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Adjacent Uniformities

By

Norman LEVINE and Louis J. NACHMAN

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1. Introduction.

A proximity space (X, δ) is totally bounded or completely bounded iff there is exactly one uniformity which generates the proximity relation δ [6]. A natural question to ask is "what properties are possessed by proximity relations which are generated by exactly n uniformities where n is finite and more than one?" Such proximities do not exist as was proved recently by Reed and Thron [4, Corollary 2.1.3]. In our attack on this problem we were led to a concept which has some interest in its own right; the concept of immediate predecessor or immediate successor in the collection of uniformities on a set X.

Given two uniformities $\mathscr{U}$ and $\mathscr{V}$ on a set X, we will say that $\mathscr{U}$ is an immediate predecessor of $\mathscr{V}$ (and $\mathscr{V}$ is an immediate successor of $\mathscr{U}$) when $\mathscr{U}\subseteq\mathscr{V}$, $\mathscr{U}\neq\mathscr{V}$ and $\mathscr{U}=\mathscr{W}$ or $\mathscr{W}=\mathscr{V}$ when $\mathscr{W}$ is a uniformity on X and $\mathscr{U}\subseteq\mathscr{W}\subseteq\mathscr{V}$. We shall write $\mathscr{U}$ imp $\mathscr{V}$ when $\mathscr{U}$ is an immediate predecessor of $\mathscr{V}$. $\mathscr{U}$ and $\mathscr{V}$ are called adjacent if $\mathscr{U}$ imp $\mathscr{V}$ or $\mathscr{V}$ imp $\mathscr{U}$.

In this paper, E will generically denote an equivalence relation on X and $\mathscr{U}(E)$ will denote the uniformity with $\{E\}$ as base. $\mathscr{U}(d)$ denotes the uniformity generated in the standard manner by the pseudo-metric d.

In §2, we show that any uniformity whose topology is not discrete has an immediate successor. An example is given to show that not every uniformity has an immediate successor, and the general construction of immediate successors is examined.

In §3, we show that every non-trivial uniformity has an immediate predecessor. A necessary and sufficient condition for $\mathscr{U}(E^*)$ imp $\mathscr{U}(E)$ is given and used to construct an example to show that the construction used in §2 is not the only way to generate immediate successors.

In §4 we discuss some questions about the relations between "imp" defined for uniformities, proximities, and completely regular topologies.

We conclude with §5 in which some unanswered questions are listed.

2. Immediate successors.

We first prove that for any uniformity $\mathscr{U}$ whose topology is not discrete, there is a uniformity $\mathscr{V}$ such that $\mathscr{U}$ imp $\mathscr{V}$. The following lemma is useful in proving this theorem. Notationally, if $x \in X$, we denote by $E(x)$ the equivalence relation $(\{x\} \times \{x\}) \cup (\mathscr{C}\{x\} \times \mathscr{C}\{x\})$ where $\mathscr{C}$ denotes the complement operator.

LEMMA 2.1 *Suppose that $\mathscr{U}$ is a uniformity for X and $x^* \in X$. Suppose further that $\mathscr{V}$ is a uniformity for X and $\mathscr{U} \subseteq \mathscr{V} \subseteq \mathscr{U} \vee \mathscr{U}(E(x^*))$. Then* (1) $\mathscr{V} = \mathscr{U} \vee \mathscr{U}(E(x^*))$ *or* (2) *there exists a $y \neq x^*$ such that $(x^*, y) \in U \cap V$ whenever $U = U^{-1} \in \mathscr{U}$ and $V = V^{-1} \in \mathscr{V}$ and $U \cap E(x^*) \subseteq V$.*

PROOF: Suppose (2) does not hold. Then there exist $U = U^{-1} \in \mathscr{U}$, $V = V^{-1} \in \mathscr{V}$, $U \cap E(x^*) \subseteq V$, but $(x^*, y) \in U \cap V$ implies that $x^* = y$. Clearly $U \cap E(x^*) \subseteq V \cap U$. We show now that $V \cap U \subseteq U \cap E(x^*)$. Let $(a, b) \in U \cap V$. If $a = x^*$ or $b = x^*$, then $a = b$ and $(a, b) \in U \cap E(x^*)$. If $a \neq x^* \neq b$, then $(a, b) \in U \cap E(x^*)$. Thus $V \cap U = U \cap E(x^*)$. But $\mathscr{U} \subseteq \mathscr{V}$ and hence $U \cap E(x^*) \in \mathscr{V}$ and therefore $E(x^*) \in \mathscr{V}$. Thus $\mathscr{U} \vee \mathscr{U}(E(x^*)) \subseteq \mathscr{V}$ and (1) holds.

THEOREM 2.2 *Let $\mathscr{U}$ be a uniformity for X and $\{x^*\} \notin \mathscr{T}(\mathscr{U})$. Then $\mathscr{U}$ imp $\mathscr{U} \vee \mathscr{U}(E(x^*))$.*

PROOF: Clearly $\mathscr{U} \subseteq \mathscr{U} \vee \mathscr{U}(E(x^*))$. If $E(x^*) \in \mathscr{U}$ then it follows that $\mathscr{U} \neq \mathscr{U} \vee \mathscr{U}(E(x^*))$.

Next, suppose that $\mathscr{U} \subseteq \mathscr{V} \subseteq \mathscr{U} \vee \mathscr{U}(E(x^*))$ and that $\mathscr{V} \neq \mathscr{U} \vee \mathscr{U}(E(x^*))$. We will show that $\mathscr{U} = \mathscr{V}$. It suffices to show that $V \circ V \in \mathscr{U}$ when $V = V^{-1} \in \mathscr{V}$. Now there exists a $U \in \mathscr{U}$ such that $U \cap E(x^*) \subseteq V$ and there exists a pseudo metric d for X such that $U \in \mathscr{U}(d) \subseteq \mathscr{U}$ [2]. Hence there exists an $\varepsilon > 0$ such that $W_\varepsilon = \{(x, y): d(x, y) < \varepsilon\} \subseteq U$. Thus $W_\varepsilon = W_\varepsilon^{-1}$. By lemma 2.1, there exists a $y \neq x^*$ such that $(x^*, y) \in W_\varepsilon \cap V$. Thus $0 \leqq d(x^*, y) < \varepsilon$; let $\delta = \varepsilon - d(x^*, y)$. It follows then that $\varepsilon \geqq \delta > 0$ and $W_\delta \subseteq W_\varepsilon$. It suffices to show that $W_\delta \subseteq V \circ V$. Let $(a, b) \in W_\delta$. If $a = x^* = b$, then $(a, b) \in V \circ V$. If $a \neq x^* \neq b$, then $(a, b) \in W_\varepsilon \cap E(x^*) \subseteq V \circ V$. If $a = x^* \neq b$, then $d(y, b) \leqq d(y, x^*) + d(x^*, b) = d(y, x^*) + d(a, b) < d(y, x^*) + \delta = \varepsilon$. Thus $(y, b) \in W_\varepsilon \cap E(x^*) \subseteq V$. Since $(x^*, y) \in V$ and $(y, b) \in V$, it follows that $(x^*, b) \in V \circ V$ and $(a, b) \in V \circ V$. The case $a \neq x^* = b$ is similarly treated.

In [4], a uniformity $\mathscr{U}$ on X is m-bounded (m an infinite cardinal) if every uniformly discrete subspace of $(X, \mathscr{U})$ has cardinality less then m. A subset D of X is called uniformly discrete with respect to $\mathscr{U}$ iff the gage of $\mathscr{U}$ contains a pseudo metric d such that for some $\varepsilon > 0$, the d-distance between distinct points of D is at least ε. (See def. 1.1 and theorem 1.2 of [4].)

It is shown in corollary 2.1.2 of [4] that if $\mathscr{U}$ is a uniformity on X, n an infinite cardinal, n^+ its successor, $\aleph_0 < m \leqq n^+$ and $\mathscr{U}$ is not n-bounded, then there are at least 2^n distinct uniformities $\mathscr{V}$ such that $\mathscr{V} \subseteq \mathscr{U}$ and $\delta(\mathscr{V}) = \delta(\mathscr{U})$.

We apply this result to show that not every uniformity has an immediate successor.

Example 2.3 Let $\mathscr{U}$ be the totally bounded uniformity which generates the discrete proximity on R, the reals. Let $\mathscr{U} \subseteq \mathscr{V}$ and $\mathscr{U} \neq \mathscr{V}$ where $\mathscr{V}$ is a uniformity for R. Then $\delta(\mathscr{U}) = \delta(\mathscr{V})$, $\mathscr{V}$ is not totally bounded and hence not $\aleph_0$-bounded. Letting $n^+ = \aleph_0^+ = m$ in the above result in [4], we have at least $2^{\aleph_0}$ distinct uniformities $\mathscr{W}$ such that $\mathscr{W} \subseteq \mathscr{V}$ and $\delta(\mathscr{W}) = \delta(\mathscr{V})$. Thus $\delta(\mathscr{W}) = \delta(\mathscr{U})$ and hence $\mathscr{U} \subseteq \mathscr{W}$. Hence there is at least one $\mathscr{W}$ such that $\mathscr{U} \subseteq \mathscr{W} \subseteq \mathscr{V}$ and $\mathscr{U} \neq \mathscr{W} \neq \mathscr{V}$. Thus $\mathscr{U}$ has no immediate successor.

In example 2.3, we show that $\mathscr{U} \vee \mathscr{U}(E(x))$ is not the only way to generate immediate successors.

Along these lines, we have the following results.

Theorem 2.4 *Let* $\mathscr{U}$ *imp* $\mathscr{V}$ *where* $\mathscr{U}$ *and* $\mathscr{V}$ *are uniformities for* X. *There exists a pseudo metric* d *on* X *such that* $\mathscr{V} = \mathscr{U} \vee \mathscr{U}(d)$.

Proof: Let $\mathscr{D}(\mathscr{U})$ and $\mathscr{D}(\mathscr{V})$ be the gages of $\mathscr{U}$ and $\mathscr{V}$ respectively. Then $\mathscr{D}(\mathscr{U}) \subseteq \mathscr{D}(\mathscr{V})$, but $\mathscr{D}(\mathscr{U}) \neq \mathscr{D}(\mathscr{V})$. Let $d \in \mathscr{D}(\mathscr{V}) - \mathscr{D}(\mathscr{U})$. Then $\mathscr{U} \subseteq \mathscr{U} \vee \mathscr{U}(d) \subseteq \mathscr{V}$ and $\mathscr{U} \neq \mathscr{U} \vee \mathscr{U}(d)$. It follows then that $\mathscr{V} = \mathscr{U} \vee \mathscr{U}(d)$.

Generally, $\mathscr{U} \vee \mathscr{U}(d)$ is not an immediate successor to $\mathscr{U}$. For consider

Example 2.5 Let $X = R$, the reals and $\mathscr{U} = \{X \times X\}$. Let d be the metric for X defined by $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ if $x \neq y$. Clearly, $\mathscr{U}$ imp $\mathscr{U} \vee \mathscr{U}(d)$ is false.

3. Immediate predecessors.

To show that every non-trivial uniformity has an immediate predecessor we use the following construction. Notationally, if $\mathscr{A}$ is a family of set $\cap \mathscr{A} = \cap \{A : A \in \mathscr{A}\}$. As is usual if $S \subseteq X \times X$, $S[x] = \{y : (x, y) \in S\}$.

Suppose $\mathscr{U}$ is a uniformity for X and (x, y) is a point in $X \times X$ which is not in $\cap \mathscr{U}$. For each symmetric $U \in \mathscr{U}$ let $U^{\#} = U \cup (U[x] \times U[y]) \cup (U[y] \times U[x])$. Then routine computation shows that the collection of all such $U^{\#}$'s is a base for a uniformity for X which we denote by $\mathscr{U}^{\#}$. Clearly $\mathscr{U}^{\#}$ is a proper subsed of $\mathscr{U}$ and $(x, y) \in \cap \mathscr{U}^{\#}$.

We let $\mathscr{V}=\sup\{\mathscr{V}': \mathscr{V}'\subseteq\mathscr{U}$ and $(x, y)\in\bigcap\mathscr{V}'\}$.

THEOREM 3.1 *If $\mathscr{U}$ is a non-trivial uniformity for X then $\mathscr{U}$ has an immediate predecessor.*

PROOF: Since $\mathscr{U}$ is non-trivial there is an $(x, y)\in X\times X$ which is not in $\bigcap\mathscr{U}$. Letting $\mathscr{U}^{\#}$ and $\mathscr{V}$ be as above and noting that $(x, y)\in\bigcap\mathscr{V}$ we have $\mathscr{U}^{\#}\subseteq\mathscr{V}\subseteq\mathscr{U}$ and $\mathscr{V}\neq\mathscr{U}$. We claim $\mathscr{V}$ imp $\mathscr{U}$.

Suppose $\mathscr{V}\subseteq\mathscr{W}\subseteq\mathscr{U}$ and $\mathscr{W}\neq\mathscr{V}$. We will show that $\mathscr{W}=\mathscr{U}$. Since $\mathscr{V}$ is the largest subuniformity of $\mathscr{U}$ for which (x, y) is in every entourage, there is a closed symmetric W in $\mathscr{W}$ such that (x, y) is not in W.

Now suppose $U\in\mathscr{U}$. Then, since W is closed relative to $\mathscr{W}$ it is closed relative to $\mathscr{U}$. We can therefore find a symmetric U_0 in $\mathscr{U}$ such that $U_0\subseteq U$ and $[(U_0[x]\times U_0[y])\cup(U_0[y]\times U_0[x])]\cap W=\emptyset$. Then $(U_0)^{\#}\cap W=U_0\cap W\subseteq U_0\subseteq U$. But since $\mathscr{U}^{\#}\subseteq\mathscr{V}\subseteq\mathscr{W}$ and $(U_0)^{\#}\in\mathscr{U}^{\#}$, $(U_0)^{\#}\cap W$ is in $\mathscr{W}$ and thus so is U, completing the proof.

Suppose E is a non-trivial equivalence relation. In the next theorem we give necessary and sufficient conditions for $\mathscr{U}(E)$ to have an immediate predecessor which is generated by an equivalence relation.

THEOREM 3.2 *Let E and E^* be equivalence relations on X. Then $\mathscr{U}(E^*)$ imp $\mathscr{U}(E)$ iff there exist points x and y of X such that $E[x]\neq E[y]$ and $E^*=E\cup A\times A$ where $A=E[x]\cup E[y]$.*

PROOF: Suppose that $\mathscr{U}(E^*)$ imp $\mathscr{U}(E)$. Then $E\subseteq E^*$, but $E\neq E^*$. Thus $E[x]\subseteq E^*[x]$ for all $x\in X$ and there exists an x_0 such that $E[x_0]\neq E^*[x_0]$. Let $y_0\in E^*[x_0]-E[x_0]$. Then $E[x_0]\neq E[y_0]$ and $E^*[x_0]=E^*[y_0]$. Let $F=E\cup A\times A$ where $A=E[x_0]\cup E[y_0]$. Then $E\subseteq F\subseteq E^*$ and $E\neq F$. Thus $\mathscr{U}(E^*)\subseteq\mathscr{U}(F)\subseteq\mathscr{U}(E)$. Since $\mathscr{U}(F)\neq\mathscr{U}(E)$, it follows that $\mathscr{U}(E^*)=\mathscr{U}(F)$. Thus $E^*=F$.

Conversely, suppose that E^* is related to E as in the statement of the theorem. By theorem 3.1, there exists a uniformity $\mathscr{V}$ such that $\mathscr{U}(E^*)\subseteq\mathscr{V}$ and $\mathscr{V}$ imp $\mathscr{U}(E)$. We will show that $\mathscr{U}(E^*)=\mathscr{V}$ by showing that $\mathscr{V}\subseteq\mathscr{U}(E^*)$. Let $V=V^{-1}\in\mathscr{V}$; it suffices to show that $E^*\subseteq V\circ V\circ V$. We first show that $(E^*-E)\cap V\neq\emptyset$. Suppose on the contrary that $(E^*-E)\cap V=\emptyset$. Then $E=E\cap V=E^*\cap V\in\mathscr{V}$ and thus $E\in\mathscr{V}$. It follows then that $\mathscr{U}(E)\subseteq\mathscr{V}$ and hence $\mathscr{U}(E)=\mathscr{V}$, a contradiction. Hence $(E^*-E)\cap V\neq\emptyset$. Let $(a, b)\in(E^*-E)\cap V$. Then $(a, b)\in E[x]\times E[y]$ or $(a, b)\in E[y]\times E[x]$. Assume the former. Let $(c, d)\in E^*$. If $(c, d)\in E$, then $(c, d)\in V\subseteq V\circ V\circ V$ since $E\subseteq V$. If $(c, d)\notin E$, then $(c, d)\in E[x]\times E[y]$ or $(c, d)\in E[y]\times E[x]$. Again without loss of generality, assume the former. Then $(a, c)\in E$, $(b, d)\in E$ and $(a, b)\in V$.

Since $E \subseteq V$, it follows that $(c, d) \in V \circ V \circ V$.

We close this section with an example which shows that the construction used in theorem 2.2 is not the only way of generating immediate successors. This example is a direct application of theorem 3.2.

EXAMPLE 3.3 Again let $X=[0, 1]$ and let E be the equivalence relation generated by the partition $\{[0, 1/3), [1/3, 2/3], (2/3, 1]\}$ and let E^* be generated by $\{[0, 2/3], (2/3, 1]\}$. Then $\mathscr{U}(E^*)$ imp $\mathscr{U}(E)$ by theorem 3.2. Clearly there is no $x \in X$ for which $\mathscr{U}(E)=\mathscr{U}(E^*)\vee\mathscr{U}(E(x))$.

4. Adjacent uniformities, topologies, and proximities.

In this section we study the relationships between immediate sucessors in the lattice of uniformities and immediate sucessors in the corresponding lattices of proximities and completely regular topologies. We begin with an example to show that $\mathscr{U}$ imp $\mathscr{V}$ need not imply that $\mathscr{T}(\mathscr{U})$ imp $\mathscr{T}(\mathscr{V})$ in the lattice of completely regular topologies.

EXAMPLE 4.1 Let $X=[1, 2]$ and let $\mathscr{T}=\{O \mid 1 \notin O \text{ or } 1 \in O \text{ and } \mathscr{C}O \text{ is finite}\}$. Then $(X, \mathscr{T})$ is compact Hausdroff and hence completely regular. Suppose $\mathscr{T}^\#$ is a completely regular topology for X such that $\mathscr{T}\subseteq\mathscr{T}^\#$, $\mathscr{T}\neq\mathscr{T}^\#$. Then there exists a completely regular topology $\mathscr{T}^*$ for X for which $\mathscr{T}\subseteq\mathscr{T}^*\subseteq\mathscr{T}^\#$, $\mathscr{T}\neq\mathscr{T}^*\neq\mathscr{T}^\#$. To see this, let $O^\# \in \mathscr{T}^\#-\mathscr{T}$. Then $1 \in O^\#$ and $\mathscr{C}O^\#$ is not finite. Letting $\mathscr{T}(O^\#, \mathscr{C}O^\#)$ be the topology whose elements are $\emptyset$, X, $O^\#$, $\mathscr{C}O^\#$, we have $\mathscr{T}\subseteq\mathscr{T}\vee\mathscr{T}(O^\#, \mathscr{C}O^\#)\subseteq\mathscr{T}^\#$ since $O^\# \in \mathscr{T}^\#$ and $\mathscr{C}O^\# \in \mathscr{T}\subseteq\mathscr{T}^\#$. Also, $\mathscr{T}\neq\mathscr{T}\vee\mathscr{T}(O^\#, \mathscr{C}O^\#)$. Let $\mathscr{C}O^\#=A\cup B$ where $A\cap B=\emptyset$ and A and B are both infinite and let $\mathscr{T}^*=\mathscr{T}\vee\mathscr{T}(O^\#\cup A, B)$. Since $\mathscr{T}$ and $\mathscr{T}(O^\#\cup A, B)$ are completely regular, it follows that $\mathscr{T}^*$ is completely regular. $\mathscr{T}\subseteq\mathscr{T}^*$ and $\mathscr{T}\neq\mathscr{T}^*$ since $O^\#\cup A \in \mathscr{T}^*-\mathscr{T}$. Now A and B are in $\mathscr{T}$ and hence $O^\#\cup A$ and B are in $\mathscr{T}\vee\mathscr{T}(O^\#, \mathscr{C}O^\#)\subseteq\mathscr{T}^\#$. Thus $\mathscr{T}^*\subseteq\mathscr{T}^\#$. Finally, we show that $O^\# \notin \mathscr{T}^*$ and thus $\mathscr{T}^*\neq\mathscr{T}^\#$. Suppose $O^\# \in \mathscr{T}^*$. Now $1 \in O^\#$; by definition of $\mathscr{T}^*$, there exists a $U \in \mathscr{T}$ and $V \in \mathscr{T}(O^\#\cup A, B)$ such that $1 \in U\cap V\subseteq O^\#$ and hence $\mathscr{C}O^\#\subseteq\mathscr{C}U\cup\mathscr{C}V$. Since $\mathscr{C}U$ is finite and $\mathscr{C}V$ is B or $\emptyset$ and $\mathscr{C}O^\#=A\cup B$, we have a contradiction. Hence $\mathscr{T}^*\neq\mathscr{T}^\#$.

Now, since $\mathscr{T}$ is completely regular there is a uniformity $\mathscr{U}$ for X such that $\mathscr{T}(\mathscr{U})=\mathscr{T}$. Since $\mathscr{T}$ is not discrete, by theorem 2.2, $\mathscr{U}$ imp $\mathscr{U}\vee\mathscr{U}(E(1))$. But by the above argument, there is a completely regular topology $\mathscr{T}^*$ for which $\mathscr{T}(\mathscr{U})\subseteq\mathscr{T}^*\subseteq\mathscr{T}(\mathscr{U}\vee\mathscr{U}(E(1)))$, the inclusions being proper. Hence $\mathscr{T}(\mathscr{U})$ imp $\mathscr{T}(\mathscr{U}\vee\mathscr{U}(E(1)))$ is false.

As a corollary to Theorem 3.1 we can show that every non-trivial completely regular topology has an immediate predecessor in the lattice of completely regular topologies. A uniformity $\mathscr{U}$ is fine if it is the largest uniformity whose topology is $\mathscr{T}(\mathscr{U})$.

THEOREM 4.2 *Suppose $\mathscr{T}$ is a non-trivial completely regular topology for X. Then there is a completely regular topology $\mathscr{T}^*$ such that $\mathscr{T}^*$ imp $\mathscr{T}$ in the lattice of completely regular topologies on X.*

PROOF: Let $\mathscr{U}$ be the fine uniformity for $\mathscr{T}$. Then $\mathscr{U}$ is non-trivial. Let $\mathscr{V}$ be as in theorem 3.1, and let $\mathscr{T}^*=\mathscr{T}(\mathscr{V})$. Then $\mathscr{T}^*\subseteq\mathscr{T}$ and $\mathscr{T}^*\neq\mathscr{T}$ since x and y are separated in $\mathscr{T}$ but not in $\mathscr{T}^*$. We claim $\mathscr{T}^*$ imp $\mathscr{T}$.

Suppose $\mathscr{T}^*\subseteq\mathscr{T}^{\#}\subseteq\mathscr{T}$ and $\mathscr{T}^{\#}$ is completely regular. Let $\mathscr{W}$ be any uniformity which generates $\mathscr{T}^{\#}$. Then, since $\mathscr{T}^*\subseteq\mathscr{T}^{\#}$, $\mathscr{T}(\mathscr{V}\vee\mathscr{W})=\mathscr{T}^{\#}$. Clearly $\mathscr{V}\subseteq\mathscr{V}\vee\mathscr{W}\subseteq\mathscr{U}$. Hence $\mathscr{V}\vee\mathscr{W}$ is $\mathscr{U}$ or $\mathscr{V}\vee\mathscr{W}$ is $\mathscr{V}$. In the former case $\mathscr{T}^{\#}=\mathscr{T}^*$ and in the latter case $\mathscr{T}^{\#}=\mathscr{T}$. Hence $\mathscr{T}^*$ imp $\mathscr{T}$.

It is not our purpose here to study the relation δ imp η where δ and η are proximity relations on X, but merely to relate what we know about the relation $\mathscr{U}$ imp $\mathscr{V}$ to the study of proximity classes.

$\mathscr{U}(\delta)$ denotes the totally bounded uniformity whose proximity is δ, while $\delta(\mathscr{U})$ denotes the proximity of the uniformity $\mathscr{U}$ defined by $A\delta(\mathscr{U})B$ iff $U[A]\cap B\neq\emptyset$ for all $U\in\mathscr{U}$.

THEOREM 4.3 *Suppose $\mathscr{U}$ and $\mathscr{V}$ are uniformities for X and $\mathscr{U}$ imp $\mathscr{V}$. Then $\delta(\mathscr{U})$ imp $\delta(\mathscr{V})$ iff $\delta(\mathscr{U})\neq\delta(\mathscr{V})$.*

PROOF: $\delta(\mathscr{U})$ imp $\delta(\mathscr{V})$ implies that $\delta(\mathscr{U})\neq\delta(\mathscr{V})$ by definition. Conversely, $\mathscr{U}$ imp $\mathscr{V}$ implies that $\mathscr{U}\subseteq\mathscr{V}$ which implies that $\delta(\mathscr{U})\leq\delta(\mathscr{V})$. If $\delta(\mathscr{U})$ imp $\delta(\mathscr{V})$ is false, then there exists a proximity relation δ^* for which $\delta(\mathscr{U})\leqq\delta^*\leqq\delta(\mathscr{V})$ and $\delta(\mathscr{U})\neq\delta^*\neq\delta(\mathscr{V})$. Then it follows from theorem 21.23 of [6] that $\delta(\mathscr{U}\vee\mathscr{U}(\delta^*))=\delta^*$. Since $\delta^*\leqq\delta(\mathscr{V})$, we have $\mathscr{U}(\delta^*)\subseteq\mathscr{U}(\delta(\mathscr{V}))\subseteq\mathscr{V}$. Then $\mathscr{U}\vee\mathscr{U}(\delta^*)\subseteq\mathscr{V}$ since $\mathscr{U}\subseteq\mathscr{V}$ and $\mathscr{U}(\delta^*)\subseteq\mathscr{V}$. Therefore $\mathscr{U}\subseteq\mathscr{U}\vee\mathscr{U}(\delta^*)\subseteq\mathscr{V}$ and all inclusions are proper since $\delta(\mathscr{U})\neq\delta^*\neq\delta(\mathscr{V})$, a contradiction.

The following theorem is a reasonable partial converse to theorem 4.3.

THEOREM 4.4 *Suppose that δ and η are proximity relations on X. If δ imp η, then $\mathscr{U}(\delta)$ imp $\mathscr{U}(\eta)$.*

PROOF: This theorem is a direct result of the fact that the functorial isomorphism from the category of totally bounded uniform spaces to the category of proximity spaces is an order isomorphism.

The following theorems give us additional insight into the relationship between the "imp" relationship and proximity classes.

LEMMA 4.5 *If $\mathscr{U}$ imp $\mathscr{V}$, $\delta(\mathscr{U}) \neq \delta(\mathscr{V})$ and $\mathscr{U}$ is totally bounded (precompact) then $\mathscr{V}$ is totally bounded.*

PROOF. Suppose $\mathscr{V}$ is not totally bounded. Then $\mathscr{U}(\delta(\mathscr{V})) \subseteq \mathscr{V}$ and the inclusion is proper. Since $\mathscr{U} \subseteq \mathscr{V}$, $\delta(\mathscr{U}) \leq \delta(\mathscr{V})$ and therefore $\mathscr{U} \subseteq \mathscr{U}(\delta(\mathscr{V})) \subseteq \mathscr{V}$ because $\mathscr{U}$ is totally bounded. But $\mathscr{U}$ imp $\mathscr{V}$ and $\mathscr{U}(\delta(\mathscr{V})) \neq \mathscr{V}$; thus $\mathscr{U} = \mathscr{U}(\delta(\mathscr{V}))$ and hence $\delta(\mathscr{U}) = \delta(\mathscr{U}(\delta(\mathscr{V}))) = \delta(\mathscr{V})$, a contradiction.

As an immediate corollary we have

COROLLARY 4.6 *If $\mathscr{U}$ imp $\mathscr{V}$ and $(X, \mathscr{U})$ is compact, then $\mathscr{V}$ is totally bounded.*

Although the collection of uniformities generating a given proximity relation usually does not have a largest member, such a supremum does exist in some of the more interesting cases, e.g. pseudometric proximities, completely bounded proximities, etc. We examine this possibility in the next theorems.

THEOREM 4.7 *If $\mathscr{U}$ is the largest uniformity which generates $\delta(\mathscr{U})$ and $\mathscr{V}$ is the smallest uniformity which generates $\delta(\mathscr{V})$ then $\mathscr{U}$ imp $\mathscr{V}$ iff* (i) $\mathscr{U} \subseteq \mathscr{V}$ *and* (ii) $\delta(\mathscr{U})$ imp $\delta(\mathscr{V})$.

PROOF: Suppose $\mathscr{U}$ imp $\mathscr{V}$. Then $\mathscr{U} \subseteq \mathscr{V}$ and $\delta(\mathscr{U}) \neq \delta(\mathscr{V})$. Hence by 4.3, $\delta(\mathscr{U})$ imp $\delta(\mathscr{V})$.

On the other hand suppose $\mathscr{U} \subseteq \mathscr{V}$ and $\delta(\mathscr{U})$ imp $\delta(\mathscr{V})$. Since $\delta(\mathscr{U}) \neq \delta(\mathscr{V})$, $\mathscr{U} \neq \mathscr{V}$. Let $\mathscr{U} \subseteq \mathscr{W} \subseteq \mathscr{V}$. Then $\delta(\mathscr{U}) \leqq \delta(\mathscr{W}) \leqq \delta(\mathscr{V})$ and $\delta(\mathscr{U}) = \delta(\mathscr{W})$ or $\delta(\mathscr{V}) = \delta(\mathscr{W})$. If $\delta(\mathscr{U}) = \delta(\mathscr{W})$ then $\mathscr{W} \subseteq \mathscr{U}$ and hence $\mathscr{W} = \mathscr{U}$. Similarly if $\delta(\mathscr{V}) = \delta(\mathscr{W})$ then $\mathscr{V} \subseteq \mathscr{W}$ and hence $\mathscr{W} = \mathscr{V}$. Thus $\mathscr{W} = \mathscr{U}$ or $\mathscr{W} = \mathscr{V}$ and $\mathscr{U}$ imp $\mathscr{V}$.

COROLLARY 4.8 *If $\mathscr{U}$ is pseudometrizable (or $(X, \mathscr{U})$ is fine, or $\delta(\mathscr{U})$ is completely bounded) and $\mathscr{V}$ is totally bounded then $\mathscr{U}$ imp $\mathscr{V}$ iff* (i) *$\mathscr{U} \subseteq \mathscr{V}$ and* (ii) *$\delta(\mathscr{U})$ imp $\delta(\mathscr{V})$.*

We close this section with a theorem involving topological considerations. If $\mathscr{U}$ is a uniformity and E an equivalence relation, then $\mathscr{U} \vee \mathscr{U}(E)$ may or may not be adjacent to $\mathscr{U}$. This next theorem considers this question in a special case.

THEOREM 4.9 *Suppose that $\mathscr{T}(\mathscr{U})$ is connected and that $\mathscr{T}(\mathscr{V})$ is compact on the set X and suppose that $\mathscr{U}$ imp $\mathscr{V}$. Then $\mathscr{V} = \mathscr{U} \vee \mathscr{U}(E)$ for some equiva-*

lence relation E iff $\mathscr{T}(\mathscr{V})$ is disconnected.

PROOF: Suppose that $\mathscr{V}=\mathscr{U}\vee\mathscr{U}(E)$ for some equivalence relation E. Since $\mathscr{U}\neq\mathscr{V}$, it follows that $E\neq X\times X$. But E is open and closed in $(X\times X, \mathscr{T}(\mathscr{U}(E))\times\mathscr{T}(\mathscr{U}(E)))$ and hence open and closed in $(X\times X, \mathscr{T}(\mathscr{V})\times\mathscr{T}(\mathscr{V}))$. Thus $(X\times X, \mathscr{T}(\mathscr{V})\times\mathscr{T}(\mathscr{V}))$ is disconnected and hence so is $(X, \mathscr{T}(\mathscr{V}))$.

Conversely, let $(X, \mathscr{T}(\mathscr{V}))$ be disconnected. Then there exists $\emptyset\neq A\neq X$, A being both open and closed relative to $\mathscr{T}(\mathscr{V})$. Let $E=A\times A\cup\mathscr{C}A\times\mathscr{C}A$. Then E is a $\mathscr{T}(\mathscr{V})\times\mathscr{T}(\mathscr{V})$-neighborhood of Δ and hence $E\in\mathscr{V}$ since $(X, \mathscr{T}(\mathscr{V}))$ is compact. Thus $\mathscr{U}\vee\mathscr{U}(E)\subseteq\mathscr{V}$. But $E\notin\mathscr{U}$; for if $E\in\mathscr{U}$, then $(X, \mathscr{T}(\mathscr{U}))$ would be disconnected since $A=E[A]$ and $\mathscr{C}A=E[\mathscr{C}A]$. Now $\mathscr{U}\subseteq\mathscr{U}\vee\mathscr{U}(E)\subseteq\mathscr{V}$ and since $\mathscr{U}\neq\mathscr{U}\vee\mathscr{U}(E)$, it follows that $\mathscr{U}\vee\mathscr{U}(E)=\mathscr{V}$.

5. Some questions.

The reader may wish to try his hand at some of the following:

(1) If $\mathscr{U}$ imp $\mathscr{V}$, need $\mathscr{V}=\mathscr{U}\vee\mathscr{U}(E)$ for some equivalence relation E? (See theorem 4.9).

(2) Can $\mathscr{U}$ imp $\mathscr{V}$ and $\delta(\mathscr{U})=\delta(\mathscr{V})$ hold simultaneously?

(3) Study the relation δ imp η for proximity relations on a set X.

The Ohio State University,
Oakland University

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