J. Math. Tokushima Univ.
Vol. 4. (1970), 37-42

On the Logarithmic ψ-Type and Logarithmic Lower ψ-Type of the Functions Represented by the Series $\sum_{n=1}^{\infty} a_n \cdot e^{\lambda_n \cdot \psi(x)}$

By

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(Received September 30, 1970)

1. Consider the function

$$F(x) = \sum_{n=1}^{\infty} a_n \cdot e^{\lambda_n \cdot \psi(x)} \tag{1.1}$$

where

$$\limsup_{n\to\infty} \frac{n}{\lambda_n} = D < \infty, \tag{1.2}$$

$\lambda_{n+1} > \lambda_n$, $\lambda_1 \geq 0$, $\lim_{n\to\infty} \lambda_n = \infty$, $\{a_n\}$ $(n=1, 2, 3, \dots)$ is a sequence of real positive numbers and $\psi(x)$ is an increasing continuous function of real variable x, defined either for all x on the real line or for all x in the interval $\eta < x < \infty$ where $-\infty < \eta < \infty$, satisfying the following conditions:

(1.3) (i) $\psi(x)$ tends to infinity as $x \to \infty$.

(ii) $\psi(x)$ assumes every value from $-\infty$ to $+\infty$.

(iii) $\psi(x)$ has an inverse, that is, if $y = \psi(x)$, then there exists a function ψ^{-1} such that $\psi^{-1}(y) = x$.

(iv) $\psi(x) - \psi(x-k) = \phi(x) = 0(1)$ for every fixed $k > 0$.

Let x_c and x_a be the abscissa of ψ-convergence and abscissa of absolute ψ-convergence respectively. If $x_c = \infty$, and $x_a = \infty$, then sum function $F(x)$ is defined and continuous for every $x > \eta$.

2. If, in the series (1.1), we put $\psi(\sigma) = \sigma$, we get a Dirichlet series, viz., $F(\sigma) = \sum_{n=1}^{\infty} a_n \cdot e^{\lambda_n \cdot \sigma}$ $(-\infty < \sigma < \infty)$ of real variable σ.

If, in the series (1.1), we substitute $\psi(r) = \log r$ and $\lambda_n = n-1$, we get a

Taylor series, viz.,

$$F(r)=\sum_{n=0}^{\infty} a_n r^n \qquad (0<r<\infty)$$

of real variable r.

When λ_n and $\psi(x)$ are respectively replaced by $n-1$ and x, the series takes the form of Taylor-D series, viz.,

$$F(x)=\sum_{n=0}^{\infty} a_n e^{nx}.$$

Here, in this paper, we have attempted to unify the various aspects of the two theories of entire functions defined by Taylor series and Dirichlet series respectively which have so far been treated separately by different workers in the two fields. Applications given at the end of each theorem, are intended to emphasize this fact.

3. Let $F(x)$ in (1.1) be the function of zero ψ-order [1, p. 18], then,

$$\limsup_{x\to\infty} \frac{\log\log F(x)}{\psi(x)} = \limsup_{x\to\infty} \frac{\log\log \mu(x,F)}{\psi(x)} = 0. \tag{3.1}$$

For this class of functions, ψ-type [1, p. 21] of $F(x)$ cannot be defined. To overcome this difficulty, we define logarithmic ψ-order and logarithmic lower ψ-order of this, by the relation

$$\lim_{x\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log F(x)}{\log \psi(x)} = \begin{matrix}\rho^*\\ \lambda^*\end{matrix} \qquad (1\leqslant \lambda^* \leqslant \rho^* < \infty)$$

We shall call ρ^* and λ^* as the logarithmic ψ-order and logarithmic lower ψ-order of function $F(x)$ respectively. Since, [1, p. 17]

$$\log F(x) \sim \log \mu(x,F) \qquad \text{as } x\to\infty,$$

where $\mu(x,F)$ is the maximum term of rank $\nu(x,F)$ of series (1.1).

$$\lim_{x\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log F(x)}{\log \psi(x)} = \lim_{x\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log \mu(x,F)}{\log \psi(x)} = \begin{matrix}\rho^*\\ \lambda^*\end{matrix}. \tag{3.2}$$

The definitions for logarithmic ψ-order and logarithmic lower ψ-order enable us to define logarithmic ψ-type and logarithmic lower ψ-type of the function $F(x)$ of zero ψ-order by the relation

$$\lim_{x\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log F(x)}{e^{\rho^* . \log\psi(x)}} = \begin{matrix}T^*\\ t^*\end{matrix} \qquad \text{for } 1<\rho^*<\infty.$$

Again, for the functions of finite ψ-order

$$\log F(x) \sim \log \mu(x, F) \qquad \text{as } x\to\infty,$$

therefore

$$\lim_{x\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log F(x)}{[\psi(x)]^{\rho^*}} = \lim_{x\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \mu(x, F)}{[\psi(x)]^{\rho^*}} = \begin{matrix}T^*\\ t^*\end{matrix}. \tag{3.3}$$

Application: The fact, [2, p. 253] that in case of Taylor series $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $\sum |a_n| r^n$ does not differ very much from its greatest term and that $\max_{|z|=r} |f(z)|$ lies between the two, can be extended to the case of Dirichlet series also. Thus, in view of section 2, the definitions for logarithmic type and logarithmic lower type in case of Taylor and Dirichlet series, follow from the definition given in (3.3) viz.,

$$\text{(i)} \qquad \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log M(r)}{(\log r)^{\rho^*}} = \begin{matrix}T^*\\ t^*\end{matrix}, \qquad M(r) = \max_{|z|=r} |f(z)|.$$

$$\text{(ii)} \qquad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log M(\sigma, f)}{\sigma^{\rho^*}} = \begin{matrix}T^*\\ t^*\end{matrix},$$

$$(s=\sigma+it) \text{ and } M(\sigma, f) = \text{l.u.b.} |f(s)|, \quad -\infty<t<\infty.$$

In my previous papers [2, 3] we have obtained expressions for logarithmic ψ-type and logarithmic lower ψ-type in terms of the coefficients of the series in (1.1). In this paper, we have proved some theorems connecting logarithmic ψ-order, logarithmic ψ-type and logarithmic ψ-growth numbers, defined in section 4.

4. We define logarithmic ψ-growth numbers ν^* and δ^* by

$$\lim_{x\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\lambda_{\nu(x,F)}}{[\psi(x)]^{\rho^*-1}} = \begin{matrix}\nu^*\\ \delta^*\end{matrix}. \tag{4.1}$$

Theorem 1. *Let*

$$F(x) = \sum_{n=1}^{\infty} a_n \cdot e^{\lambda_n \cdot \psi(x)}$$

be a function of logarithmic ψ-order ρ^* $(1<\rho^*<\infty)$, logarithmic ψ-type T^*

and logarithmic lower ψ-type t^*, then

$$\delta^* \leqslant \rho^* t^* \leqslant \rho^* T^* \leqslant \nu^* \tag{4.2}$$

where

$$\lim_{x\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\lambda_{\nu(x,F)}}{[\psi(x)]^{\rho^*-1}} = \begin{matrix}\nu^*\\ \delta^*\end{matrix}.$$

PROOF. It is known, [1, p. 16], that

$$\log \mu(x, F) = \log \mu(x_0, F) + \int_{x_0}^{x} \lambda_{\nu(t,F)} \cdot \psi'(t)\, dt.$$

Therefore, for almost all values of x, for which $\lambda_{\nu(x,F)}$ is continuous, we have

$$\frac{\mu'(x, F)}{\mu(x, F)} = \lambda_{\nu(x,F)} \cdot \psi'(x).$$

Therefore, by hypothesis, for any $\varepsilon > 0$ and $n > n_0$

$$(\delta^* - \varepsilon)\{\psi(x)\}^{\rho^*-1} < \lambda_{\nu(x,F)} < (\nu^* + \varepsilon)\{\psi(x)\}^{\rho^*-1}.$$

Or

$$(\delta^* - \varepsilon)\{\psi(x)\}^{\rho^*-1} \cdot \psi'(x) < \frac{\mu'(x, F)}{\mu(x, F)} < (\nu^* + \varepsilon)\{\psi(x)\}^{\rho^*-1} \cdot \psi'(x).$$

Integrating between the limits x_0 to x, we have

$$\frac{\delta^* - \varepsilon}{\rho^*} + 0(1) < \frac{\log \mu(x, F)}{\{\psi(x)\}^{\rho^*}} < \frac{\nu^* + \varepsilon}{\rho^*} + 0(1).$$

On proceeding to limits, we obtain the result in (4.2).

THEOREM 2. *Let $F(x) = \sum_{n=1}^{\infty} a_n \cdot e^{\lambda_n \cdot \psi(x)}$ be a function of logarithmic ψ-order ρ^* $(1 < \rho^* < \infty)$, then*

$$\limsup_{x\to\infty} \frac{\log \mu(x, F)}{\lambda_{\nu(x,F)} \cdot \psi(x)} \leqslant 1 - (\delta^*)^{1/(\rho^*-1)} \tag{4.3}$$

PROOF. It is known [2], that

$$\limsup_{x\to\infty} \frac{\log \lambda_n}{\log\left(\dfrac{1}{\lambda_n} \cdot \log \dfrac{1}{a_n}\right)} = \rho^* - 1.$$

Hence, for all values of $n>n_0$,

$$\frac{\log \lambda_n}{\log\left(\frac{1}{\lambda_n}\log\frac{1}{a_n}\right)}<\rho^*-1+\varepsilon.$$

Or

$$\log a_n<-(\lambda_n)^{\frac{\rho^*+\varepsilon}{\rho^*-1+\varepsilon}}.$$

Also,

$$\log \mu(x, F)=\log a_{\nu(x,F)}+\lambda_{\nu(x,F)}\cdot\psi(x)$$

$$<-(\lambda_{\nu(x,F)})^{\frac{\rho^*+\varepsilon}{\rho^*-1+\varepsilon}}+\lambda_{\nu(x,F)}\cdot\psi(x).$$

Or

$$\frac{\log \mu(x, F)}{\lambda_{\nu(x,F)}\cdot\psi(x)}<1-\left[\frac{\lambda_{\nu(x,F)}}{\{\psi(x)\}^{(\rho^*-1+\varepsilon)}}\right]^{1/(\rho^*-1+\varepsilon)}.$$

On proceeding to limits we obtain the result in (4.3).

THEOREM 3. *Let* $F(x)=\sum_{n=1}^{\infty}a_n\cdot e^{\lambda_n\cdot\psi(x)}$ *be a function of logarithmic* ψ*-order* ρ^* *and logarithmic lower* ψ*-order* λ^*, *then*

$$\liminf_{x\to\infty}\frac{\lambda_{\nu(x,F)}\cdot\psi(x)}{\log \mu(x, F)}\leqslant\lambda^*\leqslant\rho^*\leqslant\limsup_{x\to\infty}\frac{\lambda_{\nu(x,F)}\cdot\psi(x)}{\log \mu(x, F)}. \tag{4.4}$$

PROOF. Let

$$\lim_{x\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\ \frac{\lambda_{\nu(x,F)}\cdot\psi(x)}{\log \mu(x, F)}=\frac{c}{d}.$$

Therefore, for every $\varepsilon>0$ and for large x and consequently for large $\psi(x)$, we have

$$d-\varepsilon<\frac{\lambda_{\nu(x,F)}\cdot\psi(x)}{\log \mu(x, F)}<c+\varepsilon.$$

Also,

$\dfrac{\mu'(x, F)}{\mu(x, F)}=\lambda_{\nu(x,F)}\cdot\psi'(x)$ for all values of x. Therefore,

$$\frac{d-\varepsilon}{\psi(x)} \cdot \psi'(x) < \frac{\mu'(x, F)}{\mu(x, F) \cdot \log \mu(x, F)} < \frac{c+\varepsilon}{\psi(x)} \psi'(x).$$

Integrating the above inequality between the limits x_0 and x, and then dividing it by $\log \psi(x)$, we have

$$(d-\varepsilon)+0(1) < \frac{\log \log \mu(x, F)}{\log \psi(x)} < 0(1)+(c+\varepsilon).$$

Since $\log \mu(x, F) \sim \log F(x)$ as x tends to infinity, [1, p. 17], therefore proceeding to limits, we obtain the result in (4.4).

APPLICATION: Results similar to those of the theorems mentioned above, hold good in case of TAYLOR series and DIRICHLET series also.

Finally I wish to thank Professor R.S.L. Srivastava, Head of Mathematics Department, Indian Institute of technology, Kanpur, for guidance and help in the preparation of this paper.

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