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A Remark on Harmonic Quasiconformal Mappings

By

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K-quasiconformal mappings of Riemann surfaces was investigated by P. J. Kiernan in [2]. One of his interesting results is that harmonic K-quasiconformal mappings of certain Riemann surfaces are distance-decreasing. We shall discuss here harmonic K-quasiconformal mappings of n-dimensional Riemannian manifolds and generalize the above Kiernan's theorem to this case. In section 1 we review the theory of harmonic forms as found in [4]. Section 2 is devoted to get some lemmas which are used to prove our theorem in the last section. Concerning quasiconformal mappings, we use the fact given by H. Wu in [5].

§1. Vector bundle valued harmonic forms

Let M be an n-dimensional Riemannian manifold and E a vector bundle over M with a metric along fibres and covariant differentiation D_X compatible with the metric for any vector field X. $C^p(E)$ is the real vector space of all E-valued differential p-forms on M. Next, an operater ∂: $C^p(E)\to C^{p+1}(E)$ is defined by

$$(\partial\theta)(X_1, \cdots, X_{p+1})=\sum_{i=1}^{p+1}(-1)^{i+1}D_{X_i}(\theta(X_1, \cdots, \hat{X}^i, \cdots, X_{p+1}))$$
$$+\sum_{i<j}^{p+1}(-1)^{i+j}\theta([X_i, X_j], X_1, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_{p+1}),$$

where X_i's denote vector fields on M. The covariant derivative $D_X\theta$ of $\theta\in C^p(E)$ is an E-valued p-form given by

$$(D_X\theta)(X_1, \cdots, X_p)=D_X(\theta(X_1, \cdots, X_p))-\sum_{i=1}^{p}\theta(X_1, \cdots, \nabla_X X_i, \cdots, X_p),$$

where $\nabla_X X_i$ represents the convariant derivative of the vector field X_i in the Riemannian manifold M. An operator ∂^* is defined as follows. Let $x\in M$ and $u_1, \cdots, u_{p-1}$ be any tangent vectors at x. We define

$$(\partial^*\theta)_x(u_1, \cdots, u_{p-1}) = -\sum_{k=1}^{n} (D_{e_k}\theta)_x(e_k, u_1, \cdots, u_{p-1}),$$

where $\{e_1, \cdots, e_n\}$ is an orthonormal base of the tangent space $T_x(M)$ at x and $\theta \in C^p(E)$. The Laplacian $\square$ for E-valued differential forms is given by $\square = \partial\partial^* + \partial^*\partial$. The scalar product of two E-valued p-forms θ and η is given by

$$<\theta, \eta>(x) = \sum_{i_1, \dots, i_p=1}^{n} <\theta(e_{i_1}, \cdots, e_{i_p}), \eta(e_{i_1}, \cdots, e_{i_p})>.$$

Now we have

THEOREM (MATSUSHIMA). *Let θ be an E-valued 1-form. Then*

$$<\square\theta, \theta> = \frac{1}{2} \Delta <\theta, \theta> + <D\theta, D\theta> + A,$$

where Δ denotes the Laplacian of the Riemannian manifold M and A denotes a smooth function in M defined as follows;

$$A(x) = \sum_{i,j} <\tilde{R}(e_i, e_j)\theta(e_j), \theta(e_i)> + \sum_i <\theta(S(e_i)), \theta(e_i)>,$$

where $\tilde{R}$ is the curvature tensor of D and S denotes the endomorphism of $T_x(M)$ defined by Ricci tensor S of M, i.e. $S(e_i) = \sum_k S_{ki} e_k$.

§2. Some preliminaries

We now list our notations. M and N denote n-dimensional Riemannian manifolds with metric connections ∇ and ∇' respectively. f is C^∞-mapping of M to N. E is the bundle induced by f from $T(N)$. Then E has the covariant differential operator D compatible with the metric deduced naturally from the Riemannian metric of N. Let $s = (s_i)$ be a frame of $T(N)$ over V where V is an open set in N. There exist 1-forms θ_{ij} on V such that $\nabla' s_i = \sum_{j=1}^{n} \theta_{ij} s_j$. If U is an open set in M with $f(U) \subset V$, then D is given as $D(s_i(f(x))) = \sum_{j=1}^{n} f^*(\theta_{ij}) s_j(f(x))$.

Obviously, the differential f_* of f is regarded as an E-valued 1-form on M. Let $x \in M$, we take orthonormal bases $\{e_1, \cdots, e_n\}$ and $\{f_1, \cdots, f_n\}$ of $T_x(M)$ and $T_{f(x)}(N)$. For the curvature tensor R of ∇ and R' of ∇', we set

$$R'_{abcd} = R'(f_a, f_b, f_c, f_d), \quad S_{ij} = \sum_{k=1}^{n} R(e_k, e_i, e_k, e_j).$$

In terms of the base $\{f_a\}$ and the dual base $\{e^i\}$ of $\{e_i\}$ the E-valued 1-form f_* is represented as

$$(f_*)_x=\sum_{a,i}\phi_i^a(x)\,e^i\otimes f_a.$$

In this situation, we get the following lemma directly from the theorem in Section 1.

LEMMA 1. (EELLS *and* SAMPSON, KIERNAN)

$$(1)\qquad <\Box f_*, f_*>(x)=<Df_*, Df_*>(x)+\frac{1}{2}\Delta_x<f, f_*>$$
$$-\Sigma R'_{abcd}\phi_i^a\phi_j^b\phi_i^c\phi_j^d+\sum S_{ij}\phi_i^a\phi_j^a.$$

Next, we shall explain some properties of K-quasiconformal mappings according to [5] of H. Wu. Suppose that f is K-quasiconformal. Let $\lambda_1\leqq\cdots\leqq\lambda_n$ be eigenvalues at x of the symmetric matrix $G(x)=(G_{ij}(x))$, where $G_{ij}(x)=\sum_{a=1}^{n}\phi_i^a(x)\phi_j^a(x)$, by definition. Then we have $\left(\frac{\lambda_n}{\lambda_1}\right)^{\frac{1}{2}}\leqq K$ at each point $x\in M$. This implies *trace* $G\leqq nK^2(\det G)^{\frac{1}{n}}$. Evidently, *det* $G(x)=|\phi_i^a(x)|^2$ and trace $G(x)=\sum_{a,i}(\phi_i^a(x))^2=<f_*, f_*>(x)=\|f_*\|_x^2$. Summing up, we have

LEMMA 2.

$$(2)\qquad \|f_*\|^2\leqq nK^2J^{\frac{2}{n}},$$

where $J=\left|\,|\phi_i^a|\,\right|$.

We shall show another inequality about *det* G without the assumption that f is K-quasiconformal. If $\phi_i=(\phi_i^1,\cdots,\phi_i^n)$, it is evident that

$$(3)\qquad \begin{vmatrix} G_{ii} & G_{ij} \\ G_{ij} & G_{jj} \end{vmatrix}=\|\phi_i\|^2\|\phi_j\|^2-(<\phi_i, \phi_j>)^2.$$

In addition we put $H=\sum_{i<j}^{n}\begin{vmatrix} G_{ii} & G_{ij} \\ G_{ij} & G_{jj} \end{vmatrix}$, then

LEMMA 3.

$$(4)\qquad nJ^{\frac{4}{n}}\leqq 2H.$$

PROOF. As $J^2=\det G$, it is sufficient to show that

$$(det\ G)^2\left(\frac{n}{2}\right)^n \leqq H^n.$$

As it is well known, the positive definite symmetric matrix G is written as

$$G=\begin{pmatrix} p_{11} & & 0 \\ \vdots & \ddots & \\ p_{n1} & \cdots & p_{nn} \end{pmatrix}\begin{pmatrix} p_{11} & \cdots & p_{n1} \\ & \ddots & \vdots \\ 0 & & p_{nn} \end{pmatrix}=PP^t,$$

that is, $G_{ij}=\sum_{k=1}^{k\leqq i} p_{ik}p_{jk}(i\leqq j)$. Hence, we have

$$Det\ G=(det\ P)^2=\prod_{i=1}^{n}(p_{ii})^2.$$

On the other hand, it holds

$$\begin{aligned}\begin{vmatrix} G_{ii} & G_{ij} \\ G_{ij} & G_{jj} \end{vmatrix} &= (p_{i1}^2+\cdots+p_{ii}^2)(p_{j1}^2+\cdots+p_{jj}^2) \\ &\quad -(p_{i1}p_{j1}+\cdots+p_{ii}p_{ji})^2 \\ &\geqq p_{ii}^2 p_{jj}^2.\end{aligned}$$

This implies

$$H^n \geqq (\sum_{i<j} p_{ii}^2 p_{jj}^2)^n.$$

The problem is reduced to compering $\Pi(p_{ii})^4$ with the right hand side of the last inequality. Now define A_k by

$$(x_1x_2+\cdots+x_{k-1}x_k)^k=A_k x_1^2 x_2^2\cdots x_k^2+\cdots.$$

Then A_k is determined inductively by

$$A_k=k(k-1)A_{k-1}+\frac{k(k-1)^2}{2}A_{k-2}(A_2=1,\ A_3=6)$$

and it satisfies

$$A_k \geqq \left(\frac{k}{2}\right)^k \qquad (k\geqq 2).$$

Thus we have

$$H^n \geqq (\sum_{i<j} p_{ii}^2 p_{jj}^2)^n \geqq A_n \Pi(p_{ii})^4 \geqq \left(\frac{n}{2}\right)^n (det\ G)^2.$$

§3. Harmonic K-quasiconformal mappings

We are now in position to prove the following theorem which is a generalization of the Kiernan's result ([2], Theorem 6).

THEOREM. *Let M and N be n-dimensional Riemannian manifolds, and let f be a harmonic K-quasiconformal mapping of M to N such that the function $\|f_*\|^2=<f_*, f_*>$ has its maximum on M. Suppose that the sectional curvature of M is everywhere $\geqq -A$ and that the sectional curvature of N is everywhere $\leqq -B$, where A and B are positive constants, then*

$$(5)\qquad \|f_*(X)\|^2 \leqq n(n-1)\frac{A}{B}K^4\|X\|^2$$

for every vector $X \in T(M)$.

PROOF. Firstly we have $\Box f^*=0$. Let p be a maximum point of $\|f_*\|^2$. Using $\Delta_p\|f_*\|^2 \geqq 0$, (1) yields

$$(6)\qquad -\sum_{\substack{a,b,c,d\\ i,j}}^{n} R'_{abcd}\phi_i^a\phi_j^b\phi_i^c\phi_j^d \leqq -\sum_{a,i,j}^{n} S_{ij}\phi_i^a\phi_j^a.$$

By assumption, for a vector $X \in T_x(M)$

$$-(n-1)A\|X\|^2 \leqq \sum_{i,k}^{n} S_{ik}\xi^i\xi^k,$$

where $X=\Sigma\xi^i e_i$ with respect to an orthonormal base $\{e_i\}$ of $T_x(M)$. Making use of this, we get

$$(7)\qquad -\sum_{a,i,j}^{n} S_{ij}\phi_i^a\phi_j^a \leqq (n-1)A\sum_{a,i}^{n}\|\phi_i^a\|^2 \leqq (n-1)A\|f_*\|^2.$$

The left hand side of the inequality (6) can be written as

$$-\sum_{\substack{a,b,c,d\\ i,j}}^{n} R'_{abcd}\phi_i^a\phi_j^b\phi_i^c\phi_j^d = -\sum_{i,j}^{n} R'(p;\phi_i,\phi_j)\|\phi_i\wedge\phi_j\|^2.$$

Here $\|\phi_i\wedge\phi_j\|$ denotes the area of the parallelogram spanned by the vectors ϕ_i and ϕ_j and $R'(p;\phi_i,\phi_j)$ denotes the sectional curvature of N at $f(p)$ along the section spanned by ϕ_i and ϕ_j. By taking account of $\|\phi_i\wedge\phi_j\|^2=\|\phi_i\|^2\|\phi_j\|^2-(<\phi_i,\phi_j>)^2$, we obtain

$$(8)\qquad -\sum_{\substack{a,b,c,d\\ i,j}} R'_{abcd}\phi_i^a\phi_j^b\phi_i^c\phi_j^d \geqq 2B\sum_{i<j}\|\phi_i\wedge\phi_j\|^2 \geqq 2BH.$$

Using (7) and (8), (6) gives

$$2H \leqq (n-1)\frac{A}{B}\|f_*\|^2.$$

Combining (2) and (4) with this, we get

$$nJ^{\frac{4}{n}} \leqq (n-1)\frac{A}{B}\|f_*\|^2 \leqq n(n-1)\frac{A}{B}K^2 J^{\frac{2}{n}}.$$

Hence we have

$$J^{\frac{2}{n}} \leqq (n-1)\frac{A}{B}K^2.$$

Thus we find

$$(9) \qquad \langle f_*, f_* \rangle (p) = \|f_*\|_p^2 \leqq nK^2 J^{\frac{2}{n}} \leqq n(n-1)\frac{A}{B}K^4.$$

Now, let X be an arbitrary vector at some point $q \in M$. Then it follows

$$\|f_*(X)\|^2 = \Sigma G_{ij}(q)\zeta^i\zeta^j,$$

where $X = \Sigma \zeta^i e_i$ with respect to an orthonormal base $\{e_i\}$ of $T_q(M)$. By an orthonormal transformation T, the symmetric matrix $G(q) = (G_{ij}(q))$ is reduced to a diagonal matrix, that is,

$${}^tTGT = \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix}.$$

If we put $TX = Y = \Sigma \eta^i e_i$, then we get $\|Y\| = \|X\|$ and

$$\|f_*(X)\|^2 = \alpha_1(\zeta^1)^2 + \cdots + \alpha_n(\zeta^n)^2 \leqq trace\ G(q)\|X\|^2.$$

As *trace* $G(q) = \|f_*\|_q^2$, this implies

$$\|f_*(X)\|^2 \leqq \|f_*\|_q^2\|X\|^2 \leqq \|f_*\|_p^2\|X\|^2.$$

The desired result follows from this and (9).

COROLLARY. *In addition to the assumption of the theorem, we suppose that M is connected. If d_M and d_N denote the metrics induced from the Riemannian metrics respectively. Then it follows*

$$d_N(f(p_1), f(p_2)) \leqq \sqrt{n(n-1)\frac{A}{B} K^4}\, d_M(p_1, p_2)$$

for every $p_1, p_2 \in M$.

Proof. For any positive constant ε, there exists a curve $\gamma(t)(0 \leqq t \leqq 1)$ on M which satisfies $\gamma(0)=p_1$, $\gamma(1)=p_2$ and

$$d_M(p_1, p_2)+\varepsilon > \int_0^1 \left\| \left(\frac{d\gamma}{dt} \right) \right\| dt.$$

Therefore we get

$$\sqrt{n(n-1)\frac{A}{B} K^4}\,(d_M(p_1, p_2)+\varepsilon) > \int_0^1 \left\| f_*\left(\frac{d\gamma}{dt} \right) \right\| dt \geqq d_N(f(p_1), f(p_2)).$$

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