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Almost Complex Projective Connections

By

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The theory of real projective connections standing on the view-point of the theory of vector bundles was investigated perfectly by T. Otsuki [2]. Its extension to generalized spaces was done by the present author and K. Eguchi [1]. On the other hand, T. Ishihara [4] studied complex projective structures and found the close relation between the complex projective structures and H-projective connections which had been introduced by T. Tashiro [5] and developed by S. Ishihara [3] and many other authors [6].

In the present paper, our main purpose is to study manifolds endowed with almost complex projective structures. First of all, in §§1 and 2, we consider an almost complex projective vector bundle introduced in [4] and several distributions which define an almost complex projective connection. This connection, however, can be determined also by an usual differential geometric method. In §3, we find a certain distribution $\mathcal{P}$ which is intrinsic to the almost complex projective connection. A mapping σ which gives a correspondence between any two fibers in the almost complex projective vector bundle is treated in §4. And we show that the mapping σ preserves the complex projective structure in each fiber invariant if the distribution $\mathcal{P}$ is integrable. The last section is devoted to study a manifold where the given distribution $\mathcal{P}$ is integrable, for example, we obtain that if $\mathcal{P}$ is integrable then the manifold is an H-projectively flat complex manifold.

§1. Almost complex projective vector bundles

Let M be a $2n$-dimensional differentiable manifold of class C^∞ and let there be given an almost complex structure f. Let P be an almost complex projective vector bundle [4], that is, P is a vector bundle over M which has R^{2n+2} as a standard fiber, and is constructed as follows;

1) The structure group G is formed by all elements of the type

$$\left(\begin{array}{c:cc} g^i_j & 0 & 0 \\ \hdashline & 1 & 0 \\ * & 0 & 1 \end{array}\right), \text{ where } (g^i_j)\in GL(2n, R).^{1)}$$

Obviously G is a subgroup of $GL(2n+2, R)$.

2) Let U and V be two coordinate neighbourhoods of M such that $U\cap V\neq\phi$ and $(x^1, x^2, \cdots, x^{2n})$, $(\bar{x}^1, \bar{x}^2, \cdots, \bar{x}^{2n})$ be the local coordinate systems valid on U and V respectively. We define the transition function G_{UV}: $U\cap V\rightarrow G$ by

$$(1.1)\qquad \left(\begin{array}{c:cc} \dfrac{\partial\bar{x}^i}{\partial x^j} & 0 & 0 \\ \hdashline b\partial_j \log\Delta+cf^m_j\partial_m\log\Delta & 1 & 0 \\ c\partial_j\log\Delta-bf^m_j\partial_m\log\Delta & 0 & 1 \end{array}\right),$$

where $\partial_j=\dfrac{\partial}{\partial x^j}$, $\Delta=\det|\partial_i\bar{x}^j|$, b and c are constants, and f^i_j is the given almost complex structure.

Let π be the projection of bundle P $(\pi: P\rightarrow M)$. If we write (x^i, w^λ) and $(\bar{x}^i, \bar{w}^\lambda)$ for the canonical coordinate systems valid on $\pi^{-1}(U)$ and $\pi^{-1}(V)$ respectively, due to the bundle structure, the w^λ and $\bar{w}^\lambda$ are related by

$$(1.2)\qquad \left\{\begin{aligned} \bar{u}^i &= \frac{\partial\bar{x}^i}{\partial x^l}u^l\,, \\ \bar{v}^1 &= (b\partial_l\log\Delta+cf^m_l\partial_m\log\Delta)u^l+v^1\,, \\ \bar{v}^2 &= (c\partial_l\log\Delta-bf^m_l\partial_m\log\Delta)u^l+v^2\,, \end{aligned}\right.$$

where we put $(w^\lambda)=(u^i, v^1, v^2)$.

Now, we define a mapping C^*, which is fiber-preserving, by

$$(1.3)\qquad C^*(w^\lambda)=\left(\begin{array}{c:cc} \alpha\delta^i_j+\beta f^i_j & 0 & \\ \hdashline 0 & \alpha & -\beta \\ & \beta & \alpha \end{array}\right)\begin{pmatrix} u^j \\ v^1 \\ v^2 \end{pmatrix}=\begin{pmatrix} \alpha u^i+\beta f^i_j u^j \\ \alpha v^1-\beta v^2 \\ \beta v^1+\alpha v^2 \end{pmatrix},$$

where $\alpha^2+\beta^2\neq 0$.

1) Throughout the paper, the Roman indices h, i, j, k etc. run over the range $1, \cdots, 2n$ and the Greek indices α, β, γ etc. over the range $1, \cdots, 2n, 2n+1, 2n+2$, and 1^* stands for $2n+1$ and 2^* for $2n+2$.

The relation $G_{UV}C^*_{(U)}=C^*_{(V)}G_{UV}$ follows at once, hence the mapping C^* is well-defined on each fiber.

In the case where $\alpha=0$ and $\beta=1$, we denote C^* by F, then

$$F=\left(\begin{array}{c:cc} f^i_j & 0 & \\ \hdashline 0 & 0 & -1 \\ & 1 & 0 \end{array}\right) \tag{1.4}$$

satisfies $F^2=-E$ and $\dfrac{\partial F^\lambda_\mu}{\partial w^\sigma}=0$, that is, F gives a complex structure on each fiber.

Now, we say that two points (w^λ) and $(\overline{w}^\lambda)$ on a fiber are mutually *complex-projective* if (w^λ) and $(\overline{w}^\lambda)$ are related by $\overline{w}^\lambda=c^*w^\lambda$ for certain constants α and β. Denoting this relation by $w^\lambda \overset{c^*}{\barwedge} \overline{w}^\lambda$, we can easily see that the relation $\overset{c^*}{\barwedge}$ satisfies an equivalence relation. Hence, if we remove the origine from R^{2n+2} and identify all points which are mutually complex-projective, we obtain an n-dimensional complex projective space $P^n(C)$. In an almost complex projective vector bundle P, if we convert each fiber into a complex projective space by means of the above mentioned method, the fiber bundle is called an almost complex projective bundle over M and is denoted by $\tilde{P}$ hereafter.

§ 2. Almost complex projective connections

Let P be an almost complex projective vector bundle over an almost complex manifold M. Let P_p be the tangent space of P at $p\in P$. If we assign to each $p\in P$ a subspace Φ^V_p of P_p formed by all vectors tangent to the fiber through p, then we have a $(2n+2)$-dimensional distribution on P. We call it the vertical distribution and denote it by Φ^V. A distribution Φ^h which is complementary to the Φ^V and satisfies the following conditions

1) $P_p=\Phi^h_p\oplus\Phi^V_p$,
2) $d\tilde{c}^*\Phi^h_p=\Phi^h_{\tilde{c}^*p}$,
3) Φ^h_p is a linear space with respect to w^λ

is called the horizontal distribution, where $\tilde{c}^*$ is the mapping $P\to P$ which maps each $p=(x^i, w^\lambda)$ into $\tilde{c}^*p=(x^i, c^*w^\lambda)$.

If we denote by p^V and p^h the projection operator of vector fields on P to Φ^V and Φ^h respectively, then from the condition 1) these projection operators should have the following forms

$$p^V=\begin{pmatrix} 0 & 0 \\ \theta^\lambda_j & \delta^\lambda_\mu \end{pmatrix}, \quad p^h=\begin{pmatrix} \delta^i_j & 0 \\ -\theta^\lambda_j & 0 \end{pmatrix}. \tag{2.1}$$

When we put

$$(2.2)\quad X_i = \frac{\partial}{\partial x^i} - \theta_i^\lambda(x, w)\frac{\partial}{\partial w^\lambda} = \frac{\partial}{\partial x^i} - \theta_i^k(x, w)\frac{\partial}{\partial u^k} - \theta_i^{1*}(x, w)\frac{\partial}{\partial v^1} - \theta_i^{2*}(x, w)\frac{\partial}{\partial v^2},$$

then $\{X_i\}$ forms a local basis on Φ^h. Calculating the condition 2) with respect to the basis $\{X_i\}$, we have

$$\begin{cases} \theta_k^j(x, c^*w) = \alpha\theta_k^j(x, w) + \beta f_m^j \theta_k^m(x, w) - \beta\partial_k f_m^j u^m, \\ \theta_k^{1*}(x, c^*w) = \alpha\theta_k^{1*}(x, w) - \beta\theta_k^{2*}(x, w), \\ \theta_k^{2*}(x, c^*w) = \alpha\theta_k^{2*}(x, w) + \beta\theta_k^{1*}(x, w). \end{cases}$$

On the other hand, from the condition 3), θ_j^λ can be written as

$$\begin{cases} \theta_k^j(x, w) = \varphi_{km}^j(x)u^m + p_k^j(x)v^1 + q_k^j(x)v^2, \\ \theta_k^{1*}(x, w) = p_{km}(x)u^m + q_k(x)v^1 + r_k(x)v^2, \\ \theta_k^{2*}(x, w) = \bar{p}_{km}(x)u^m + \bar{q}_k(x)v^1 + \bar{r}_k(x)v^2. \end{cases}$$

Thus we obtain

$$\begin{cases} \partial_k f_h^j - f_m^j \varphi_{kh}^m + \varphi_{km}^j f_h^m = 0, \\ q_k^j = f_m^j p_k^m, \\ \bar{p}_{km} = -p_{kl} f_m^l, \\ \bar{q}_k = -r_k, \\ \bar{r}_k = q_k. \end{cases}$$

Hence θ_j^λ have the form

$$(2.3)\quad \begin{cases} \theta_k^j(x, w) = \varphi_{km}^j(x)u^m + p_k^j(x)v^1 + f_m^j(x)p_k^m(x)v^2, \\ \theta_k^{1*}(x, w) = p_{km}(x)u^m + q_k(x)v^1 + r_k(x)v^2, \\ \theta_k^{2*}(x, w) = -p_{km}(x)f_l^m(x)u^l - r_k(x)v^1 + q_k(x)v^2, \end{cases}$$

where φ_{jk}^i satisfy the condition

$$(2.4)\quad \partial_k f_h^j - f_l^j \varphi_{kh}^l + \varphi_{kl}^j f_h^l = 0.$$

Since the projection operator p^V must be a (1-1)-tensor, the condition $\bar{p}^V J = J p^V$ is necessarily satisfied, where J is the Jacobian matrix of the coordinate transformation $(x^i, w^\lambda) \to (\bar{x}^i, \bar{w}^\lambda)$ on P. So the law of transformation of the θ_j^λ under the coordinate transformation of M is

$$
(2.5)\quad \begin{cases}
\bar\theta^i_m \dfrac{\partial \bar x^m}{\partial x^k} + \dfrac{\partial^2 \bar x^i}{\partial x^k \partial x^m} u^m = \dfrac{\partial \bar x^i}{\partial x^m} \theta^m_k , \\
\bar\theta^{1*}_m \dfrac{\partial \bar x^m}{\partial x^k} = \theta^{1*}_k + b\theta^m_k \partial_m \log \Delta + c\theta^m_k f^l_m \partial_l \log \Delta \\
\qquad -(b\partial_k \partial_l \log \Delta + c\partial_k f^m_l \partial_m \log \Delta + cf^m_l \partial_k \partial_m \log \Delta) u^l , \\
\bar\theta^{2*}_m \dfrac{\partial \bar x^m}{\partial x^k} = \theta^{2*}_k + c\theta^m_k \partial_m \log \Delta - b\theta^m_k f^l_m \partial_l \log \Delta \\
\qquad -(c\partial_k \partial_l \log \Delta - b\partial_k f^m_l \partial_m \log \Delta - bf^m_l \partial_k \partial_m \log \Delta) u^l .
\end{cases}
$$

By substituting (2.3) into $(2.5)_1$ and $(2.5)_2$, we obtain the law of transformation under the coordinate transformation of M with respect to the components φ^i_{jk}, p^i_j, p_{jk}, q_k and r_k as follows:

$$
(2.6)\quad \begin{cases}
\bar\varphi^j_{ml} \dfrac{\partial \bar x^m}{\partial x_k} \dfrac{\partial \bar x^l}{\partial x^j} + \dfrac{\partial^2 \bar x^i}{\partial x^k \partial x^j} = \dfrac{\partial \bar x^i}{\partial x^m} \varphi^m_{kj} - \dfrac{\partial \bar x^i}{\partial x^m} P^m_k (b\partial_j \log\Delta + cf^r_j \partial_r \log \Delta) \\
\qquad - \dfrac{\partial \bar x^i}{\partial x^m} f^m_l P^l_k (c\partial_j \log \Delta - bf^r_j \partial_r \log \Delta) , \\
\bar p^i_m \dfrac{\partial \bar x^m}{\partial x^k} = \dfrac{\partial \bar x^i}{\partial x^m} p^m_k , \\
\bar p_{ml} \dfrac{\partial \bar x^m}{\partial x^k} \dfrac{\partial \bar x^l}{\partial x^j} + \bar q_m \dfrac{\partial \bar x^m}{\partial x^k} (b\partial_j \log \Delta + cf^r_j \partial_r \log \Delta) + \bar r_m \dfrac{\partial \bar x^m}{\partial x^k} (c\partial_j \log \Delta - bf^r_j \partial_r \log \Delta) \\
\qquad = p_{kj} + b\varphi^m_{kj} \partial_m \log \Delta + c\varphi^m_{kj} f^r_m \partial_r \log \Delta \\
\qquad\qquad -(b\partial_k \partial_j \log \Delta + c\partial_k f^m_j \partial_m \log \Delta + cf^m_j \partial_k \partial_m \log \Delta) , \\
\bar q_m \dfrac{\partial \bar x^m}{\partial x^k} = q_k + bp^m_k \partial_m \log \Delta + cp^r_k f^m_r \partial_m \log \Delta , \\
\bar r_m \dfrac{\partial \bar x^m}{\partial x^k} = r_k + bp^r_k f^m_r \partial_m \log \Delta - cp^m_k \partial_m \log \Delta .
\end{cases}
$$

In analogy with the real projective connections [1], we assume that

$$
(2.7)\qquad p^i_j = \delta^i_j , \quad r_k = f^m_k q_m .
$$

It is evident from the law of transformation (2.6) that this assumption is independent of the choice of the canonical coordinate system. At this stage, for the components of θ^λ_j, we obtain

$$
(2.8)\quad \begin{cases}
\theta^j_k(x, w) = \varphi^j_{kl}(x) u^l + \delta^j_k v^1 + f^j_k(x) v^2 , \\
\theta^{1*}_k(x, w) = p_{kl}(x) u^l + q_k(x) v^1 + q_m(x) f^m_k(x) v^2 , \\
\theta^{2*}_k(x, w) = -p_{km}(x) f^m_l(x) u^l - q_m(x) f^m_k(x) v^1 + q_k(x) v^2 ,
\end{cases}
$$

where φ^i_{jk}satisfies the condition

$$\partial_k f_h^j - f_m^j \varphi_{kh}^m + \varphi_{km}^j f_h^m = 0 . \tag{2.9}$$

Again, the law of transformation for the components of θ_j^λ is given by

$$\left\{\begin{aligned}
&\bar{\varphi}_{ml}^i \frac{\partial \bar{x}^m}{\partial x^k}\frac{\partial \bar{x}^l}{\partial x^j} + \frac{\partial^2 \bar{x}^i}{\partial x^k \partial x^j} = \frac{\partial \bar{x}^i}{\partial x^m}\varphi_{kj}^m - \frac{\partial \bar{x}^i}{\partial x^k}(b\partial_j \log \Delta + c f_j^r \partial_r \log \Delta) \\
&\qquad\qquad\qquad - \frac{\partial \bar{x}^i}{\partial x^m} f_k^m (c\partial_j \log \Delta - b f_j^r \partial_r \log \Delta) , \\
&\bar{p}_{ml}\frac{\partial \bar{x}^m}{\partial x^k}\frac{\partial \bar{x}^l}{\partial x^j} + \bar{q}_m \frac{\partial \bar{x}^m}{\partial x^k}(b\,\partial_j \log \Delta + c f_j^r \partial_r \log \Delta) \\
&\qquad + \bar{q}_h \bar{f}_m^h \frac{\partial \bar{x}^m}{\partial x^k}(c\partial_j \log \Delta - b f_j^r \partial_r \log \Delta) \\
&\quad = p_{kj} + b\varphi_{kj}^m \partial_m \log \Delta + c\varphi_{kj}^m f_m^r \partial_r \log \Delta \\
&\qquad - (b\partial_k \partial_j \log \Delta + c\partial_k f_j^m \partial_m \log \Delta + c f_j^m \partial_k \partial_m \log \Delta) , \\
&\bar{q}_m \frac{\partial \bar{x}^m}{\partial x^k} = q_k + b\partial_k \log \Delta + c f_k^m \partial_m \log \Delta .
\end{aligned}\right. \tag{2.10}$$

Conversely, after some calculation, we can easily verify that if the θ_j^λ given by (2.8) satisfies (2.9) and (2.10), then θ_j^λ satisfies $(2.5)_{1,2,3}$. Henceforce we call the θ_j^λ an *almost complex projective connection.* Thus we obtain

PROPOSITION 2.1. *In an almost complex projective vector bundle over an almost complex manifold, if the quantity θ_j^λ is given by* (2.8) *and satisfies* (2.9) *and* (2.10), *then the θ_j^λ defines an almost complex projective connection.*

REMARK. Henceforce we shall restrict ourselves to the case where almost complex projective connections are given by (2.8) together with (2.10) and satisfy (2.9).

THEOREM 2.1. *When an almost complex projective connection is given in an almost complex projective vector bundle over an almost complex manifold M, then*

$$\Pi_{jk}^i \equiv \varphi_{jk}^i + q_m(\delta_j^i \delta_k^m - f_j^i f_k^m) \tag{2.11}$$

gives an affine connection on M and satisfies

$$\nabla_k f_j^i \equiv \partial_k f_j^i - f_m^i \Pi_{kj}^m + \Pi_{km}^i f_j^m = 0 . \tag{2.12}$$

Moreover

$$\Pi_{lm} \equiv p_{lm} + \partial_l q_m - q_t \varphi_{lm}^t \tag{2.13}$$

is a tensor field on M.

PROOF. From the definition, it follows directly that Π^i_{jk} is a function of x^i and is independent of w^λ. For the law of transformation, by virtue of (2.10) and (2.11), it is easily seen that

$$\bar{\Pi}^i_{lm}\frac{\partial \bar{x}^l}{\partial x^j}\frac{\partial \bar{x}^m}{\partial x^k} = \bar{\varphi}^i_{lm}\frac{\partial \bar{x}^l}{\partial x^j}\frac{\partial \bar{x}^m}{\partial x^k}+\bar{q}_m\frac{\partial \bar{x}^i}{\partial x^j}\frac{\partial \bar{x}^m}{\partial x^k}-\bar{q}_r\bar{f}^r_m\bar{f}^i_l\frac{\partial \bar{x}^l}{\partial x^j}\frac{\partial \bar{x}^m}{\partial x^k}$$

$$= -\frac{\partial^2 \bar{x}^i}{\partial x^j \partial x^k}+\frac{\partial \bar{x}^i}{\partial x^m}[\varphi^m_{jk}+q_k\delta^m_j-q_r f^r_k f^m_j]\,.$$

Hence we obtain $\bar{\Pi}^i_{lm}\dfrac{\partial \bar{x}^l}{\partial \partial^j}\dfrac{\partial \bar{x}^m}{\partial x^k}+\dfrac{\partial^2 \bar{x}^i}{\partial x^j \partial x^k}=\dfrac{\partial \bar{x}^i}{\partial x^m}\Pi^m_{jk}$, which implies Π^i_{jk} is an affine connection on M. Moreover it follows that

$$\nabla_k f^i_j = \partial_k f^i_j - f^i_l[\varphi^l_{kj}+q_r(\delta^r_j\delta^l_k - f^r_j f^l_k)]+[\varphi^i_{kl}+q_r(\delta^r_l\delta^i_k-f^r_l f^i_k)]f^l_j$$

$$= \partial_k f^i_j - f^i_l\varphi^l_{kj}+\varphi^i_{kl}f^l_j\,.$$

Thus (2.9) leads us to $\nabla_k f^i_j=0$.

For Π_{lm}, due to (2.10) and (2.13), we see

$$\bar{\Pi}_{ml}\frac{\partial \bar{x}^m}{\partial x^k}\frac{\partial \bar{x}^l}{\partial x^j} = [\bar{p}_{ml}-\bar{q}_t\bar{\varphi}^t_{ml}+\bar{\partial}_m\bar{q}_l]\frac{\partial \bar{x}^m}{\partial x^k}\frac{\partial \bar{x}^l}{\partial x^j}$$

$$= p_{kj}-(b\partial_k\partial_j \log\Delta + c\partial_k f^m_j\partial_m \log\Delta + cf^m_j\partial_k\partial_m \log\Delta)$$

$$+\frac{\partial}{\partial x^k}[q_j+b\partial_j\log\Delta+cf^r_j\partial_r\log\Delta]-q_r\varphi^r_{kj}$$

$$=p_{kj}-q_r\varphi^r_{kj}+\partial_k q_j\,.$$

Hence Π_{kj} is a tensor field on M.

Besides, direct calculation shows us that the tensor Π_{lm} takes the form of

$$\Pi_{lm} = p_{lm}+\partial_l q_m - q_t\Pi^t_{lm}+q_l q_m - q_t f^t_l q_r f^r_m\,. \tag{2. 13'}$$

REMARK. An almost complex manifold M admits an almost complex projective connection if and only if M admits a tensor field Π_{lm}, an affinen connection Π^i_{jk} satisfying $\nabla f=0$ and a quantity q_k whose law of transformation is given by

$$\bar{q}_m\frac{\partial \bar{x}^m}{\partial x^k} = q_k+b\partial_k\log\Delta+cf^m_k\partial_m\log\Delta\,.$$

§ 3. The distribution $\mathcal{P}$

In § 2 we take $\{X_i\}$, which is defined by (2.2), as a local basis of the distribution Φ^h. In our case, the X_i are expressed as

(3.1) $$X_i = \frac{\partial}{\partial x^i} - (\varphi^k_{im} u^m + \delta^k_i v^1 + f^k_i v^2)\frac{\partial}{\partial u^k} - (p_{im} u^m + q_i v^1 + q_m f^m_i v^2)\frac{\partial}{\partial v^1}$$
$$-(-p_{im} f^m_l u^l - q_m f^m_i v^1 + q_i v^2)\frac{\partial}{\partial v^2}.$$

Now we put

(3.2) $$\begin{cases} Y_i = \dfrac{\partial}{\partial u^i} + q_i \dfrac{\partial}{\partial v^1} - q_t f^t_i \dfrac{\partial}{\partial v^2}, \\ Z_{1^*} = \dfrac{\partial}{\partial v^1}, \\ Z_{2^*} = \dfrac{\partial}{\partial v^1}. \end{cases}$$

From (2.1), the laws of transformation with respect to these, under the coordinate transformation, are given as

(3.3) $$X_k = \frac{\partial \bar{x}^m}{\partial x^k} \bar{X}_m, \quad Y_k = \frac{\partial \bar{x}^m}{\partial x^k} \bar{Y}_m, \quad Z_{1^*} = \bar{Z}_{1^*}, \quad Z_{2^*} = \bar{Z}_{2^*}.$$

And obviously the set $\{Y_i, Z_{1^*}, Z_{2^*}\}$ forms a local basis of Φ^V.

If $V = v^i \dfrac{\partial}{\partial x^i}$ is a vector field on M, (3.3) shows us that $V^{(h)} = v^i X_i$ and $V^{(V)} = v^i Y_i$ are vector fields on P. Hence $V^{(h)}$ and $V^{(V)}$ are called a horizontal lift and a vertical lift respectively.

Next we put

(3.4) $$\begin{cases} Y = u^k \dfrac{\partial}{\partial u^k} + v^1 \dfrac{\partial}{\partial v^1} + v^2 \dfrac{\partial}{\partial v^2}, \\ Y' = F(Y) = f^k_m u^m \dfrac{\partial}{\partial u^k} - v^2 \dfrac{\partial}{\partial v^1} + v^1 \dfrac{\partial}{\partial v^2}. \end{cases}$$

Then the Y and Y' are also vector fields on P.

Moreover we put

(3.5) $$\mathcal{P}(\theta) = \Phi^h \oplus Y \oplus Y'.$$

Then $\mathcal{P}(\theta)$ is a $(2n+2)$-dimensional distribution in P. Of course, the distribution Φ^h is dependent upon the connection θ^λ_j, so that the distribution $\mathcal{P}(\theta)$ is also dependent upon the θ^λ_j.

PROPOSITION 3.1. *The vector fields Y and Y' are invariant under the mapping $\tilde{C}^*$ and so is the distribution $\mathcal{P}(\theta)$.*

PROOF. We find, from the definition of the mapping C^*,

$$\begin{aligned} d\tilde{C}^* Y_p &= u^i(\alpha\delta_i^j+\beta f_i^j)\frac{\partial}{\partial \bar{u}^j}+v^1\left(\alpha\frac{\partial}{\partial \bar{v}^1}+\beta\frac{\partial}{\partial \bar{v}^2}\right)+v^2\left(-\beta\frac{\partial}{\partial \bar{v}^1}+\alpha\frac{\partial}{\partial \bar{v}^2}\right) \\ &= \bar{u}^j\frac{\partial}{\partial \bar{u}^j}+\bar{v}^1\frac{\partial}{\partial \bar{v}^1}+\bar{v}^2\frac{\partial}{\partial \bar{v}^2} \\ &= Y_{\tilde{C}^*(p)}\,. \end{aligned}$$

Similarly we find $d\tilde{C}^* Y'_p = Y'_{\tilde{C}^*(p)}$. Thus we have

$$d\tilde{C}^* Y = Y\,, \quad d\tilde{C}^* Y' = Y'\,. \tag{3.6}$$

By the definition, we also have $d\tilde{C}^*\Phi_p^h = \Phi_{\tilde{C}^*(p)}^{h}$. Thus we obtain

$$d\tilde{C}^*\mathcal{P}(\theta) = \mathcal{P}(\theta)\,.$$

Remark. The Proposition 3.1 shows us that the distribution $\mathcal{P}(\theta)$ is well-defined not only on P but also on $\tilde{P}$.

Let there be given two almost complex projective connections θ and $\hat{\theta}$. Then we have two distributions $\mathcal{P}(\theta)$ and $\mathcal{P}(\hat{\theta})$. For these, it is apparent that a relation $\mathcal{P}(\theta)=\mathcal{P}(\hat{\theta})$ holds if and only if $\hat{X}_i = X_i - P_i Y - Q_i Y'$ for certain covector fields P_i and Q_i. This condition is rewritten as

$$\begin{cases} \hat{\varphi}_{i\,j}^{\,k} = \varphi_{i\,j}^{\,k} + P_i\delta_j^k + Q_i f_j^k\,, \\ \hat{P}_{ij} = P_{ij}\,, \\ \hat{q}_i = q_i + P_i \\ \hat{q}_m f_i^m = q_m f_i^m - Q_i\,. \end{cases}$$

Hence we have $Q_i = -P_m f_i^m$, which leads us to

$$\begin{cases} \hat{\varphi}_{i\,j}^{\,k} = \varphi_{i\,j}^{\,k} + P_i\delta_j^k - P_m f_i^m f_j^k\,, \\ \hat{q}_i = q_i + P_i\,, \\ \hat{P}_{ij} = P_{ik}^{\,i}\,. \end{cases}$$

For Π_{jk}^i and Π_{lm}, we have

$$\begin{cases} \hat{\Pi}_{jk}^i = \varphi_{jk}^i + q_m(\delta_j^i\delta_k^m - f_j^i f_k^m) + P_j\delta_k^i + P_k\delta_j^i \\ \qquad\qquad - P_m f_j^m f_k^i - P_m f_k^m f_j^i\,, \\ \hat{\Pi}_{lm} = P_{lm} - q_t\varphi_{lm}^t + \partial_l q_m - q_m P_l + q_t f_m^t P_r f_l^r \\ \qquad\qquad - P_t\varphi_{lm}^t - P_m P_l + P_r f_l^r P_t f_m^t + \partial_l P_m\,. \end{cases}$$

Consequently we obtain

Theorem 3.1. *In order that any two almost complex projective connections θ and $\hat{\theta}$ have the distribution $\mathcal{P}(\theta)$ in common, it is necessary and sufficient that the* *almost complex projective connections θ and $\hat{\theta}$ are related by*

$$(3.7)\qquad \begin{cases} \hat{\varphi}^k_{ij} = \varphi^k_{ij} + P_i\delta^k_j - P_m f^m_i f^k_j, \\ \hat{q}_i = q_i + P_i, \\ \hat{P}_{ij} = P_{ij}, \end{cases}$$

for a certain covector field P_i. Further, in this case, the following holds

$$(3.8)\qquad \begin{cases} \hat{\Pi}^i_{jk} = \Pi^i_{jk} + P_j\delta^i_k + P_k\delta^i_j - P_m f^m_j f^i_k - P_m f^m_k f^i_j, \\ \hat{\Pi}_{lm} = \Pi_{lm} - P_t\Pi^t_{lm} - P_l P_m + \partial_l P_m + P_r f^r_l P_s f^s_m. \end{cases}$$

In this theorem, of course, the relation $\hat{\nabla}_k f^i_j = \nabla_k f^i_j = 0$ is true. Because

$$\begin{aligned} \hat{\nabla}_k f^i_j &= \partial_k f^i_j - f^i_l(\varphi^l_{kj} + P_k\delta^l_j - P_m f^m_k f^l_j) + (\varphi^i_{kl} + P_k\delta^i_l - P_m f^m_k \delta^i_l) f^l_j \\ &= \nabla_k f^i_j . \end{aligned}$$

Any two almost complex projective connections θ and $\hat{\theta}$ satisfying $\mathcal{P}(\theta) = \mathcal{P}(\hat{\theta})$ are called mutually *complex-projectively related.* Prof. Y. Tashiro called the linear connections satisfying $(3.8)_1$ *HP-related* [5]. The HP-related connections are linear connections which have all HP-curves in common. Thus

COROLLARY. *If linear connections Π^i_{jk} and $\hat{\Pi}^i_{jk}$ are determined respectively by two almost complex projective connections which are mutually complex-projectively related, then they are mutually HP-related.*

It is obvious that (3.8) is a generalization of a law of change of frames in infinite hyperplane in a theory of projectively connected real manifold.

§4 Holonomy mappings

Let us consider a C^1-curve C: $x^i = x^i(t)$ on M. A curve $\tilde{C}$ on P is called the horizontal lift of C if it satisfies following two conditions, that is (1) $\pi(\tilde{C}) = C$ and (2) the tangent vector to $\tilde{C}$ at each point of $\tilde{C}$ belongs to the horizontal distribution Φ^h. The condition (2) can be restated as follows: the tangent vector field along $\tilde{C}$ is given by $\frac{dx^i}{dt}X_i$. And so, a curve $\tilde{C}$: $x^i = x^i(t)$, $u^i = u^i(t)$, $v^1 = v^1(t)$, $v^2 = v^2(t)$ on P is the horizontal lift of C : $x^i = x^i(t)$ on M if and only if the condition

$$(4.1)\qquad \begin{cases} \dfrac{du^k}{dt} + \varphi^k_{ml}\dfrac{dx^m}{dt}u^l + \dfrac{dx^k}{dt}v^1 + f^k_m\dfrac{dx^m}{dt}v^2 = 0, \\ \dfrac{dv^1}{dt} + P_{ml}\dfrac{dx^m}{dt}u^l + q_m\dfrac{dx^m}{dt}v^1 + q_m f^m_r\dfrac{dx^r}{dt}v^2 = 0, \\ \dfrac{dv^2}{dt} - P_{mr}\dfrac{dx^m}{dt}f^r_l u^l - q_m f^m_r\dfrac{dx^r}{dt}v^1 + q_m\dfrac{dx^m}{dt}v^2 = 0 \end{cases}$$

is satisfied.

Let there be given a curve C joining two points A and B in M. As the equation (4.1) is a simultaneous linear homogeneous differential equation with respect to (u^i, v^1, v^2), the horizontal lift of C which passes through a point P in $\pi^{-1}(A)$ is uniquely determined. Further, $\pi^{-1}(B)\cap\tilde{C}$ becomes an only point Q. Then we can define a mapping $\sigma_c\colon \pi^{-1}(A)\to\pi^{-1}(B)$ by $\sigma_c(P)=Q$. It is obvious that σ_c is a diffeomorphism from $\pi^{-1}(A)$ to $\pi^{-1}(B)$. Thus we call the mapping σ_c *a holonomy mapping* corresponding to a curve C.

THEOREM 4.1. *By a holonomy mapping σ_c corresponding to a curve C joining two points A and B in a manifold M endowed with an almost complex projective connection, mutually complex-projective points in $\pi^{-1}(A)$ are mapped into mutually complex-projective points in $\pi^{-1}(B)$. That is to say, if $P\overset{c^*}{\barwedge}Q$ in $\pi^{-1}(A)$, then $\sigma_c(P)\overset{c^*}{\barwedge}\sigma_c(Q)$ in $\pi^{-1}(B)$.*

PROOF. With respect to a local canonical coordinate system, the equation of a horizontal lift $\tilde{C}$ of C which passes through a point P in $\pi^{-1}(A)$ can be represented by $(x^i(t), u^i(t), v^1(t), v^2(t))$. Any points on $\tilde{C}$ satisfies the equation (4.1). On the other hand, by the definition of $\tilde{C}^*$, we find

$$\tilde{C}^*P=(x^i(t),\ \alpha u^i(t)+\beta f^i_j u^j(t),\ \alpha v^1(t)-\beta v^2(t),\ \beta v^1(t)+\alpha v^2(t))\,. \tag{4.2}$$

Direct calculation will show us that these points satisfy also the equation (4.1), for example, the left hand of the first term of (4.1) is reduced to

$$\begin{aligned}&\frac{d}{dt}[\alpha u^k+\beta f^k_j u^j]+\varphi^k_{ml}\frac{dx^m}{dt}(\alpha u^l+\beta f^l_j u^j)+\frac{dx^k}{dt}(\alpha v^1-\beta v^2)+f^k_m\frac{dx^m}{dt}(\beta v^1+\alpha v^2)\\ &=\beta\frac{dx^m}{dt}u^l[\partial_m f^k_l-f^k_r\varphi^r_{ml}+\varphi^k_{mr}f^r_l]\\ &=0\,.\end{aligned}$$

The rest terms are verified by similar calculation. Hence the Theorem 4.1 is proved.

From the Theorem 4.1, the holonomy mapping σ_c can be induced in the almost complex projective bundle $\tilde{P}$. We denote it by $\tilde{\sigma}_c$. Then it follows that the mapping $\tilde{\sigma}_c$ maps each point of the complex projective space $\widetilde{\pi^{-1}(A)}$ into $\widetilde{\pi^{-1}(B)}$ holomorphically.

Now, let us define a 2-dimensional distribution $\mathfrak{D}$ in P by $\mathfrak{D}=Y\oplus Y'$. Then

LEMMA *The distribution $\mathfrak{D}$ is always integrable, and its integral manifold D*

passing through a point P_0 is a set of all points which are complex-projective to P_0 in a fiber.

Proof. Infact, from (3.4), it follows directly that

$$[Y, Y'] = 0 . \tag{4.3}$$

Hence the distribution $\mathfrak{D}$ is integrable. Therefore, each point of the integral manifold D of the distribution $\mathfrak{D}$, which passes through a point $P_0(x_0^k, u_0^k, v_0^1, v_0^2)$, is written, by solving its differential equations, as

$$\begin{cases} x^k = x_0^k , \\ u^k = u_0^k(e^t \cos s) + u_0^m f_m^k(e^t \cos s) , \\ v^1 = v_0^1(e^t \cos s) - v_0^2(e^t \sin s) , \\ v^2 = v_0^1(e^t \sin s) + v_0^2(e^t \cos s) , \end{cases} \tag{4.4}$$

where t and s are parameters. It is clear that each point given by (4.4) is complex-projective to P_0.

Theorem 4.2. *Let C be a C^1-curve joining two points A and B in a manifold M endowed with an almost complex projective connection θ, and $\sigma_{(\theta)}$ be a holonomy mapping corresponding to the curve C and the connection θ, that is $\sigma_{(\theta)}: \pi^{-1}(A) \to \pi^{-1}(B)$. If $\hat{\theta}$ is another almost complex projective connection admitting the distribution $\mathcal{P}(\theta)$ in common with θ, then $\sigma_{(\theta)}(P)$ and $\sigma_{(\hat{\theta})}(P)$ are mutually complex-projective for each point P in $\pi^{-1}(A)$.*

Proof. The local basis $\{X_i\}$ of the horizontal distribution $\Phi^h(\theta)$ is given by (3.1). If $\{\hat{X}_i\}$ is the local basis of the distribution $\Phi^h(\hat{\theta})$ with respect to the $\hat{\theta}$, the relations (3.7) and (3.4) give us

$$\hat{X}_i = X_i - P_i Y + P_m f_i^m Y' . \tag{4.5}$$

On the other hand (3.1) and (3.4) lead us to

$$[X_i, Y] = 0 \quad \text{and} \quad [X_i, Y'] = 0 , \tag{4.6}$$

by virtue of (2.9). In the manifold M, however, we may consider only the points on the given curve C. Hence we denote by $P(C)$ a submanifold of P whose projection to M is the given curve C. Put $X = \dfrac{dx^i}{dt} X_i$ and $\mathcal{H} = X \oplus Y \oplus Y'$ in the $P(C)$, then we have that X is a vector field on $P(C)$ and $\mathcal{H}$ is a 3-dimensional distribution on $P(C)$. And we have that $\mathcal{H}$ is integrable because of (4.3) and (4.6). Hence we denote by H an integral manifold of $\mathcal{H}$ passing through a point P which exists in $\pi^{-1}(A)$. Since the vector field X is horizontal, Y and

Y' are vertical, $H \cap \pi^{-1}(B)$ coincides with a 2-dimensional submanifold D which is given by the preceding Lemma and is defined in $\pi^{-1}(B)$.

Now it is clear that $\sigma_{(\theta)}(P)$ exists in the D. The relation (4.5) shows us that $\sigma_{(\hat{\theta})}(P)$ exists also in the same D. Hence the proof is complete from the Lemma.

THEOREM 4.3. *Let M be a manifold endowed with an almost complex projective connection, A and B be two points in M, C_1 and C_2 be arbitrary C^1-curves joining A and B in M. If the distribution $\mathcal{P}(\theta)$ is integrable on the almost complex projective vector bundle over M, then holonomy mappings σ_{c_1} and σ_{c_2} map any point P_0 in $\pi^{-1}(A)$ into the points which are mutually complex-projective in $\pi^{-1}(B)$, that is to say, $\sigma_{c_1}(P_0) \overset{c*}{\wedge} \sigma_{c_2}(P_0)$ for each point P_0 in $\pi^{-1}(A)$.*

PROOF. Since the distribution $\mathcal{P}(\theta)$ includes the horizontal distribution $\Phi^h_{(\theta)}$ and is integrable, a horizontal lift of any curve joining A and B is included in an integral manifold S_0 of $\mathcal{P}(\theta)$ passing through the point P_0. Hence, we have $\sigma_{c_1}(P_0) \in S_0 \cap \pi^{-1}(B)$ and $\sigma_{c_2}(P_0) \in S_0 \cap \pi^{-1}(B)$. On the other hand $S_0 \cap \pi^{-1}(B) \subset D$ where D is a certain integral manifold of the distribution $\mathfrak{D} = Y \oplus Y'$ which is a vertical part of $\mathcal{P}(\theta)$. Thus the lemma shows us that

$$\sigma_{c_1}(P_0) \overset{c*}{\wedge} \sigma_{c_2}(P_0) .$$

§ 5. An algegraic condition for the distribution $\mathcal{P}(\theta)$ to be integrable

Let us consider a condition for the distribution $\mathcal{P}(\theta)$ to be integrable on an almost complex projective vector bundle over the manifold M endowed with an almost complex projective connection. The definition (3.5) and the relation (4.3) and (4.6) show us that a condition for the $\mathcal{P}(\theta)$ to be integrable is $[X_i, X_j] \in \mathcal{P}(\theta)$. For $[X_i, X_i]$, after some complicated calculations, we obtain

$$\left\{\begin{aligned} [X_i, X_j] = &\{R_{jik}{}^h + \delta^h_j \Pi_{ik} - \delta^h_i \Pi_{jk} + f^h_i f^m_k \Pi_{jm} - f^h_j f^m_k \Pi_{im} \\ &\qquad + (\Pi^r_{ij} - \Pi^r_{ji})(\delta^h_r \delta^m_k - f^h_r f^m_k) q_m\} u^k \frac{\partial}{\partial u^h} \\ &+ (\Pi^h_{ji} - \Pi^h_{ij}) v^1 \frac{\partial}{\partial u^h} + f^h_r (\Pi^r_{ji} - \Pi^r_{ij}) v^2 \frac{\partial}{\partial u^h} \\ &+ [\nabla_j \Pi_{ik} - \nabla_i \Pi_{jk} - \Pi_{mk}(\Pi^m_{ij} - \Pi^m_{ji}) + q_m q_k (\Pi^m_{ij} - \Pi^m_{ji}) \\ &- q_r f^r_m q_l f^l_k (\Pi^m_{ij} - \Pi^m_{ji}) + q_r \{R_{jik}{}^r - \delta^r_i \Pi_{jk} + \delta^r_j \Pi_{ik} + f^r_i f^m_k \Pi_{jm} \\ &- f^r_j f^m_k \Pi_{im} + \delta^r_k (\Pi_{ij} - \Pi_{ji}) - f^r_k (\Pi_{im} f^m_j - \Pi_{jm} f^m_i)\}] u^k \frac{\partial}{\partial v^1} \end{aligned}\right.$$

$$
(5.1)\quad \left\{
\begin{aligned}
&-\{(\Pi_{ij}-\Pi_{ji})+q_r(\Pi^r_{ij}-\Pi^r_{ji})\}v^1\frac{\partial}{\partial v^1}\\
&-\{(\Pi_{im}f^m_j-\Pi_{jm}f^m_i)+q_r f^r_m(\Pi^m_{ij}-\Pi^m_{ji})\}v^2\frac{\partial}{\partial v^1}\\
&-f^t_k[\nabla_j\Pi_{it}-\nabla_i\Pi_{jt}-\Pi_{mt}(\Pi^m_{ij}-\Pi^m_{ji})+q_m q_t(\Pi^m_{ij}-\Pi^m_{ji})\\
&\qquad -q_r f^r_m q_l f^l_t(\Pi^m_{ij}-\Pi^m_{ji})\\
&\qquad +q_r\{R_{jit}{}^r-\delta^r_i\Pi_{jt}+\delta^r_j\Pi_{it}+f^r_i f^m_t\Pi_{jm}-f^r_j f^m_t\Pi_{im}\\
&\qquad\qquad +\delta^r_t(\Pi_{ij}-\Pi_{ji})-f^r_t(\Pi_{im}f^m_j-\Pi_{jm}f^m_i)\}]u^k\frac{\partial}{\partial v^2}\\
&+\{(\Pi_{im}f^m_j-\Pi_{jm}f^m_i)+q_r f^r_m(\Pi^m_{ij}-\Pi^m_{ji})\}v^1\frac{\partial}{\partial v^2}\\
&-\{(\Pi_{ij}-\Pi_{ji})+q_r(\Pi^r_{ij}-\Pi^r_{ji})\}v^2\frac{\partial}{\partial v^2}\,,
\end{aligned}\right.
$$

where

$$
(5.2)\qquad R_{jik}{}^h=\partial_j\Pi^h_{ik}-\partial_i\Pi^h_{jk}+\Pi^h_{jm}\Pi^m_{ik}-\Pi^h_{im}\Pi^m_{jk}\,,
$$

and ∇ implies a covariant derivative with respect to the linear connection Π^i_{jk}. The relation (5.1) implies that $[X_i, X_j]\in\Phi^V$. Hence the condition $[X_i, X_j]\in \mathcal{P}(\theta)$ is equivalent to $[X_i, X_j]=\alpha_{ij}Y+\beta_{ij}Y'$ for certain α_{ij} and β_{ij}. This is rewritten as

$$
(5.3)\quad [X_i, X_j]=(\alpha_{ij}\delta^h_k+\beta_{ij}f^h_k)u^k\frac{\partial}{\partial u^h}+(\alpha_{ij}v^1-\beta_{ij}v^2)\frac{\partial}{\partial v^1}+(\alpha_{ij}v^2+\beta_{ij}v^1)\frac{\partial}{\partial v^2}\,.
$$

Hence, comparing (5.3) with (5.1), we get $\Pi^h_{ij}=\Pi^h_{ji}$, $\alpha_{ij}=\Pi_{ji}-\Pi_{ij}$ and $\beta_{ij}=\Pi_{im}f^m_j-\Pi_{jm}f^m_i$. From these, we obtain

THEOREM 5.1. *Let M be a manifold endowed with an almost complex projective connection. The distribution $\mathcal{P}(\theta)$ defined by* (3.5) *is integrable on the almost complex projective vector bundle over M if and only if the following conditions for the connection are satisfied*

$$
(5.4)\qquad \left\{
\begin{aligned}
&\Pi^h_{ij}=\Pi^h_{ji}\,,\\
&\nabla_j\Pi_{ik}=\nabla_i\Pi_{jk}\,,\\
&R_{jik}{}^h+\delta^h_j\Pi_{ik}-\delta^h_i\Pi_{jk}+f^h_i f^m_k\Pi_{jm}-f^h_j f^m_k\Pi_{im}\\
&\qquad\qquad +\delta^h_k(\Pi_{ij}-\Pi_{ji})-f^h_k(\Pi_{im}f^m_j-\Pi_{jm}f^m_i)=0\,.
\end{aligned}\right.
$$

From the Theorem 5.1, Theorem 2.1 together with the theorem of Tashiro [5], we obtain directly

THEOREM 5.2. *Let M be a manifold stated in the Theorem* 5.1. *If the distribution $\mathcal{P}(\theta)$ is integrable, then the manifold M is a complex manifold and is HP-flat, and the connection Π^i_{jk} is a symmetric f-connection.*

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