\Big\},$$

$$N_3^2 = e^{i(\varphi_2+\varphi_3)}\Big\{\sqrt{\frac{(1-x_1)(1-x_2)}{2(1+x_2)}}(1+x_1)\frac{\partial}{\partial x_1}-\sqrt{\frac{(1-x_1)(1-x_2^2)}{2}}\frac{\partial}{\partial x_2}$$
$$+\frac{1}{i}\Big(\sqrt{\frac{1-x_2}{2(1-x_1)(1+x_2)}}\frac{\partial}{\partial\varphi_2}-\sqrt{\frac{(1-x_1)(1+x_2)}{8(1-x_2)}}\frac{\partial}{\partial\varphi_3}\Big)\Big\},$$

$$\bar{N}_2^3 = e^{-i(\varphi_2+\varphi_3)}\Big\{-\sqrt{\frac{(1-x_1)(1-x_2)}{2(1+x_2)}}(1+x_1)\frac{\partial}{\partial x_1}+\sqrt{\frac{(1-x_1)(1-x_2^2)}{2}}\frac{\partial}{\partial x_2}$$
$$+\frac{1}{i}\Big(\sqrt{\frac{1-x_2}{2(1-x_1)(1+x_2)}}\frac{\partial}{\partial\varphi_2}-\sqrt{\frac{(1-x_1)(1+x_2)}{8(1-x_2)}}\frac{\partial}{\partial\varphi_3}\Big)\Big\},$$

$$M_\pm = e^{\mp i(\varphi_1-\varphi_2)}\Big\{\mp\sqrt{1-x_1^2}\frac{\partial}{\partial x_1}-\frac{1}{2i}\Big(\sqrt{\frac{1-x_1}{1+x_1}}\frac{\partial}{\partial\varphi_1}+\sqrt{\frac{1+x_1}{1-x_1}}\frac{\partial}{\partial\varphi_2}\Big)\Big\},$$

$$N_\pm = e^{\mp i(\varphi_1+\varphi_2)}\Big\{\pm\sqrt{1-x_1^2}\frac{\partial}{\partial x_1}+\frac{1}{2i}\Big(\sqrt{\frac{1-x_1}{1+x_1}}\frac{\partial}{\partial\varphi_1}-\sqrt{\frac{1+x_1}{1-x_1}}\frac{\partial}{\partial\varphi_2}\Big)\Big\}.$$

Acknowledgements

The author would like to express his sincere thanks to Professor Minoru Hamaguchi for his careful reading of the manuscript and for his kind encouragement during the present work. The author also wishes to express his sincere thanks to Professor Mineo Ikeda (Kyoto University) for his valuable comments.

Department of Physics, Faculty of Education, Tokushima University

References

[1] Kayama, T. (1968), (1969): On a generalization of the spherical functions for the unitary group I, II, Math. Japonicae **13**, 159–174, **14**, 159–176.

[2] Ikeda, M. (1967): On spherical functions for the unitary group I, II, III, Mem. Fac. Engi. Hiroshima Univ. **3**, 17–29, 31–53, 55–75.
Ikeda, M. and Kayama, T. (1967): On spherical functions for the unitary group IV, Mem. Fac. Engi. Hiroshima Univ. **3**, 77–100.

[3] Ikeda, M. and Kayama, T. (1968): On the generating function of spherical functions for unitary group, Math. Japonicae **12**, 159–175.

[4] Condon, E. U. and Shortley, G. H. (1935): The theory of atomic spectra, Cambridge Univ. Press.

[5] Gel'Fand, I. M. and Tseitlin, M. L. (1950): Конечномерные представления группы унимодуярных матриц, Dokl. Akad. Nauk SSSR **71**, 825–828.
Racah. G. (1962): Lectures on Lie groups (Group theoretical concepts and methods in elementary particle physics. Edited by F. Gürsey, Gordon and Breach 1962, 1–36).
Elliott, J. P. and Harvey, M. (1963): Collective motion in the nuclear shell model III. The calculation of spectra, Proc. Roy. Soc. **A272**, 557–577.
Pursey, D. L. (1963): Irreducible representations of the 'unitary symmetry' group, Proc. Roy. Soc. **A275**, 284–294.