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A Characterization of G_δ Graphs

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§1. Introduction

Let f be a mapping on X into Y where X and Y are topological spaces. The graph G of f is said to be G_δ provided G is a G_δ set in $X\times Y$. F. Burton Jones and E. S. Thomas, Jr [1], [2] obtained some results on connected G_δ graphs which are interesting to the study of almost continuous mappings.

We first, in this paper, give a necessary and sufficient condition in order that the graph of mapping be a G_δ graph. Next we show by means of an example that the converse of a result in [1] is not true.

Notations. Let A be a set in X. Let denote by $A^{(1)}$ the derived set of A, that is the set of all accumulation points of A, by $A^{(2)}$ the derived set of $A^{(1)}, \cdots$, by $A^{(n)}$ the derived set of $A^{(n-1)}$ $(n=1, 2, \cdots)$. Here we note that the sequence $\{A^{(1)}, A^{(2)}, \cdots\}$ is decreasing.

Let f be a mapping on X into Y and let $Z\subset X$. Then $f|Z$ denotes the mapping which, for $x\in Z$, assigns $f(x)$ to x.

Most of terms can be extracted from Kuratowski's Topology [3].

§2. The characterization theorem

THEOREM. *Let f be a mapping on X into Y where X and Y are metrizable topological spaces. Let N denote the set of all points x of X such that f is discontinuous at x. Then the condition that there exists a positive integer n for which the graph of $f|N^{(n)}$ is a G_δ set in $X\times Y$ is necessary and sufficient in order that f have a G_δ graph.*

The following lemmas are useful in proving this theorem.

LEMMA 1. *If G_i $(i=1, 2, \cdots, n)$ are G_δ sets in X, then the union $\bigcup_{i=1}^{n} G_i$ is a G_δ set in X.*

It is well-known that Lemma 1 is true. On the other hand, the union of an arbitrary number of G_δ sets is not necessarily a G_δ set, but the next result is true.

LEMMA 2. *Let T be an arbitrary index set. If $G_t=\bigcap_{i=1}^{\infty} G_{ti}(t \in T)$ where G_{ti} are open sets in X and $G_{t1} \cap G_{s1}=\phi$ for every pair of indexes $t \neq s$ in T, then the union $\bigcup_{t \in T} G_t$ is a G_δ set in X.*

PROOF. Without loss of generality, we may assume that the sequences $\{G_{t1}, G_{t2}, \cdots\}$ are decreasing. This and the condition in the lemma imply that if $t \neq s$, then for every pair i, j, $G_{ti} \cap G_{sj} \subset G_{t1} \cap G_{s1}=\phi$ and hence $G_{ti} \cap G_{sj} =\phi$.

To prove this lemma, it is sufficient to show that $\bigcup_{t \in T} G_t=\bigcap_{i=1}^{\infty}(\bigcup_{t \in T} G_{ti})$. Let $p \in \bigcap_{i=1}^{\infty}(\bigcup_{t \in T} G_{ti})$. Then for each i there is an integer $t(p, i)$ such that $p \in G_{t(p,i)i}$. By the preceding fact, $t(p, i)$ is independent on i. Let t_p denote $t(p, i)$. Then we have

$$p \in \bigcap_{i=1}^{\infty} G_{t_p i} \subset \bigcup_{t \in T} G_t \text{ and hence } \bigcap_{i=1}^{\infty}(\bigcup_{t \in T} G_{ti}) \subset \bigcup_{t \in T} G_t.$$

On the other hand, we have

$$G_t=\bigcap_{i=1}^{\infty} G_{ti} \subset \bigcap_{i=1}^{\infty}(\bigcup_{t \in T} G_{ti}) \text{ and hence } \bigcup_{t \in T} G_t \subset \bigcap_{i=1}^{\infty}(\bigcup_{t \in T} G_{ti}).$$

Thus we have $\bigcup_{t \in T} G_t=\bigcap_{i=1}^{\infty}(\bigcup_{t \in T} G_{ti})$, which completes the proof.

LEMMA 3. *Let f, X, Y and N be as in Theorem. Then the graph of $f|(X-\bar{N})$ is a G_δ set in $X \times Y$.*

PROOF. Let G denote the graph of $f|(X-\bar{N})$. Since $f|(X-\bar{N})$ is continuous by the definition of N, it follows that $\bar{G} \cap \{(X-\bar{N}) \times Y\}=G$ where $\bar{G}$ is the closure of G in $X \times Y$. Since in a metric space each closed set is a G_δ set, it follows that $\bar{G}=\bigcap_{i=1}^{\infty} G_i$ where G_i are open in $X \times Y$. Hence we have

$$G=(\bigcap_{i=1}^{\infty} G_i) \cap \{(X-\bar{N}) \times Y\}=\bigcap_{i=1}^{\infty}[G_i \cap \{(X-\bar{N}) \times Y\}].$$

Thus the graph of $f|(X-\bar{N})$ is a G_δ set in $X \times Y$.

PROOF of THEOREM. To prove the sufficiency, suppose that there exists a positive integer n such that the graph of $f|N^{(n)}$ is a G_δ set in $X \times Y$.

First, since by Lemma 3, the graph $f|(X-\bar{N})$ is a G_δ set in $X \times Y$, then by Lemma 1 it is sufficient to show that the graph of $f|\bar{N}$ is a G_δ set in $X \times Y$.

Now $\bar{N}$ is the union of the derived set $N^{(1)}$ and the set N_1 of all isolated points of N. Since each point of N_1 is isolated, so is each point of the graph of $f|N_1$. Therefore there exists a family $\boldsymbol{F}$ of open sets in $X\times Y$ such that every element of $\boldsymbol{F}$ contains just one point of the graph of $f|N_1$ and any two elements of $\boldsymbol{F}$ are mutually separated. Moreover, each point of the graph of $f|N_1$ is a G_δ set in $X\times Y$. Therefore, by Lemma 2 the graph of $f|N_1$ is a G_δ set in $X\times Y$. Hence by Lemma 1, it is sufficient to show that the graph of $f|N^{(1)}$ is a G_δ set in $X\times Y$.

Next, since $N^{(1)}$ is closed, $N^{(1)}$ is the union of $N^{(2)}$ and the set N_2 of all isolated points of $N^{(1)}$. Then, by the same method as in the case of $f|N_1$ we can prove that the graph of $f|N_2$ is a G_δ set in $X\times Y$. Hence by Lemma 1, it is sufficient to show that the graph of $f|N^{(2)}$ is a G_δ set in $X\times Y$.

After the n-th step of the above argument, it is sufficient to show that a graph of $f|N^{(n)}$ is a G_δ set in $X\times Y$. This is the hypothesis. Thus the sufficiency is proved.

To prove the necessity, suppose that the graph of f is an intersection $\bigcap_{i=1}^{\infty} G_i$ of open sets G_i in $X\times Y$. Let n be a positive integer. Since $N^{(n)}$ is closed in X, it follows that $N^{(n)}=\bigcap_{i=1}^{\infty} H_i$ where H_i are open sets in X. Then we have

$$\text{the graph of } f|N^{(n)}=\bigcap_{i=1}^{\infty}\{G_i\cap(H_i\times Y)\},\ \text{and hence}$$

the graph of $f|N^{(n)}$ is a G_δ set in $X\times Y$. Thus Theorem is completey proved.

COROLLARY. *Let f, X, Y and N be as in Theorem. Then if there exists a positive integer n such that $N^{(n)}=\phi$, then f has a G_δ graph.*

PROOF. We may assume that n is the smallest integer such that $N^{(n)}=\phi$. Then every point of $N^{(n-1)}$ is isolated, and hence the graph of $f|N^{(n-1)}$ is a G_δ set in $X\times Y$. Therefore, by Theorem f has a G_δ graph.

§ 3. The example

In this section, let f be a function on I into I where I is the unit interval and let G be the graph of f. F. B. Jones and E. S. Thomas, Jr. have proved the following fact:

If G is a connected G_δ graph, then G is nowhere dense in I^2.

In the following, we give an example to show that a connected and nowhere dense graph G is not necessarily a G_δ set in I^2.

LEMMA 4. *Let $S=M_1\cup M_2$ be a decomposition of a complete metric space S into M_1 and M_2 such that both M_1 and M_2 are dense in S and M_1 is countable. Then any G_δ set containing M_1 contains uncountable points of M_2.*

PROOF. The countable set M_1 is not a G_δ set. For suppose, on the contrary, that M_1 is a G_δ set. Then the complement M_2 of M_1 is a F_σ set. Hence let $M_2=\bigcup_{i=1}^{\infty} F_i$ where F_i are closed sets. Let $\boldsymbol{F}$ be the family consisting of all F_i and all points of M_1. Then $\boldsymbol{F}$ is a countable family of closed sets, and the union of elements of $\boldsymbol{F}$ is S. Since by the hyopthesis, S is a complete metric space, then by Baire theorem, some element of $\boldsymbol{F}$ contains an open set in S. This contradicts to the hypothesis that M_1 and M_2 are dense in S. Therefore, M_1 is not a G_δ set.

Next let D be any G_δ set containing M_1 and let $D=\bigcap_{i=1}^{\infty} D_i$ where D_i are open sets. Suppose that D contains at most countable points $\{a_1, a_2, \cdots\}$ of M_2. Then it follows that

$M_1=\bigcap_{i=1}^{\infty}(D_i-a_i)$, and hence M_1 is a G_δ set, contradicting to the preceding result. Thus Lemma 4 is proved.

LEMMA 5. *Let A be a totally disconnected and dense in itself subset of I. Then the graph is connected provided f satisfies the condition as follows:*

If C is any component of $I-A$, then the graph of $f|\bar{C}$ is connected and meets both $I\times\{0\}$ and $I\times\{1\}$.

PROOF. Suppose, on the contrary, that G is not connected. Let $G=G_1\cup G_2$ be a separation of G. Let I_1 and I_2 be the projections of G_1 and G_2 into the domain I of f, respectively. Then without loss of generality, there exist a sequence $\{a_1, a_2, \cdots\}$ of points in I_1 and a point b of I_2 such that $\{a_1, a_2, \cdots\}$ converges to b. Moreover, we may assume that any point of $\{a_1, a_2, \cdots\}$ is not in A. For when a_n is in A, then a_n can be replaced by the following point. Let U_n be an ε-neighborhood of the point $(a_n, f(a_n))$ such that $U_n\cap G_2=\phi$ and $\varepsilon<\frac{1}{n}$. Then by hypothesis, the projection of U_n into the domain of f contains at least one component C of $I-A$, and the graph of $f|C$ meets with U_n. The projection of a point belonging to both U_n and the graph of $f|C$ is a point desired.

Now if some component D of $I-A$ contains an infinite number of points of $\{a_1, a_2, \cdots\}$, then we have $b\in\bar{D}$. Hence the point $(b, f(b))$ belongs to G_1 which contains the graph of $f|\bar{D}$, contradicting to $(b, f(b))\in G_2$. On the other hand, if every component of $I-A$ contains at most a finite number of points of $\{a_1, a_2, \cdots\}$, then to each n we assign a point P_n of the intersection of $I\times\{f(b)\}$ and the graph of f on the component of $I-A$ containing a_n. Then the sequence $\{P_1, P_2, \cdots\}$ is in G_1 and converges to $(b, f(b))$, contradicting to $(b, f(b)) \in G_2$. Thus Lemma 5 is proved.

EXAMPLE. Let K be the Cantor ternary set in I and let $\{(r_i, s_i)\,|\,i=1, 2, \cdots\}$ be the family of pairwise disjoint open intervals whose union is $I-K$. For each n, let B_n be an arc in $[r_n, s_n]\times I$ such that the following conditions hold:

(1) The points $(r_n, 0)$ and $(s_n, 0)$ are the end points of B_n.
(2) For each x in $[r_n, s_n]$, $B_n\cap(\{x\}\times I)$ is a point.
(3) B_n meets $I\times\{1\}$.

Let $H=K-\bigcup_{i=1}^{\infty}[r_i, s_i]$ and consider the set G defined as follows:

$$G=\bigcup_{n=1}^{\infty}B_n\cup\{(x, 1)\,|\,x \in H\}.$$

Then the set of all discontinuous points of the function defined by G is the set K. Here we note that K is a totally disconnected and dense in itself subset of I, and that both H and $K-H$ are dense in K and $K-H$ is countable.

Hence by Lemma 5 G is a connected graph, and by the theorem, p. 142, of [3] G is nowhere dense.

On the other hand, by Lemma 4 any G_δ set containing the graph of $f|K$ contains uncountable points of $H\times\{0\}$, and hence G is not a G_δ set in I^2. Thus by Theorem in §2, G is not a G_δ graph.

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References

[1] F. Burton Jones and E. S. Thomas, Jr., Connected G_δ graphs, Duke Math. J. 33 (1966), 341-346.
[2] E. S. Thomas, Jr., Some characterizations of functions of Baire class 1, Proc. Amer. Math. Soc. 17 (1966), 456-461.
[3] K. Kuratowski, Topology I, Academic Press, New York, 1966.