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On Almost Complex Projective Structures

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In the present paper, we shall study an almost complex projective structure which is closely related to the concept of the holomorphically projective change of affine connections.

After some preliminaries, in section 2 we introduce the notion of the almost complex projective structure and examine its relationship with affine connections of some type.

In section 3, we construct a so-called almost complex projective tangent bundle by giving transition functions in explicit form. Also, it is shown that this bundle is regarded as an associated bundle of the almost complex projective structure.

An exposition of an almost complex projective connection is given in section 4, and section 5 contains a definition of a canonical form on the almost complex projective structure and investigations of some properties concerning it. Finally, the last section is devoted to consideration on a normal connection and the local flatness of the almost complex projective structure.

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§1. Complex projective spaces

Let N be an n-dimensional complex projective space with a complex homogeneous coordinate system $(\xi^1, \dots, \xi^n, \xi^0)$. Let $PL(n, C)$ be the complex projective transformation group of N and $H(n, C)$ the isotropy subgroup of $PL(n, C)$ which leaves invariant the point $O=(0, \dots, 0, \xi^0)$, $\xi^0 \neq 0$. The action of $GL(n+1, C)$ on C^{n+1} induces the action on the complex projective space N and we can regard $PL(n, C)$ as $GL(n+1, C)$ modulo its center. If (s^A_B)[1] $\in GL(n+1, C)$, we shall denote the induced complex projective transformation by $\{(s^A_B)\}$. It is obvious that we can take $(a^a; a^a_b; a_b)$ as a local complex

1) In the paper the indices A, B run over the range $1, \dots, n, 0$; a, b, c the range $1, \dots, n$; h, i, j, k, l the range $1, \dots, 2n$; a^*, b^*, c^* the range $n+1, \dots, 2n$; and for example, a^* means $a+n$.

coordinate system in the neighborhood of the identity of $PL(n, C)$ which consists of $\{\{(s_B^A)\}; (s_B^A) \in GL(n+1, C)\ s_0^0 \neq 0\}$, where we set

$$a^a = \frac{s_0^a}{s_0^0}, \quad a_b^a = \frac{s_b^a}{s_0^0}, \quad a_b = \frac{s_b^0}{s_0^0}. \tag{1.1}$$

The above neighborhood contains $H(n, C)$ and we can regard $H(n, C)$ as the following

$$H(n, C) = \left\{ \left(\begin{array}{c|c} a_b^a & 0 \\ \hline a_b & 1 \end{array} \right) \in GL(n+1, C) \right\}, \tag{1.2}$$

and the operation of $H(n, C)$ on N is given, in terms of the homogeneous coordinate system $(\xi^1, \dots, \xi^n, \xi^0)$, by

$$\begin{cases} \bar{\xi}^a = \sum{}^{2)} a_b^a \xi^b, \\ \bar{\xi}^0 = \sum a_b \xi^b + \xi^0. \end{cases} \tag{1.3}$$

If we set $z^a = \frac{\xi^a}{\xi^0}$ for $(\xi^1, \dots, \xi^n, \xi^0) \in N$, $\xi^0 \neq 0$, we can take $(z^1, \dots, z^n)$ as a local coordinate in the neighborhood of $O \in N$ defined by $\xi^0 \neq 0$. Then, the operation of $H(n, C)$ may be written by

$$\bar{z}^a = \frac{\sum a_b^a z^b}{1 + \sum a_b z^b} = \sum a_b^a z^b - \frac{1}{2} \sum (a_b^a a_c + a_c^a a_b) z^b z^c + \cdots. \tag{1.4}$$

Put

$$\xi^\lambda = \eta^\lambda + \sqrt{-1}\, \zeta^\lambda, \quad \bar{\xi}^\lambda = \bar{\eta}^\lambda + \sqrt{-1}\, \bar{\zeta}^\lambda,$$
$$a_b^a = p_b^a + \sqrt{-1}\, q_b^a, \quad a_b = p_b + \sqrt{-1}\, q_b,$$
$$z^a = x^a + \sqrt{-1}\, y^a, \quad \bar{z}^a = \bar{x}^a + \sqrt{-1}\, \bar{y}^a,$$

then (1.2), (1.3) and (1.4) can be represented as the following form

$$H(n, C) = \left\{ \left(\begin{array}{cc|cc} p_b^a & -q_b^a & \multicolumn{2}{c}{0} \\ q_b^a & p_b^a & & \\ \hline p_b & -q_b & 1 & 0 \\ q_b & p_b & 0 & 1 \end{array} \right) \in GL(2n+2, R) \right\}, \tag{1.2'}$$

$$\begin{pmatrix} \bar{\eta}^1 \\ \vdots \\ \bar{\eta}^n \\ \bar{\zeta}^1 \\ \vdots \\ \bar{\zeta}^n \\ \bar{\eta}^0 \\ \bar{\zeta}^0 \end{pmatrix} = \left(\begin{array}{cc|cc} p_b^a & -q_b^a & \multicolumn{2}{c}{0} \\ q_b^a & p_b^a & & \\ \hline p_b & -q_b & 1 & 0 \\ q_b & p_b & 0 & 1 \end{array} \right) \begin{pmatrix} \eta^1 \\ \vdots \\ \eta^n \\ \zeta^1 \\ \vdots \\ \zeta^n \\ \eta^0 \\ \zeta^0 \end{pmatrix}, \tag{1.3'}$$

2) In this section, $\sum$ means the summation from 1 to n over the repeated indices.

and

$$(1.4)' \quad \begin{cases} \bar{x}^a = \sum(p_b^a x^b - q_b^a y^b) - \dfrac{1}{2}\sum\{A_{bc}^a x^b x^c - B_{bc}^a(x^b y^c + x^c y^b) - A_{bc}^a y^b y^c\} + \cdots \\ \bar{y}^a = \sum(q_b^a x^b + p_b^a y^b) - \dfrac{1}{2}\sum\{B_{bc}^a x^b x^c + A_{bc}^a(x^b y^c + x^c y^b) - B_{bc}^a y^b y^c\} + \cdots \end{cases}$$

where $A_{bc}^a = p_b^a p_c - q_b^a q_c + p_c^a p_b - q_c^a q_b$, $B_{bc}^a = p_b^a q_c + q_b^a p_c + p_c^a q_b + q_c^a p_b$.

§2. Almost complex projective structures

Let M be a $2n$-dimensional manifold[3]. Let U and V be neighborhoods of origin O in R^{2n}. Two mappings f: $U \to M$ and g: $V \to M$ determine the same *k-jet* at O if they have the same partial derivatives up to order k at O. The k-jet determined by f is denoted by $j_o^k(f)$. If f is a diffeomorphism of a neighborhood of O onto an open subset of M, then the 2-jet $j_o^2(f)$ is called a *2-frame* at $p=f(O)$. The set of 2-frames of M will be denoted by $P^2(M)$.

Let $G^2(2n)$ be the set of 2-frames $j_o^2(g)$ at $O \in R^{2n}$, where g is a diffeomorphism from a neighborhood of O in R^{2n} onto a neighborhood in R^{2n}. Then, $G^2(2n)$ is a Lie group with multiplication defined by the composition of jets, i.e,

$$j_o^2(g) \circ j_o^2(g') = j_o^2(g \circ g').$$

In fact, it is represented as

$$(2.1) \qquad G^2(2n) = \{(S_j^i, S_{jk}^i); (S_j^i) \in GL(2n, R), S_{jk}^i = S_{kj}^i\},$$

and multiplication is given by the formula

$$(2.2) \qquad (S_j^i, S_{jk}^i) \circ (\bar{S}_j^i, \bar{S}_{jk}^i) = (\textstyle\sum^{4)} S_l^i \bar{S}_j^l, \sum S_l^i \bar{S}_{jk}^l + \sum S_{lh}^i \bar{S}_j^l \bar{S}_k^h).$$

It is an easy matter to verify that $P^2(M)$ is a principal fibre bundle over M with structure group $G^2(2n)$ and natural projection π^2, $\pi^2(j_o^2(f)) = f(O)$.

Let $P(M)$ be the usual frame bundle over M with natural projection π. It is well known that any local coordinate ststem $(u^1, \dots, u^{2n})$ on an open subset U of M induces a local coordinate system (u^i, u_j^i) on $\pi^{-1}(U)$. Furthermore, we can give a local coordinate system (u^i, u_j^i, u_{jk}^i) on $(\pi^2)^{-1}(U)$ so that the operation of $G^2(2n)$ on $(\pi^2)^{-1}(U)$ is represented as

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3) In the present paper we shall restrict attention to manifolds which are $2n$-dimensional and of class C^∞. We assume further in the paper that any geometric object, for example, any mapping or any tensor field, is of class C^∞.

4) In this and next sections, $\sum$ means summation from 1 to $2n$ over the repeated indices, unless otherwise indicated.

$$(2.3)\qquad (u^i,\ u^i_j,\ u^i_{jk})\circ(S^i_j,\ S^i_{jk})=(u^i,\ \sum u^i_l S^l_j,\ \sum u^i_l S^l_{jk}+\sum u^i_{lh} S^l_j S^h_k),$$

and $\pi^2_1((u^i, u^i_j, u^i_{jk}))=(u^i, u^i_j)$, where π^2_1 is the natural projection of $P^2(M)$ onto $P(M)$ (that is, π^2_1 projects $j^2_o(f)$ on $j^1_o(f)$). The local coordinate system (u^i, u^i_j, u^i_{jk}) is called *a local coordinate system on* $(\pi^2)^{-1}(U)$ *induced by* (u^i). In this paper, we sometimes employ these local coordinate systems without notice.

If we put

$$(2.4)\qquad H^2(n, C)=\left\{(S^i_j, S^i_{jk})\in G^2(2n)\ \middle|\ \begin{array}{l} (S^i_j)=\begin{pmatrix} p^a_b & -q^a_b \\ q^a_b & p^a_b \end{pmatrix}, \\ S^a_{bc}=-S^a_{b^*c^*}=S^{a^*}_{bc^*}=S^{a^*}_{b^*c}=-A^a_{bc}, \\ S^{a^*}_{bc}=-S^{a^*}_{b^*c^*}=S^a_{bc^*}=S^a_{b^*c^*}=B^a_{bc}, \\ A^a_{bc}=p^a_bp_c-q^a_bq_c+p^a_cp_b-q^a_cq_b, \\ B^a_{bc}=p^a_bq_c+q^a_bp_c+p^a_cq_b+q^a_cp_b. \end{array}\right\}$$

then, it is obvious that $H^2(n, C)$ is a closed Lie subgroup of $G^2(2n)$.

It is shown in the following that there exsists a natural isomorphism between $H(n, C)$ and $H^2(n, C)$. Since $H(n, C)$ acts on N, we can consider that $H(n, c)$ acts on R^{2n} by imbedding R^{2n} into N naturally and that a transformation given by (1.4)′ is a local diffeomorphism of a neighborhood of origin $O\in R^{2n}$ onto a neighborhood of O which sends O to O. Then we assign to each element of $H(n, C)$ the 2-jet $j^2_o(g)$, where g is the local diffeomorphism given by (1.4)′ with correspondence to the element. Now, it is easy to show that each 2-jet defined above is contained in $H^2(n, C)$ and the assignment is an isomorphism. In the sequel, we shall sometimes identify $H(n, C)$ and $H^2(n, C)$, and the following definition may be adequate.

Definition 1. *An almost complex projective structure on* M (simply, *a.c.p.-structure*) is a sub-bunddle $Pc(M)$ of $P^2(M)$ with structure group $H^2(n, C)$.

If an *a. c. p.*-structure $Pc(M)$ on M is given, we can regard the image $\pi^2_1(Pc(M))$ as an almost complex structure on M. We call this almost complex structure as *the underlying almost complex structure of* $Pc(M)$. Conversely, if M has an almost complex structure, we shall show that an *a. c. p.*-structure can be constructed on M naturally.

Lemma 1. *Let* M *have an almost complex sturucture* φ *and* ∇ *be a symmetric connection on* M. *Then, we can construct an a.c.p.-structure on* M *naturally.*

Proof. Let $\{U_\alpha\}_{\alpha\in I}$ be an open covering of M with such a property that there exists a local section s_α of the almost complex structure φ defined on each U_α. We may assume that each U_α is a local coordinate neighborhood with local coordinates $u_\alpha^1, \ldots, u_\alpha^{2n}$. Then, s_α is given by

$$s_\alpha(p)=(u_\alpha^i(p),\ u_{\alpha j}^{\ i}(p)), \tag{2.5}$$

Now, we define a local section $\bar{s}_\alpha$ of $P^2(M)$ on U_α by

$$\bar{s}_\alpha(p)=(u_\alpha^i(p),\ u_{\alpha j}^{\ i}(p),\ -\sum \Gamma_{\alpha\, e h}^{\ i}\, u_{\alpha j}^{\ e}(p) u_{\alpha k}^{\ h}(p)), \tag{2.6}$$

where $\nabla_{\frac{\partial}{\partial u_\alpha^i}} \dfrac{\partial}{\partial u_\alpha^j}=\Gamma_{\alpha\, i j}^{\ k} \dfrac{\partial}{\partial u_\alpha^k}$.

We can verify with ease that $\bar{s}_\alpha$ has no relation to the local coordinates $u_\alpha^1, \ldots, u_\alpha^{2n}$ and only depends on s_α. As we have $s_\alpha=s_\beta a_{\beta\alpha}$, $a_{\beta\alpha} \in GL(n, C)$ on $U_\alpha \cap U_\beta \neq \emptyset$, we can show $\bar{s}_\alpha=\bar{s}_\beta \bar{a}_{\beta\alpha}$, where $\bar{a}_{\beta\alpha}=(a_{\beta\alpha}, 0) \in H^2(n, C)$. This implies that there exists an *a.c.p.*-structure on M, and also it is easy to show that the *a.c.p.*-structure defined above does not depend on the choice of local sections s_α.

Definition 2. Let $Pc(M)$ be an *a.c.p.*-structure with an underlying almost complex structure φ and ∇ a symmetric connection on M. We call that ∇ belongs to $Pc(M)$ if, for any local section of the almost complex structure φ and ∇, a local section of $P^2(M)$ given by (2.6) is a local section of $Pc(M)$.

In the case of Lemma 1, ∇ also belongs to $Pc(M)$. Now, we shall describe a relation of two symmetric connections which belong to the same *a.c.p.*-structure.

Lemma 2. *Let $Pc(M)$ be an a.c.p.-structure with an underlying almost complex structure φ and ∇, ∇^* be symmetric connections on M. We assume furthermore that ∇ belongs to $Pc(M)$. Then, ∇^* belongs to $Pc(M)$ if and only if there exists a 1-form λ on M such that*

$$\nabla_X^* Y-\nabla_X Y=\lambda(X) Y+\lambda(Y) X-\{\lambda(\varphi(X)) \varphi(Y)+\lambda(\varphi(Y)) \varphi(X)\}, \tag{2.7}$$

for any vector fields X, Y.

Proof. Firstly, we assume that ∇^* belongs to $Pc(M)$. Let U be an open set in M over which there is a local section s of the almost complex structure φ. Then, we can construct the local section $\bar{s}$ from s and ∇ as in the proof of Lemma 1 and $\bar{s}^*$ from s and ∇^*.

From the assumption, $\bar{s}$ and $\bar{s}^*$ are local sections of $Pc(M)$ over U. So,

$$\bar{s}^*(p)=\bar{s}(p) S(p), \quad p \in U, \quad S(p) \in H^2(n, C), \tag{2.8}$$

where from (2.2) and the definition of s and s^*, S has the form

$$(2.9)\quad \begin{cases} S=(I_{2n},\ S^i_{jk}), \\ S^a_{bc}=-S^a_{b^*c^*}=S^{a^*}_{bc^*}=S^{a^*}_{b^*c}=-\delta^a_b p_c-\delta^a_c p_b, \\ S^{a^*}_{b^*c^*}=-S^{a^*}_{bc}=S^a_{b^*c}=S^a_{bc^*}=\delta^a_b q_c+\delta^a_c q_b, \\ I_{2n};\ 2n\textit{-dimensional identity matrix.} \end{cases}$$

We may assume that there is a coordinate system $(u^1, \ldots, u^{2n})$ on U. Let $\nabla_{\frac{\partial}{\partial u^i}}\frac{\partial}{\partial u^j}=\Gamma^k_{ij}\frac{\partial}{\partial u^k}$, $\nabla^*_{\frac{\partial}{\partial u^i}}\frac{\partial}{\partial u^j}=\Gamma^{*k}_{ij}\frac{\partial}{\partial u^k}$ and $s(p)=(u^i(p),\ u^i_j(p))$, $p\in U$, then (2.8) is represented as

$$(2.10)\qquad \Gamma^{*i}_{j\,k}-\Gamma^i_{jk}=-\sum u^i_l(p)S^l_{hm}(p)u^{-1h}_{\ \ j}(p)u^{-1m}_{\ \ k}(p).$$

It is easy to show that in the above relation, $S^l_{hm}(p)$ do not depend on the choice of the local section s. Therefore we obtain from (2.10)

$$(2.11)\qquad \Gamma^{*i}_{jk}-\Gamma^i_{jk}=-\sum u^i_l S^l_{hm}u^{-1h}_{\ \ j}u^{-1m}_{\ \ k}.$$

By means of (2.9), it follows from (2.11)

$$\Gamma^{*i}_{jk}-\Gamma^i_{jk}=\sum_{c=1}^n\delta^i_j(p_c u^{-1c}_{\ \ k}-q_c u^{-1c^*}_{\ \ k})+\sum_{c=1}^n\delta^i_k(p_c u^{-1c}_{\ \ j}-q_c u^{-1c^*}_{\ \ j})$$
$$+\sum_{c=1}^n\varphi^i_j(p_c u^{-1c^*}_{\ \ k}+q_c u^{-1c}_{\ \ k})+\sum_{c=1}^n\varphi^i_k(p_c u^{-1c^*}_{\ \ j}+q_c u^{-1c}_{\ \ j}).$$

On putting

$$(2.12)\qquad \lambda_j=\sum_{c=1}^n\{p_c u^{-1c}_{\ \ j}+(-q_c)u^{-1c^*}_{\ \ j}\},$$

it follows

$$\sum\varphi^j_k\lambda_j=-\sum_{c=1}^n(p_c u^{-1c^*}_{\ \ k}+q_c u^{-c}_{\ \ k}).$$

Hence

$$(2.13)\qquad \Gamma^{*i}_{jk}-\Gamma^{ik}_{j}=\delta^i_j\lambda_k+\delta^i_k\lambda_j-\sum\lambda_l(\varphi^i_j\varphi^l_k+\varphi^i_k\varphi^l_j).$$

From the expression of λ_j given by (2.12), λ_j define a 1-form on M and ∇^* is related to ∇ as (2.7) on M.

Conversely, if ∇^* is related to ∇ as (2.7), then for any local section s of the almost complex structure φ, local sections of $P^2(M)$ $\bar{s}$ and $\bar{s}^*$ constructed from ∇ and ∇^* respectively satisfy the relation (2.8). Of course $\bar{s}$ is a local section of $Pc(M)$, $\bar{s}^*$ is also a local section of $Pc(M)$. This implies that ∇^* belongs to $Pc(M)$.

From Lemma 1 and 2, we shall show the following.

THEOREM 1. *Let M be a manifold with an almost complex structure φ and ∇ a symmetric connection on M. Then, we can construct an a.c.p.-structure on M which depends φ and also ∇.*

Conversely, if we assume that M is a paracompact manifold with an a.c.p.-structure $Pc(M)$, we can consider that $Pc(M)$ is constructed from the underlying almost complex structure (see the statement after Definition 1) and a certain symmetric connection ∇, like the method of the first part of this theorem.

PROOF. The first part of the theorem is the same as Lemma 1, so we shall show the second part.

Let $\{U_\alpha\}_{\alpha\in I}$ be a locally finite open covering of M so that there exist local coordinate systems $(U_\alpha, (u_\alpha^1, \dots, u_\alpha^{2n}))$ of M and $\{f_\alpha\}_{\alpha\in I}$ be a partition of unity subordinated to $\{U_\alpha\}_{\alpha\in I}$. Moreover, we assume there is a local section s_α of $Pc(M)$ over each U_α. Then we can represent s_α as

$$s_\alpha = (u_l^i,\ u_{\alpha l}^{\ i},\ -\sum \Gamma_{\alpha\, jk}^{\ i} u_{\alpha\, l}^{\ j} u_{\alpha\, h}^{\ k}),$$

so there exists a symmetric connection ∇_α on U_α defined by $\Gamma_{\alpha\, jk}^{\ i}$. Now, we define a symmetric connection ∇ on M by $\nabla = \sum_\alpha f_\alpha \nabla_\alpha$.

Let φ be a (1,1)-tensor which defines the underlying almost complex structure of $Pc(M)$. Let p be any point of M and U_α be a neighborhood of p. By means of Lemma 2, for any $\beta \in I$ such that $U_\alpha \cap U_\beta = \emptyset$, the definition of ∇_α and ∇_β implies that ∇_α and ∇_β satisfy a relation

$$\nabla_{\beta X} Y - \nabla_{\alpha X} Y = \lambda_{\beta\alpha}(X) Y + \lambda_{\beta\alpha}(Y) X - \{\lambda_{\beta\alpha}(\varphi(X))\varphi(Y) + \lambda_{\beta\alpha}(\varphi(Y))\varphi(X)\}$$

on $U_\alpha \cap U_\beta$, where $\lambda_{\beta\alpha}$ is a certain 1-form on $U_\alpha \cap U_\beta$ and X, Y are any vector fields.

Now, we have on U_α

$$\nabla - \nabla_\alpha = \sum_{\beta\in I} f_\beta (\nabla_\beta - \nabla_\alpha).$$

If we put $\bar\lambda_{\beta\alpha} = \lambda_{\beta\alpha}$ for $U_\alpha \cap U_\beta \neq \emptyset$ and $\bar\lambda_{\gamma\alpha} = 0$ for $U_\alpha \cap U_\gamma = \emptyset$, then we can write the above as

$$\nabla_X Y - \nabla_{\alpha X} Y = \sum f_\beta \{\bar\lambda_{\beta\alpha}(X) Y + \bar\lambda_{\beta\alpha}(Y) X - (\bar\lambda_{\beta\alpha}(\varphi(X))\varphi(Y) + \bar\lambda_{\beta\alpha}(\varphi(Y))\varphi(X))\}.$$

So, put $\lambda_\alpha = \sum f_\beta \bar\lambda_{\beta\alpha}$, then λ_α is a 1-form on M and we find on U_α

$$\nabla_X Y - \nabla_{\alpha X} Y = \lambda_\alpha(X) Y + \lambda_\alpha(Y) X - \{\lambda_\alpha(\varphi(X))\varphi(Y) + \lambda_\alpha(\varphi(Y))\varphi(X)\}.$$

This implies that ∇ belongs to $Pc(M)$ and we have proved the theorem.

DEFINITION 3. Let M be a manifold with an almost complex structure φ and ∇ a connection on M. Let $\alpha(t)$ and $\beta(t)$ be certain functions of the parameter t. Then we call such a curve in M *a holomorphically planar curve* (simply, *h.p.-curve*) that is defined by means of differential equation of the form

$$\frac{d^2u^k}{dt^2}+\sum\Gamma^k_{ij}\frac{du^i}{dt}\frac{du^j}{dt}=\alpha(t)\frac{du^k}{dt}+\beta(t)\sum\varphi^k_i\frac{du^i}{dt}$$

where ∇ and φ have components Γ^k_{ij} and φ^i_j respectively with respect to some local coordinates $u^1, \dots, u^{2n}$.

The following lemma is well knowm (see [2]), so we shall describe only the result.

Lemma 3. *Let M be a manifold with an almost complex structure φ. Let ∇ and ∇^* be connections on M. Then, ∇ and ∇^* have all h.p.-curves in common if and only if ∇ and ∇^* satisfy the following relation*

$$\begin{aligned}(2.14)\qquad \nabla^*_X Y-\nabla_X Y=&\lambda(X)Y+\lambda(Y)X+\rho(X)\varphi(Y)+\rho(Y)\varphi(X)\\ &+(\nabla^*_X Y-\nabla^*_Y X)-(\nabla_X Y-\nabla_Y X),\end{aligned}$$

where λ and ρ are 1-forms on M.

From Lemma 3, we shall show the following lemma, which will be applied to the proof of Theorem 2.

Lemma 4. *Two connections ∇ and ∇^* have the same torsion and all h. p.-curves in common, and satisfy the relation $\nabla\varphi=\nabla^*\varphi$ when and only when the relation*

$$(2.15)\qquad \nabla^*_X Y-\nabla_X Y=\lambda(X)Y+\lambda(Y)X-\{\lambda(\varphi(X))\varphi(Y)+\lambda(\varphi(Y))\varphi(X)\}$$

holds for a certain 1-form λ and any vector fields X, Y.

Proof. We assume that ∇ and ∇^* satisfy the relation (2.15). Then from Lemma 3, ∇ and ∇^* have all *h.p.*-curves in common and also satisfy $\nabla^*_X Y-\nabla^*_Y X-(\nabla_X Y-\nabla_Y X)=0$, that is, ∇ and ∇^* have the same torsion. We shall show $\nabla\varphi=\nabla^*\varphi$. In fact, we have

$$(\nabla^*_X\varphi)Y-(\nabla_X\varphi)Y=\nabla^*_X\varphi(Y)-\nabla_X\varphi(Y)-\varphi(\nabla^*_X Y-\nabla_X Y).$$

Hence, from (2.15) it follows

$$(\nabla^*_X\varphi)Y-(\nabla_X\varphi)Y=0.$$

Conversely, if ∇ and ∇^* have all *h.p*-curves in common, then from Lemma 3, we have (2.14). If, in addition they have the same torsion, then

(2.16) $$\nabla^*_X Y - \nabla_X Y = \lambda(X)Y + \lambda(Y)X + \rho(X)\varphi(Y) + \rho(Y)\varphi(X).$$

From (2.16) and $(\nabla^*_X\varphi)-(\nabla_X\varphi)=0$, we obtain

$$\{\lambda(\varphi(Y))+\rho(Y)\}X+\{\rho(\varphi(Y))-\lambda(Y)\}\varphi(X)=0.$$

Let $u^1, \ldots, u^{2n}$ be any local coordinates and λ, ρ and φ have components λ_i, ρ_i and φ^i_j, respectively with this local coordinates. Now, we can write the above equation as

$$(\sum \lambda_k \varphi^k_i + \rho_i)\delta^l_j + (\sum \rho_k \varphi^k_i - \lambda_i)\varphi^l_j = 0.$$

Since $\sum_{j=1}^{2n} \varphi^j_j = 0$, we have $\rho_i = -\sum \lambda_k \varphi^k_i$ or $\rho = -\lambda\varphi$.

If we substitute the above into (2.16), we have (2.15).

Let ∇ be a connection on M and T be the torsion of ∇. If we put $\bar{\nabla} = \nabla - \frac{1}{2}T$, then we have easily that $\bar{\nabla}$ is a symmetric connection.

Definition 4. Let ∇ be a connection on M with torsion T and $Pc(M)$ an *a.c.p.*-structure on M. We call that ∇ *belongs to* $Pc(M)$ if and only if $\bar{\nabla} = \nabla - \frac{1}{2}T$ belongs to $Pc(M)$ in the sense of Definition 2.

From the preparation described above, we shall show the following theorem.

Theorem 2. *Let M be a manifold with as almost complex structure φ, ∇ and ∇^* be connections on M with the same torsion T and $Pc(M)$ be an a.c.p.-structure on M to which ∇ belongs. Then, ∇^* belongs to $Pc(M)$ if and only if ∇ and ∇^* have all h.p.-curves in common and satisfy the relation $\nabla\varphi = \nabla^*\varphi$.*

Proof. We assume that ∇^* belongs to $Pc(M)$. Let $\bar{\nabla} = \nabla - \frac{1}{2}T$ and $\bar{\nabla}^* = \nabla^* - \frac{1}{2}T$, then from Lemma 2, it follows that

$$\bar{\nabla}^*_X Y - \bar{\nabla}_X Y = \lambda(X)Y + \lambda(Y)X - \{\lambda(\varphi(X))\varphi(Y) + \lambda(\varphi(Y))\varphi(X)\}.$$

Hence, the relation

$$\nabla^*_X Y - \nabla_X Y = \lambda(X)Y + \lambda(Y)X - \{\lambda(\varphi(X))\varphi(Y) + \lambda(\varphi(Y))\varphi(X)\}$$

holds. This implies together with Lemma 4 that ∇ and ∇^* have all $h.p.$-curves in common and satisfy $\nabla\varphi = \nabla^*\varphi$.

Conversely, if we assume ∇ and ∇^* have all *h.p.*-curves in common and satisfy $\nabla\varphi=\nabla^*\varphi$, it may be shown that ∇^* belongs to $Pc(M)$, by reversing the order of the previous discussion.

By this time, we have dealt with general *a.c.p.*-structures, but in sections 5 and 6, we shall treat rather limited *a.c.p.*-structures whith are defined in the following.

Let φ be a (1,1)-tensor giving an almost complex structure and ∇ be a φ-connection, that is, it satisfies the relation $\nabla\varphi=0$. The *a.c.p.*-structure with the underlying almost complex structure φ and the belonging connection ∇ is called *an a.c.p.-structure of restricted type*. Specially, it is called *a standard a.c.p.-structure* when φ-connection ∇ has the torstion $\frac{1}{8}E$, where E is the Nijenhuis tensor of φ (it is well known there exists such a connection [4]).

§3. Complex projective tangent bundles

Let M be a manifold with an almost complex structure and N be a complex projective space an in §1. We shall construct a fibre bundle over M with fibre N, associated with φ, and study a relation between this bundle and an *a.c.p.*-structure on M.

Let $\{U_\alpha, (u_\alpha^1, \dots, u_\alpha^{2n})\}_{\alpha\in I}$ be an atlas of M. For each ordered pair α, β such that $U_\alpha\cap U_\beta\neq\emptyset$, we define a mapping $g_{\alpha\beta}$ of $U_\alpha\cap U_\beta$ to $GL(2n+2, R)$ by

$$g_{\alpha\beta}=\left(\begin{array}{c|cc} \dfrac{\partial u_\alpha^i}{\partial u_\beta^j} & \multicolumn{2}{c}{0} \\ \hline b\dfrac{\partial \log\Delta}{\partial u_\beta^j}+c\sum\dfrac{\partial\log\Delta}{\partial u_\beta^k}\varphi_j^k(\beta) & 1 & 0 \\ c\dfrac{\partial \log\Delta}{\partial u_\beta^j}-b\sum\dfrac{\partial\log\Delta}{\partial u_\beta^k}\varphi_j^k(\beta) & 0 & 1 \end{array}\right)\in GL(2n+2, R), \tag{3.1}$$

where Δ is $\left|\dfrac{\partial u_\alpha^i}{\partial u_\beta^j}\right|$, $\varphi_j^i(\beta)$ are components of φ with respect to $u_\beta^1, \dots, u_\beta^{2n}$ and b, c are arbitrary constants. Then $g_{\alpha\beta}$ satisfies the relation $g_{\alpha\beta}\, g_{\beta\gamma}=g_{\alpha\gamma}$ for $U_\alpha\cap U_\beta\cap U_\gamma\neq\emptyset$ and $g_{\alpha\alpha}=I$ (identity matrix). Hence, we can construct a vector bundle $T^*(M)$ over M with transition functions $g_{\alpha\beta}$ and fibre R^{2n+2}.

Now, we shall show that the vector bundle $T^*(M)$ is a complex vector bundle with a global section. To see $T^*(M)$ a complex vector bundle it is sufficient to find a bundle isomorphism J which maps each fiber onto itself and satisfies $J^2=-I$ (identity mapping).

Let π^* be a natural projection of $T^*(M)$ onto M and $X^1, \dots, X^{2n}, R, I$ be natural coordinates of R^{2n+2}. For each U_α, there is a coordinate function φ_α which is a diffeomorphism of $U_\alpha \times R^{2n+2}$ onto $\pi^{*-1}(U_\alpha)$. We put

$$\bar{J}(X^1, \dots, X^{2n}, R, I) = \left(\begin{array}{c|cc} \varphi^i_j(\alpha) & \multicolumn{2}{c}{0} \\ \hline 0 & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \end{array}\right) \begin{pmatrix} X^1 \\ \vdots \\ X^{2n} \\ R \\ I \end{pmatrix}, \tag{3.2}$$

where $\varphi^i_j(\alpha)$ are components of φ with respect to $u^1_\alpha, \dots, u^{2n}_\alpha$. Then, we define a bundle isomorphism J_α of the restricted vector bundle $\pi^{-1}_\alpha(U_\alpha)$ over U_α by

$$J_\alpha \varphi_\alpha\{p, (X^1, \dots, X^{2n}, R, I)\} = \varphi_\alpha\{p, \bar{J}_\alpha(X^1, \dots, X^{2n}, R, I)\}, \quad p \in U_\alpha.$$

We shall show that these J_α piece together to form a bundle isomorphism J of $T^*(M)$. For this purpose, it is sufficient to see $J_\alpha = J_\beta$ on $\pi^{*-1}(U_\alpha \cap U_\beta)$ for each pair α, β with $U_\alpha \cap U_\beta \neq \emptyset$. Now, from the definition of J_α

$$\begin{aligned} &\varphi_\alpha{}^{-1} J_\beta \varphi_\alpha\{p, (X^1, \dots, X^{2n}, R, I)\} \\ &= g_{\alpha\beta} \varphi_\beta{}^{-1} J_\beta \varphi_\beta g_{\beta\alpha}\{p, (X^1, \dots, X^{2n}, R, I)\} \\ &= \{p, g_{\alpha\beta} \bar{J}_\beta g_{\beta\alpha}(X^1, \dots, X^{2n}, R, I)\}. \end{aligned}$$

Hence, it follows from the relation

$$\bar{J}_\alpha = g_{\alpha\beta} \bar{J}_\beta g_{\beta\alpha}, \tag{3.3}$$

that $J_\alpha = J_\beta$ on $\pi^{-1}(U_\alpha \cap U_\beta)$, while (3.3) is checked by means of (3.1) and (3.2). It is evident that J has the required property, because $J^2_\alpha = -I$ for any $\alpha \in I$.

Now we shall show that $T^*(M)$ has a section. For each $\alpha \in I$, we define a local section s_α over U_α by

$$s_\alpha(p) = \varphi_\alpha\{p, (0, \dots, 0, 1, 0)\}, \quad p \in U_\alpha.$$

Then, from (3.1) the relation $s_\alpha = s_\beta$ holds on $U_\alpha \cap U_\beta (\neq \emptyset)$. So, these s_α fit together to give a section.

Let C^* be a multiplicative group of non-zero complex numbers. Then we shall see that C^* can be regarded as a transformation group of $T^*(M)$ which maps each fibre onto itself. Firstly, we define the action of C^* on R^{2n+2} by

$$c(X^1, \dots, X^{2n}, R, I) = \left(\begin{array}{c|cc} a\delta^i_j + b\varphi^i_j(\alpha) & \multicolumn{2}{c}{0} \\ \hline 0 & \begin{matrix} a & -b \\ b & a \end{matrix} \end{array}\right) \begin{pmatrix} X^1 \\ \vdots \\ X^{2n} \\ R \\ I \end{pmatrix},$$

where $\varphi^i_j(\alpha)$ are components of φ with respect to local coordinates $u^1_\alpha, \ldots, u^{2n}_\alpha$ and $c=a+\sqrt{-1}\,b \in C^*$. Then, we can define the action of C^* on $T^*(M)$ by

$$c\varphi_\alpha\{p, (X^1, \ldots, X^{2n}, R, I)\}=\{p, c(X^1, \ldots, X^{2n}, R, I)\}, \quad p \in U_\alpha,$$

because this definition does not depend on the choice of a coordinate function φ_α. It is evident from the definition that C^* maps each fibre onto itself and the action of C^* on each fibre defined above is nothing but natural multiplication of C^* on each fibre that is a vector space with the complex structure defined by J.

Now, Let $\tilde{T}(M)$ be the identification space obtained by identifying in each fibre of $T^*(M)$, points which correspond under the action of C^*. Then, we can regard $\tilde{T}(M)$ as the fibre bundle over M with fibre N. In addition, $\tilde{T}(M)$ has the section induced from the section of $T^*(M)$.

Definition 5. We call $\tilde{T}(M)$ builded above *the complex projective tangent bundle associated with* φ.

We shall describe a relation between a complex projective tangent bundle and an *a.c.p.*-structure on M.

Theorem 3. *Let $Pc(M)$ be an a.c.p.-structure on M with an underlying almost complex structure φ of $Pc(M)$. Then, the associated bundle $N(M)$ of $Pc(M)$ with fibre N is isomorphic to the complex projective tangent bundle associated with* φ.

Proof. Let $\bar{N}(M)$ be the associated bundle of $Pc(M)$ with fibre R^{2n+2} and natural projection $\bar{\pi}$. As well known, $\bar{N}(M)$ consists of equivalent classes $\{(P, V)\}$, where $P \in Pc(M)$, $V \in R^{2n+2}$ and the equivalent relation is defined so as (P, V) is equivalent to $(PS, S^{-1}V)$, if $S \in H(n, C)$. Let $\{U_\alpha, (u^1_\alpha, \ldots, u^{2n}_\alpha)\}_{\alpha\in I}$ be an atlas of M, then points of $Pc(M)$ may be represented as $(u^i_\alpha, u_\alpha{}^i_j, u_\alpha{}^i_{jk})$, $i, j, k=1, \ldots, 2n$. Then, coordinate functions $\bar{\varphi}_\alpha$: $U_\alpha \times R^{2n+2} \to \bar{\pi}^{-1}(U_\alpha)$ can be given by

$$\bar{\varphi}_\alpha^{-1}(\{(P, V)\})=\{\bar{\pi}(P), (X^1, \ldots, X^{2n}, R, I)\},$$

where we put

$$V=(x^1, \ldots, x^{2n}, x^{2n+1}, x^{2n+2}) \in R^{2n+2},$$

$$P=(u^i_\alpha, u_\alpha{}^i_j, u_\alpha{}^i_{jk}),$$

$$X^i=\sum u_\alpha{}^i_j x^j,$$

$$R=b\sum T_j x^j+c\sum_{a=1}^{n}(T_{a^*}x^a-T_a x^a)-2(n+1)\,(bx^{2n+1}-cx^{2n+2}),$$

$$I=-b\sum_{a=1}^{n}(T_{a^*}x^a-T_a x^{a^*})+c\sum T_j x^j-2(n+1)(bx^{2n+2}+cx^{2n+1}),$$
$$T_j=\sum u_{\alpha}{}^{i}_{jk}u^{-1}{}^{i}_{\alpha k},$$

for arbitrary constants b, c because these φ_α do not depend on elements $(P, V)\in\pi^{-1}(U_\alpha)\times R^{2n+2}$ but classes $\{(P, V)\}$.

Let $T^*(M)$ be the vector bundle with fibre R^{2n+2} and transition functions $g_{\alpha\beta}$ given by (3.1), corresponding to φ. Then, we can find that each $\bar{g}_{\alpha\beta}=\bar{\varphi}_\alpha\bar{\varphi}_\beta^{-1}$ is the same as $g_{\alpha\beta}$. Hence, $\bar{N}(M)$ is isomorphic to $T^*(M)$. Moreover, we can see that just as $\tilde{T}(M)$ is constructed from $T^*(M)$, the associated bundle $N(M)$ of $Pc(M)$ is constructed from $\bar{N}(M)$. Therefore, $N(M)$ is isomorphic to $\tilde{T}(M)$.

§4. Almost complex projective connections

As described in §1, we take the complex local coordinate system (a^a, a^a_b, a_b) in the neighborhood of the identity in complex projective transformation group $PL(n, C)$.

If we put

$$a^a=p^a+\sqrt{-1}\ q^a,\quad a^a_b=p^a_b+\sqrt{-1}\ q^a_b,\quad a_b=q_b+\sqrt{-1}\ q_b,$$

then we can take $(p^a, q^a, p^a_b, q^a_b, p_b, q_b)$ as a real coordinate system.

Let ω^a, θ^a, ω^a_b, θ^a_b, ω_b, θ_b, $a, b=1, \dots, n$ be left invariant 1-forms on $PL(n, C)$ which coincide with dp^a, dq^a, dp^a_b, dq^a_b, dp_b, dq_b at the identity. A calculation shows that the equations of Maurer-Cartan of $PL(n, C)$ are given by

$$(4.1)\quad \begin{cases} d\omega^a=-\sum^{5)}(\omega^a_c\wedge\omega^c-\theta^a_c\wedge\theta^c), \\ d\theta^a=-\sum(\theta^a_c\wedge\omega^c+\omega^a_c\wedge\theta^c), \\ d\omega^a_b=-\sum(\omega^a_c\wedge\omega^c_b-\theta^a_c\wedge\theta^c_b)-(\omega^a\wedge\omega_b-\theta^a\wedge\theta_b)+\delta^a_b\sum(\omega_c\wedge\omega^c-\theta_c\wedge\theta^c), \\ d\theta^a_b=-\sum(\theta^a_c\wedge\omega^c_b+\omega^a_c\wedge\theta^c_b)-(\theta^a\wedge\omega_b+\omega^a\wedge\theta_b)+\delta^a_b\sum(\theta_c\wedge\omega^c+\omega_c\wedge\theta^c), \\ d\omega^a_b=-\sum(\omega_c\wedge\omega^c_b-\theta_c\wedge\theta^c_b), \\ d\theta_b=-\sum(\theta_c\wedge\omega^c_b+\omega_c\wedge\theta^c_b). \end{cases}$$

Let $\mathfrak{pl}(n, C)$ be the Lie algebra of $PL(n, C)$ and $\boldsymbol{e}_a$, $\boldsymbol{f}_a$, $\boldsymbol{E}^a_b$, $\boldsymbol{F}^a_b$, $\boldsymbol{e}^b$, $\boldsymbol{f}^b$, the base of $\mathfrak{pl}(n, C)$ which is dual to ω^a, θ^a, ω^a_b, θ^a_b, ω_b, θ_b, that is,

$$\omega^a(\boldsymbol{e}_a)=1,\ \theta^a(\boldsymbol{f}_a)=1,\ \omega^a_b(\boldsymbol{E}^b_a)=1,\ \theta^a_b(\boldsymbol{F}^b_a)=1,\ \omega_b(\boldsymbol{e}^b)=1,\ \theta_b(\boldsymbol{f}^b)=1$$

5) In this sections, $\sum$ means summation from 1 to n over the repeated indices, unless otherwise indicated.

and other pairs equal to zero.

Let S be an arbitrary element of $H(n, C)$ and S^{-1} the inverse of S. Then, as described in §1, S and S^{-1} are represented as

$$S=\left(\begin{array}{cc|cc} p_b^a & -q_b^a & & \\ q_b^a & p_b^a & \multicolumn{2}{c}{0} \\ \hline p_b & -q_b & 1 & 0 \\ q_b & p_b & 0 & 1 \end{array}\right), \qquad S^{-1}=\left(\begin{array}{cc|cc} p^{*a}_{\ b} & -q^{*a}_{\ b} & & \\ q^{*a}_{\ b} & p^{*a}_{\ b} & \multicolumn{2}{c}{0} \\ \hline p_b^* & -q_b^* & 1 & 0 \\ q_b^* & p_b^* & 0 & 1 \end{array}\right).$$

With above notations, the adjoint representation of $H(n, C)$ on $\mathfrak{pl}(n, C)$ is formulated as

$$(4.2)\quad \left\{\begin{aligned} ad(S^{-1})\boldsymbol{e}_e &= \sum(p^{*d}_{\ e}\boldsymbol{e}_d + q^{*d}_{\ e}\boldsymbol{f}_d) \\ &\quad + \sum\{(p^{*d}_{\ e}p_c - q^{*d}_{\ e}q_c - \delta_c^d p_e^*\boldsymbol{E}_d^c + (p^{*d}_{\ e}q_c + q^{*d}_{\ e}p_c - \delta_c^d q_e^*)\boldsymbol{F}_d^c\} \\ &\quad + \sum\{p_e^* p_c - q_e^* p_c)\boldsymbol{e}^c + (q_e^* p_c + p_e^* q_c)\boldsymbol{f}^c\}, \\ ad(S^{-1})\boldsymbol{f}_e &= -\sum(q^{*d}_{\ e}\boldsymbol{e}_d - p^{*d}_{\ e}\boldsymbol{f}_d) \\ &\quad - \sum\{q^{*d}_{\ e}p_c + p^{*d}_{\ e}q_c - \delta_c^d q_e^*)\boldsymbol{E}_d^c + (q^{*d}_{\ e}q_c - p^{*d}_{\ e}P_c + \delta_c^d p_e^*)\boldsymbol{F}_d^c\} \\ &\quad - \sum\{q_e^* p_c + p_e^* q_c)\boldsymbol{e}^c - (p_e^* p_c - q_e^* q_c)\boldsymbol{f}^c\}, \\ ad(S^{-1})\boldsymbol{E}_e^f &= \sum\{(p^{*d}_{\ e}p_c^f - q^{*d}_{\ e}q_c^f)\boldsymbol{E}_d^c + (p^{*d}_{\ e}q_c^f + q^{*d}_{\ e}p_c^f)\boldsymbol{F}_d^c\} \\ &\quad + \sum\{(p_e^* p_c^f - q_e^* q_c^f)\boldsymbol{e}^c + (q_e^* p_c^f + p_e^* q_c^f)\boldsymbol{f}^c\}, \\ ad(S^{-1})\boldsymbol{F}_e^f &= -\sum\{(q^{*d}_{\ e}p_c^f + p^{*d}_{\ e}q_c^f)\boldsymbol{E}_d^c - (p^{*d}_{\ e}p_c^f - q^{*d}_{\ e}q_c^f)\boldsymbol{F}_d^c\} \\ &\quad - \sum\{(q_e^* p_c^f + p_e^* q_c^f)\boldsymbol{e}^c + (p_e^* p_c^f - q_e^* q_c^f)\boldsymbol{f}^c\}. \end{aligned}\right.$$

Let $\mathfrak{m}^*$ be the subspace of $\mathfrak{pl}(n, C)$ which is spanned by $\boldsymbol{e}^b, \boldsymbol{f}^b$, $b=1, \dots, n$. As we have $ad(S^{-1})\mathfrak{m}^* \subset \mathfrak{m}^*$, $S \in H(n, C)$, the operation $ad(S^{-1})$ on the coset space $\mathfrak{pl}(n, C)/\mathfrak{m}^*$ is canonically induced. This operation is given by neglecting the terms of $\boldsymbol{e}^c$, $\boldsymbol{f}^c$ in each equation of (4.2).

Let M be a manifold and $Pc(M)$ an *a.c.p.*-structure on M. Let $\mathfrak{h}$ be the Lie algebra of $H(n, C)$. We shall define a connection in $Pc(M)$ in the sence of Cartan connection.

Definition 6. *An almost complex projective connection in* $Pc(M)$ (simply, *a.c.p.-connection*) is a 1-form ω on $Pc(M)$ with values in the Lie algebra $\mathfrak{pl}(n, C)$ which satisfies the following conditions,

(a) $\omega(A^*)=A \qquad A \in \mathfrak{h}$,

(b) $R_S^*\omega = ad(S^{-1})\omega \qquad S \in H(n, C)$, (*c*) $\omega(X)\neq 0$ for every non-zero vector X of $Pc(M)$,

where A^* is the fundamental vector field corresponding to $A\in\mathfrak{h}$.

An *a.c.p.*-connection ω can be represented as

$$\omega=\sum(\omega^a\boldsymbol{e}_a+\theta^a\boldsymbol{f}_a)+\sum(\omega^a_b\boldsymbol{E}^b_a+\theta^a_b\boldsymbol{F}^b_a)+\sum(\omega_b\boldsymbol{e}^b+\theta_b\boldsymbol{e}^b),$$

with the base $\boldsymbol{e}_a, \boldsymbol{f}_a, \boldsymbol{E}^a_b, \boldsymbol{F}^a_b, \boldsymbol{e}^b, \boldsymbol{f}^b$.

PROPOSITION 1. Let

$$\omega=\sum(\omega^a\boldsymbol{e}_a+\theta^a\boldsymbol{f}_a)+\sum(\omega^a_b\boldsymbol{E}^b_a+\theta^a_b\boldsymbol{F}^b_a)+\sum(\omega_a\boldsymbol{e}^a+\theta_a\boldsymbol{f}^a)$$

be an a.c.p.-connection in $Pc(M)$. Then structure equations of ω are given by

$$(1')\quad d\omega^a=-\sum(\omega^a_c\wedge\omega^c-\theta^a_c\wedge\theta^c)+\Omega^a,$$

$$(2')\quad d\theta^a=-\sum(\theta^a_c\wedge\omega^c+\omega^a_c\wedge\theta^c)+\Phi^a,$$

$$(3')\quad d\omega^a_b=-\sum(\omega^a_c\wedge\omega^c_b-\theta^a_c\wedge\theta^c_b)+\delta^a_b\sum(\omega_c\wedge\omega^c-\theta_c\wedge\theta^c)+\omega_b\wedge\omega^a-\theta_b\wedge\theta^a+\Omega^a_b,$$

$$(4')\quad d\theta^a_b=-\sum(\theta^a_c\wedge\omega^c_b+\omega^a_c\wedge\theta^c_b)+\delta^a_b\sum(\theta_c\wedge\omega^c+\omega_c\wedge\theta^c)-\theta^a\wedge\omega_b-\omega^a\wedge\theta_b+\Phi^a_b,$$

$$(5')\quad d\omega_b=-\sum(\omega_c\wedge\omega^c_b-\theta_c\wedge\theta^c_b)+\Omega_b,$$

$$(6')\quad d\theta_b=-\sum(\theta_c\wedge\omega^c_b+\omega_c\wedge\theta^c_b)+\Phi_b,$$

where Ω^a, Φ_a, Ω^a_b, Φ^a_b, Ω_b, Φ_b are 2-forms generated by ω^a, θ^a.

We call Ω^a and Φ^a the torsions of ω, and also Ω^a_b, Φ^a_b and Ω_b, Φ_b the curvatures of ω.

PROOF. By the condition (*c*), ω^a, θ^a, ω^a_b, θ^a_b, ω_b, θ_b give rise to a base of the cotangent space at each point of $P_c(M)$. So, we can take dual vector fields X_a, Y_a, X^b_a, Y^b_a, X^b, Y^b, that is,

$$\omega(X_a)=\boldsymbol{e}_a,\ \omega(Y_a)=\boldsymbol{f}_a,\ \omega(X^b_a)=\boldsymbol{E}^b_a,\ \omega(Y^b_a)=\boldsymbol{F}^b_a,\ \omega(X^b)=\boldsymbol{e}^b,\ \omega(Y^b)=\boldsymbol{f}^b.$$

By the condition (*a*), X_a, Y_a, X^b_a, Y^b_a are fundamental vector fields corresponding to $\boldsymbol{e}_a, \boldsymbol{f}_a, \boldsymbol{E}^b_a, \boldsymbol{F}^b_a$ respectively, hence we have

$$(4.4)\quad \begin{cases} \omega[X_a, X_b]=[\boldsymbol{e}_a, \boldsymbol{e}_b],\ \omega[X_a, Y_b]=[\boldsymbol{e}_a, \boldsymbol{f}_b], \\ \omega[X_a, X^b_c]=[\boldsymbol{e}_a, \boldsymbol{E}^b_c],\ \omega[X_a, Y^b_c]=[\boldsymbol{e}_a, \boldsymbol{F}^b_c], \\ \omega[Y_a, X^b_c]=[\boldsymbol{f}_a, \boldsymbol{E}^b_c],\ \omega[Y_a, Y^b_c]=[\boldsymbol{f}_a, \boldsymbol{F}^b_c], \\ \omega[X^a_b, Y^c_d]=[\boldsymbol{E}^a_b, \boldsymbol{F}^c_b]. \end{cases}$$

By the condition (c), it follows

$$(4.5)\qquad \begin{cases} \omega[X_a, Y^b]=[\boldsymbol{e}_a, \boldsymbol{f}^b], \ \omega[Y_a, X^b]=[\boldsymbol{f}_b, \boldsymbol{e}^b], \\ \omega[X_a, X^b]=[\boldsymbol{e}_a, \boldsymbol{e}^b], \ \omega[Y_a, Y^b]=[\boldsymbol{f}_a, \boldsymbol{f}^b]. \end{cases}$$

Generally, for 1-form θ, we have

$$(4.6)\qquad 2d\theta(X, Y)=-\theta[X, Y].$$

From (4.1,) (4.4), (4.5) and (4.6), the relation (4.3) follows with ease.

§5. The canonical form on $Pc(M)$

In the sequel, we confine our attention to $a.c.p.$-structures of restricted type.

Let $Pc(M)$ be an $a.c.p.$-structure with an underlying almost comlpex structure φ. Then, there exists a connection ∇ with torsion T which belongs to $Pc(M)$ and satisfies the relation $\nabla\varphi=0$.

Let $(u^1, \dots, u^{2n})$ be a local coordinate system of M, then points of $Pc(M)$ are represented as (u^i, u^i_j, u^i_{jk}). Let φ^i_j and T^i_{jk} be components of φ and T with respect to $(u^1, \dots, u^{2n})$. If we put

$$(5.1)\qquad \bar{T}^r_{pq}=\sum{}^{6)}(\varphi^k_i T^l_{jk}\varphi^t_l+T^t_{ji})u^{-1}{}^r_t u^j_p u^i_q$$

then $\bar{T}^r_{pq}$ are funtions on $Pc(M)$, since they do not depend on the choice of local coordinate systems.

In addition, we shall show that the functions $\bar{T}^r_{pq}$ have no relation to the choice of connections which belong to $Pc(M)$ and satisfy $\nabla\varphi=0$. In fact, let ∇' be another connection of this kind with torsion T', then symmetric connections $\bar{\nabla}=\nabla-\frac{1}{2}T$ and $\bar{\nabla}'=\nabla'-\frac{1}{2}T$ satisfy the condition of Lemma 2. Hence Lemma 4 leads us to $\bar{\nabla}\varphi=\bar{\nabla}'\varphi$, or

$$T(X, \varphi(Y))-\varphi(T(X, Y))=T'(X, \varphi(Y))-\varphi(T'(X, Y))$$

from which the assertion follows.

Let (θ^i, θ^i_j), $i, j=1, \dots, 2n$ be the canonical form on 2-frame bundle $P^2(M)$ (see [3]). Then, θ^i, θ^i_j satisfy

$$(5.2)\qquad d\theta^i=-\sum\theta^i_j\wedge\theta^j.$$

We denote the restriction of θ^i, θ^i_j to $Pc(M)$ by $\bar{\theta}^i$, $\bar{\theta}^i_j$ and define 1-forms $\tilde{\theta}^i_j$ on

6) In this section, $\sum$ means summation from 1 to $2n$, unless otherwise indicated.

$Pc(M)$ by

(5.3) $$\tilde{\theta}^i_j = \bar{\theta}^i_j + \sum \bar{T}^i_{kj} \bar{\theta}^k.$$

LEMMA 5. *For the 1-form $\tilde{\theta}^i_j$, we have the relation*

(5.4) $$\tilde{\theta}^a_b = \tilde{\theta}^{a^*}_{b^*}, \quad \tilde{\theta}^a_{b^*} = -\tilde{\theta}^{a^*}_b.$$

PROOF. It is sufficient to prove in a coordinate system. Let points of $Pc(M)$ be represented as (u^i, u^i_j, u^i_{jk}) with respect to a local coordinate system $(u^1, \dots, u^{2n})$ of M. Let Γ^i_{jk}, φ^i_j and T^i_{jk} be components of ∇, φ and T.

Now, $\nabla\varphi = 0$ is written as

(5.5) $$\frac{\partial \varphi^l_i}{\partial u^j} + \sum \varphi^k_i \Gamma^l_{jk} - \sum \varphi^l_k \Gamma^k_{ji} = 0.$$

As ∇ belongs to $Pc(M)$,

(5.6) $$\Gamma^k_{ij} - \frac{1}{2} T^k_{ij} = -\sum u^l_{pq} u^{-1p}_j u^{-1q}_k.$$

As well known, φ^i_j can be represented as

(5.7) $$\varphi^i_j = -\sum_{a=1}^{n} (u^i_a u^{-1a^*}_j - u^i_{a^*} u^{-1a}_j).$$

By the exterior differentiation of (5.5), we have

(5.8) $$\begin{cases} du^i_{a^*} - \sum \varphi^i_l du^l_a = \sum \dfrac{\partial \varphi^i_j}{\partial u^l} u^j_a du^l, \\[2ex] du^i_a + \sum \varphi^i_l du^l_{a^*} = -\sum \dfrac{\partial \varphi^i_j}{\partial u^l} u^j_{a^*} du^l. \end{cases}$$

In addition, θ^i, θ^i_j have the form

(5.9) $$\begin{cases} \theta^i = \sum u^{-1i}_j du^j, \\ \theta^i_j = \sum u^{-1i}_k du^k_j - \sum u^{-1i}_k u^k_{hj} u^{-1h}_l du^l. \end{cases}$$

Well, (5.4) follows from (5.5), (5.6), (5.7), (5.8) and (5.9) after some calculations.

We remark that $\bar{T}^p_{qr}$ have properties

(5.10) $$\bar{T}^a_{qb} = -\bar{T}^{a^*}_{qb^*}, \quad \bar{T}^a_{qb^*} = \bar{T}^{a^*}_{qb},$$

which are shown by (5.7).

Now we put

(5.11) $$\omega^{*a}_{b}=\bar{\theta}^{a}_{b},\quad \theta^{*a}_{b}=\bar{\theta}^{a^*}_{b},\quad \theta^{*i}=\bar{\theta}^{i},$$

and define the canonical form on $Pc(M)$.

Definition 7. Set

$$\theta^*=\sum_{a=1}^{n}\theta^{*a}\boldsymbol{e}_a+\sum_{a=1}^{n}\theta^{*a^*}\boldsymbol{f}_a+\sum_{a,b=1}^{n}(\omega^{*a}_{b}\boldsymbol{E}^b_a+\theta^{*a}_{b}\boldsymbol{F}^b_a).$$

Then we call this θ^* *the canonical form on* $Pc(M)$.

From (5.2) and Lemma 5, the following proposition is straightforward.

Proposition 2. The equations

(5.12) $$\begin{cases} d\theta^{*a}=-\sum(\omega^{*a}_{b}\wedge\theta^{*b}-\theta^{*a}_{b}\wedge\theta^{*b^*})+\dfrac{1}{2}\sum T^{*a}_{ij}\theta^{*i}\wedge\theta^{*j}, \\ d\theta^{*a^*}=-\sum(\theta^{*a}_{b}\wedge\theta^{*b}+\omega^{*a}_{b}\wedge\theta^{*b^*})+\dfrac{1}{2}\sum T^{*a^*}_{ij}\theta^{*i}\wedge\theta^{*j}, \end{cases}$$

hold, where

$$T^{*i}_{jk}=\frac{1}{2}(\bar{T}^i_{jk}-\bar{T}^i_{kj}).$$

We obtain by calculations the following formulas

(5.13) $$\begin{cases} R^*_S\bar{\theta}^i=\sum S^{-1}{}^i_j\bar{\theta}^j, \\ R^*_s\bar{\theta}^i_j=\sum S^{-1}{}^i_l\bar{\theta}^l_k S^k_j-S^{-1}{}^i_t S^t_{kj}S^{-1}{}^k_l\bar{\theta}^l, \\ R^*_S\bar{T}^i_{kj}\bar{\theta}^k=\sum S^{-1}{}^i_l\bar{T}^l_{kh}\bar{\theta}^k S^h_j, \end{cases}$$

and

(5.14) $$\theta^*(\boldsymbol{E}^{a^*}_b)=\boldsymbol{E}^a_b,\quad \theta^*(\boldsymbol{F}^{a^*}_b)=\boldsymbol{F}^a_b,$$

where $S=(S^i_j, S^i_{jk})\in H(n, C)$ and $\boldsymbol{E}^{a^*}_b$, $\boldsymbol{F}^{a^*}_b$ are fundamental vector fields corresponding to $\boldsymbol{E}^a_b$, $\boldsymbol{F}^a_b$.

From (4.2), (5.13) and (5.14), we verify the following proposition.

Proposition 3. *For the canonical form* θ^* *on* $Pc(M)$, *we have*

$$R^*_S\theta^*=ad(S^{-1})\theta^*,\quad S\in H(n, C),$$

$$\theta^*(\boldsymbol{E}^{a^*}_b)=\boldsymbol{E}^a_b,\quad \theta^*(\boldsymbol{F}^{a^*}_b)=\boldsymbol{F}^a_b,$$

where we consider $ad(S^{-1})$ *to act on the coset space* $\mathfrak{pl}(n, C)/\mathfrak{m}^*$.

We remark the following. If $Pc(M)$ is a standard *a.c.p.*-structure (defined at the end of the section 2), then $\bar{T}^r_{pq}$ and T^{*r}_{pq} equal to

$$\sum E^i_{jk} u^{-1r}_{\ i} u^j_p u^k_q,$$

where E^i_{jk} are components of the Nijenhuis tensor E corresponding to φ.

§6. A normal a.c.p.-connection and a flat a.c.p.-structure

In this section, we discuss the unique existence of a certain connection which may be called *a normal a.c.p.-connection in Pc(M)*, and the local flatness of a standard *a.c.p.*-structure.

Since the method of proofs is parallel to the method employed by S. Kobayashi and T. Nagano for the corresponding theorems in [3], we omit proofs and only describe results.

Thus, from Proposition 2 and 3, we can prove the following.

THEOREM 4. *Let $Pc(M)$ be an a.c.p.-structure on M and $\theta^* = \sum(\theta^{*a}\boldsymbol{e}_a + \theta^{*a^*}\boldsymbol{f}_a) + \sum(w^{*a}_{\ b}\boldsymbol{E}^b_a + \theta^{*a}_{\ b}\boldsymbol{F}^b_a)$ the canonical form on $Pc(M)$, then there exists a unique a.c.p.-connection ω which has the form $\omega = \theta^* + \sum(\boldsymbol{\theta}^*_a e^a + \boldsymbol{\omega}^*_a f^a)$ and the following properties.*

$$\sum_{a=1}^{n} K^a_{bac} = 0, \qquad \sum_{a=1}^{n} K^a_{ba^*c^*} = 0,$$

$$\sum_{a=1}^{n} R^a_{bac} = 0, \qquad \sum_{a=1}^{n} R^a_{ba^*c^*} = 0,$$

where the quantities $\Omega^a_b = \frac{1}{2}\sum K^a_{bij}\theta^{*i}\wedge\theta^{*j}$ and $\Phi^a_b = \frac{1}{2}\sum R^a_{bij}\theta^{*i}\wedge\theta^{*j}$ are defined by the equations (3) *and* (4) *of* (4.3) *for the condition ω.*

The above connection, we call *a normal a.c.p.-connection in Pc(M)*.

Next, we consider a standard *a.c.p.*-structure $Pc(M)$. If the normal connection in $Pc(M)$ has null torsions, then by virture of the remark at the end of the section 5, it is easily verfied that M is a complex manifold and $Pc(M)$ is a holomorphic fibre bundle. Hence the following is established.

THEOREM 5. *Let $Pc(M)$ be a standard a.c.p.-structure on a $2n$-dimensional manifold M. Then, M is locally, holomorphically homeomorphic to a complex projective space if and only if all torsions and curvatures of the normal a.c.p.-connection in $Pc(M)$ vanish.*

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