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A Note on Higher Deflections of a Local Ring

By

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Introduction.

Let R be a commutative Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field K. To R we associate homological invariants of R so called "deflections" $\varepsilon_i(i=1, 2,\dots)$. It is well known that $\varepsilon_1=0$ (resp. $\varepsilon_2=0$) if and only if R is regular (resp. complete intersection). ε_1 and ε_2 are computed in terms of the homology algebra $H(E)$ of the Koszul complex E of R: $\varepsilon_1=\dim_K H_1(E)$ and $\varepsilon_2=\dim H_2(E)/H_1(E)^2$.

In this note, after proving some lemmas (§1), we will calculate ε_3 and, in some restricted case, ε_4 by means of $H(E)$. As an application we give in §3 an expression of the form of Betti series of R assuming its embedding dimension is 3 and that $H(E)$ has trivial multiplication. This gives an alternating proof of a theorem due to Golod [2].

Unless otherwise specified, we shall use the same notations and the same terminology which appeared in [5].

§1. Preliminary lemmas.

Let $(R, \mathfrak{m})$ be a local ring of embedding dimension n and residue field K and let $\{t_1, \dots, t_n\}$ be a minimal system of generators of $\mathfrak{m}$. By the method of killing cycles, we have a following sequence of R-algebras $X^{(i)}(i=0, 1, 2,\dots)$ [5, §1];

$$X^{(0)}=R,\ X^{(1)}=E=R<T_1, \dots, T_n>;\ dT_i=t_i,$$

$$X^{(2)}=X^{(1)}<S_1, \dots, S_{\varepsilon_1}>;\ dS_i=s_i,\ X^{(3)}=X^{(2)}<U_1, \dots, U_{\varepsilon_2}>;\ dU_i=u_i,$$

$$X^{(4)}=X^{(3)}<V_1, \dots, V_{\varepsilon_3}>;\ dV_i=v_i, \cdots$$

where $T_i, S_i, U_i, V_i, \dots$ are variables of degree 1, 2, 3, 4, ... which kill cycles $t_i, s_i, u_i, v_i, \dots$ respectively. The i-th deflection $\varepsilon_i(i=1, 2, \dots)$ is defined by $\dim_K H_i(X^{(i)})$ but this is also equal to $\dim_K H_i(X^{(i-1)})$, if $i\geq 3$, as we see in the following lemma.

LEMMA 1. *Let X be an R-algebra and $\rho \geq 3$ be an integer such that $H_0(X)=K$ and $H_i(X)=0$ for $0<i\leq\rho-2$. If $s \in Z_{\rho-1}(X)$ is not a boundary, then $Y=X<S>$; $dS=s$,* deg $S=\rho$, *satisfies $H_{\rho+\mu}(X)\approx H_{\rho+\mu}(Y)$ for $\rho-3\geq\mu\geq 0$. Consequently, $\varepsilon_i=\dim_K H_i(X^{(i-1)})$, if $i\geq 3$.*

PROOF. ρ odd: From the exact sequence $0\rightarrow X_\lambda \rightarrow Y_\lambda \rightarrow X_{\lambda-\rho}\rightarrow 0$, we obtain the exact sequence

$$\cdots \rightarrow H_{\mu+1}(X) \rightarrow H_{\rho+\mu}(X) \rightarrow H_{\rho+\mu}(Y) \rightarrow H_\mu(X) \rightarrow \cdots$$
$$\cdots \rightarrow H_1(X) \rightarrow H_\rho(X) \rightarrow H_\rho(Y) \rightarrow H_0(X) \overset{d_{0*}}{\rightarrow} \cdots.$$

Since $H_0(X)=K$ and d_{0*} is the multiplication by σ, the homology class of s, d_{0*} is injective so that $H_\rho(X)\approx H_\rho(Y)$. If $\rho-3\geq\mu>0$, we have $H_{\mu+1}(X)=H_\mu(X)=0$ and hence $H_{\rho+\mu}(X)\approx H_{\rho+\mu}(Y)$.

ρ even: In this case the sequence $0\rightarrow X_\lambda\rightarrow Y_\lambda\rightarrow Y_{\lambda-\rho}\rightarrow 0$ is exact, hence

$$\cdots \rightarrow H_1(Y) \rightarrow H_\rho(X) \rightarrow H_\rho(Y) \rightarrow H_0(Y) \overset{d_{0*}}{\rightarrow} H_{\rho-1}(X) \rightarrow H_{\rho-1}(Y) \rightarrow 0 \quad \text{(exact)}.$$

On one hand, since $d_{0*}i_*$: $H_0(X)\rightarrow H_0(Y)\rightarrow H_{\rho-1}(X)$ is obtained by the multiplication by σ, d_{0*} is injective so that $H_\rho(X)\approx H_\rho(Y)$ since clearly we have $H_1(Y)=0$. If $\rho-3\geq\mu>0$, we have $H_\mu(Y)\approx H_\mu(X)=0$ and $H_{\mu+1}(Y)\approx H_{\mu+1}(X)=0$. Whence, the exact sequence

$$\cdots \rightarrow H_{\mu+1}(Y) \rightarrow H_{\rho+\mu}(X) \rightarrow H_{\rho+\mu}(Y) \rightarrow H_\mu(Y) \rightarrow \cdots$$

implies $H_{\rho+\mu}(X)\approx H_{\rho+\mu}(Y)$.

We need also the following lemma for calculating higher deflections.

LEMMA 2. *Let $w_1, \ldots, w_{\varepsilon_i}$ be a set of ε_i cycles in $X^{(i)}$ whose homology classes constitute a base of the K-vector space $H_i(X^{(i)})$ $(i=2,3,\ldots)$. Then, w_j $(j=1,2,\ldots,\varepsilon_i)$ can be selected in $X^{(i-1)}$.*

PROOF. Let $X^{(i)}=X^{(i-1)}<\Pi_1, \ldots, \Pi_{\varepsilon_{i-1}}>$; $d\Pi_j=\pi_j$, deg $\Pi_j=i$. Put $w_j=w$ and $\varepsilon_{i-1}=\varepsilon$. Write $w=w'+\sum_{k=1}^{\varepsilon} r_k\Pi_k$, where $w'\in X^{(i-1)}$ and $r_k\in R$. Then, $0=dw=dw'+\sum r_k\pi_k$. Since $\pi_1, \ldots, \pi_\varepsilon$ are linearly independent cycles (modulo $B(X^{(i-1)})$), each $r_k \in \mathfrak{m}$. Take $P_k\in X_1^{(i-1)}$ such that $r_k=dP_k$. Then, $\sum r_k\Pi_k=\sum(dP_k)\Pi_k=d(\sum P_k\Pi_k)+\sum P_k\pi_k$. Hence, $w=w'+\sum P_k\pi_k+d(\sum P_k\Pi_k)$ and consequently $w'+\sum P_k\pi_k$ is a cycle in $X^{(i-1)}$ and is homologous to w.

COROLLARY. u_i $(i=1, 2, \ldots, \varepsilon_2)$ can be selected in $Z_2(E)$.
v_i $(i=1, 2, \ldots, \varepsilon_3)$ can be selected in $Z_3(X^{(2)})$.

First we construct special cycles in $Z_3(X^{(2)})$. For this we fix a set I of pairs of integers (p, q) $(1\leq p<q\leq\varepsilon_1)$ such that homology classes of s_ps_q,

$(p, q) \in I$, constitute a base of the vector space $H_1(E)^2$. And, we put $J=\{(i, j) \mid 1 \leq i<j \leq \varepsilon_1, (i, j) \notin I\}$.

Now, for any set $\{x_1^{(j)} \in Z_1(E) \mid j=1, \ldots, \varepsilon_1\}$, we can find $r_{pq} \in R$, $(p, q) \in I$, and $x_3 \in E_3$ such that

$$v_x = x_3 + \sum_{i=1}^{\varepsilon_1} x_1^{(i)} S_i - \sum_{(p,q)\in I} r_{pq} s_p S_q$$

belongs to $Z_3(X^{(2)})$ and moreover $x_3 - \sum_{(p,q)\in I} r_{pq} s_p S_q$ is defined uniquely up to modulo $B_3(X^{(2)})+Z_3(E)$. In fact, we have $\sum x_1^{(i)} s_i = \sum_I r_{pq} s_p s_q + dx_3$, for some $r_{pq} \in R$ and $x_3 \in E_3$ and hence $v_x \in Z_3(X^{(2)})$. To see the second part, it is enough to show that the relation, $\sum_I r_{pq} s_p s_q + dx_3 = 0$, implies $x_3 - \sum_I r_{pq} s_p S_q \in B_3(X^{(2)})+Z_3(E)$. Now, by the definition of I, each $r_{pq} \in \mathfrak{m}$ and hence $r_{pq} = dP_{pq}$ for some $P_{pq} \in E_1$ and whence $\sum_I r_{pq} s_p s_q + dx_3 = d(\sum_I P_{pq} s_p s_q + x_3)$. Therefore $\sum_I P_{pq} s_p s_q + x_3 \in Z_3(E)$ and, consequently, $x_3 - \sum_I r_{pq} s_p S_q = x_3 - \sum (dP_{pq}) s_p S_q = x_3 - d(\sum P_{pq} s_p S_q) + \sum P_{pq} s_p s_q \in Z_3(E) + B_3(X^{(2)})$.

We remark that, if $x_1^{(i)} \in B_1(E)$ $(i=1, \ldots, \varepsilon_1)$, then $v_x \in Z_3(E)+B_3(X^{(2)})$. For, we put $x_1^{(i)} = dy_2^{(i)}$ with $y_2^{(i)} \in E_2$. Then, the relation, $0 = dv_x = d\{x_3 + \sum (dy_2^{(i)}) S_i - \sum r_{pq} s_p S_q\} = d(x_3 - \sum y_2^{(i)} s_i) + \sum r_{pq} s_p s_q$, implies that each $r_{pq} \in \mathfrak{m}$. Take $P_{pq} \in E_1$ such that $dP_{pq} = r_{pq}$. Then, we see easily that $x_3 - \sum_I y_2^{(i)} s_i + \sum P_{pq} s_p s_q \in Z_3(E)$. Hence, we have $v_x = x_3 + \sum (dy_2^{(i)}) S_i - \sum (dP_{pq}) s_p S_q = x_3 + \{d(\sum y_2^{(i)} S_i) - \sum y_2^{(i)} s_i\} - \{d(\sum P_{pq} s_p S_q) - \sum P_{pq} s_p s_q\} = \{x_3 - \sum y_2^{(i)} s_i + \sum P_{pq} s_p s_q\} + d(\sum y_2^{(i)} S_i - \sum P_{pq} s_p S_q) \in Z_3(E) + B_3(X^{(2)})$.

In particular, if $\{x_1^{(1)}, \ldots, x_1^{(\varepsilon_1)}\} = \{0, \ldots, 0, \overset{j}{\breve{s_i}}, 0, \ldots, 0\}$, then

$$v_{ij} = w_{ij} + s_i S_j - \sum_{(p,q)\in I} r_{pq}^{(ij)} s_p S_q$$

belongs to $Z_3(X^{(2)})$ for some w_{ij} in E_3 and $r_{pq}^{(ij)}$ in R. And, moreover, these v_{ij} can be imposed on the following conditions:

$$v_{ij} + v_{ji} = d(S_i S_j) \text{ if } i \neq j \quad \text{and} \quad v_{ii} = d(S_i^{(2)}).$$

For, by the above construction of v_{ij}, w_{ij} and $r_{pq}^{(ij)}$ can be selected so as to $s_i s_j = \sum_I r_{pq}^{(ij)} s_p s_q + dw_{ij}$ holds and therefore we can assume w_{ij} and $r_{pq}^{(ij)}$ satisfy the relations, $w_{ij} + w_{ji} = 0$, $w_{ii} = 0$, $r_{pq}^{(ij)} + r_{pq}^{(ji)} = 0$ and $r_{pq}^{(ii)} = 0$, by virtue of $s_i s_j = -s_j s_i$ and $s_i s_i = 0$. Consequently, $v_{ij} + v_{ji} = s_i S_j + s_j S_i = d(S_i S_j)$ and $v_{ii} = s_i S_i = d(S_i^{(2)})$.

Lemma 3. $Z_3(X^{(2)}) = B_3(X^{(2)}) + Z_3'(E) + \sum_{(i,j)\in J} R v_{ij}$, *where* $Z_3'(E)$ *is an* R-*submodule of* $Z_3(E)$ *generated by cycles in* E_3 *whose homology classes constitute a base of* $H_3(E)$ *modulo* $H_1(E) H_2(E)$.

PROOF. We first remark that $Z_1(E)Z_2(E) \subset B_3(X^{(2)})$. In fact, let $x \in Z_1(E)$ and $y \in Z_2(E)$. Then, $x = \sum \lambda_i s_i + x'$ where $\lambda_i \in R$ and $x' \in B_1(E)$. Fix x'' in E_2 such that $dx'' = x'$. Then, $xy = (\sum \lambda_i s_i + x')y = (\sum \lambda_i (dS_i))y + (dx'')y = d((\sum \lambda_i S_i)y + x''y) \in B_3(X^{(2)})$. Hence, to prove the lemma, it is enough to show that $Z_3(X^{(2)}) \subset B_3(X^{(2)}) + Z_3(E) + \sum_J R v_{ij}$. Let $v = x_3 + \sum_{i=1}^{\varepsilon_1} x_1^{(i)} S_i \in Z_3(X^{(2)})$, then $x_1^{(i)} \in Z_1(E)$ $(i=1, \dots, \varepsilon_1)$ and hence each $x_1^{(i)}$ is an R-linear combination of $s_1, \dots, s_{\varepsilon_1}$ modulo $B_1(E)$. Therefore, subtracting suitable R-linear combinations of v_{ij} from v, we can assume each $x_1^{(i)}$ is contained in $B_1(E)$. Hence our conclusion follows from the remarks stated before the lemma.

§2. The calculation of higher deflections.

By making use of the lemmas of the preceding section, we can prove the following theorem.

THEOREM 1. $\varepsilon_3 = \dim_K H_3(E)/H_1(E)H_2(E) + \binom{\varepsilon_1}{2} - \dim_K H_1(E)^2$.

In particular, if $H_1(E)^2 = H_1(E)H_2(E) = 0$, *then*

$$\varepsilon_3 = \dim_K H_3(E) + \binom{\varepsilon_1}{2}.$$

PROOF. Let $h = \dim_K H_3(E)/H_1(E)H_2(E)$ and let $\pi_1, \dots, \pi_h$ be cycles in $Z_3(E)$ such that whose homology classes constitute a base of the vector space $H_3(E)$ modulo $H_1(E)H_2(E)$. To prove our theorem, in view of lemma 1 and 3, it is enough to show that π_i $(i=1, \dots, h)$ and v_{ij}, $(i, j) \in J$, are linearly independent over K modulo $B_3(X^{(2)})$. For this, suppose that

$$x = \sum_{i=1}^{h} \alpha_i \pi_i + \sum_{(ij) \in J} \beta_{ij} v_{ij} \in B_3(X^{(2)}) = d(E_4 + \sum_{k=1}^{\varepsilon_1} E_2 S_k + \sum_{1 \le i < j \le \varepsilon_1} R S_i S_j + \sum_{i=1}^{\varepsilon_1} R S_i^{(2)}),$$

where α_i and β_{ij} are elements in R and we contend that $\alpha_i, \beta_{ij} \in \mathfrak{m}$.

Now,

$$\begin{aligned} x &= d(x_4 + \sum x_2^{(k)} S_k + \sum \mu_{ij} S_i S_j + \sum \nu_i S_i^{(2)}) \quad (x_4 \in E_4,\ x_2^{(k)} \in E_2 \text{ and } \mu_{ij}, \nu_i \in R) \\ &= (dx_2^{(1)} + \nu_1 s_1 + \mu_{12} s_2 + \cdots + \mu_{1\varepsilon_1} s_{\varepsilon_1}) S_1 + (dx_2^{(2)} + \mu_{12} s_1 + \nu_2 s_2 + \mu_{23} s_3 + \cdots \\ &\quad + \mu_{2\varepsilon_1} s_{\varepsilon_1}) S_2 + \cdots + (dx_2^{(\varepsilon_1)} + \mu_{1\varepsilon_1} s_1 + \cdots + \mu_{\varepsilon_1 - 1, \varepsilon_1} s_{\varepsilon_1 - 1} + \nu_{\varepsilon_1} s_{\varepsilon_1}) S_{\varepsilon_1} + dx_4 \\ &\quad + \sum x_2^{(k)} s_k. \end{aligned}$$

Since

$$\beta_{ij} v_{ij} = \beta_{ij} (w_{ij} + s_i S_j - \sum_I r_{pq}^{(ij)} s_p S_q),$$

we have

$$\sum \alpha_i \pi_i = \{dx_2^{(1)} + \nu_1 s_1 + \mu_{12} s_2 + \cdots + \mu_{1\varepsilon_1} s_{\varepsilon_1}\} S_1 + \{dx_2^{(2)} + (\mu_{12} - \beta'_{12}) s_1 + \nu_2 s_2 + \mu_{23} s_3 + \cdots + \mu_{2\varepsilon_1} s_{\varepsilon_1}\} S_2 + \cdots + \{(dx_2^{(\varepsilon_1)} + (\mu_{1\varepsilon_1} - \beta'_{1\varepsilon_1}) s_1 + \cdots + (\mu_{\varepsilon_1 - 1, \varepsilon_1} - \beta'_{\varepsilon_1 - 1, \varepsilon_1}) s_{\varepsilon_1 - 1} + \nu_{\varepsilon_1} s_{\varepsilon_1}\} S_{\varepsilon_1} + dx_4 + \sum x_2^{(k)} s_k - \sum_J \beta_{ij} w_{ij},$$

where

$$\beta'_{kh} = \begin{cases} \beta_{kh} & \text{if } (k, h) \in J, \\ -\sum_{(ij) \in J} r_{kh}^{(ij)} \beta_{ij} & \text{if } (k, h) \in I. \end{cases}$$

Considering the coefficients of $S_i (i = 1, 2, \ldots, \varepsilon_1)$, we get

$$\begin{cases} \nu_1 s_1 + \mu_{12} s_2 + \cdots + \mu_{1\varepsilon_1} s_{\varepsilon_1} = -dx_2^{(1)} \in B_1(E) \\ (\mu_{12} - \beta'_{12}) s_1 + \nu_2 s_2 + \cdots + \mu_{2\varepsilon_1} s_{\varepsilon_1} = -dx_2^{(2)} \in B_1(E) \\ \cdots \\ (\mu_{1\varepsilon_1} - \beta'_{1\varepsilon_1}) s_1 + \cdots + (\mu_{\varepsilon_1 - 1, \varepsilon_1} - \beta'_{\varepsilon_1 - 1, \varepsilon_1}) s_{\varepsilon_1 - 1} + \nu_{\varepsilon_1} s_{\varepsilon_1} = -dx_2^{(\varepsilon_1)} \in B_1(E). \end{cases}$$

Hence, $\nu_i (i = 1, \ldots, \varepsilon_1)$, μ_{ij} and $\beta'_{ij} (1 \leq i < j \leq \varepsilon_1)$ each belongs to $\mathfrak{m}$, since $s_i (i = 1, \ldots, \varepsilon_1)$ are linearly independent over K modulo $B_1(E)$. In particular, $\beta_{ij} \in \mathfrak{m}$ for any $(i, j) \in J$.

Take P_i, Q_{ij} and R_{ij} in E_1 such that

$$\nu_i = dP_i, \quad \mu_{ij} = dQ_{ij}, \quad \beta'_{ij} = dR_{ij}.$$

It is clear that $R_{ij} (1 \leq i < j \leq \varepsilon_1)$ can be imposed on the following relation:

$$R_{pq} = -\sum_{(ij) \in J} R_{ij} r_{pq}^{(ij)} \quad \text{for } (p, q) \in I.$$

Then, obviously,

$$\nu_i s_i = d(P_i s_i), \quad \mu_{ij} s_k = d(Q_{ij} s_k), \quad \beta'_{ij} s_k = d(R_{ij} s_k).$$

Hence

$$\begin{cases} x_2^{(1)} + P_1 s_1 + Q_{12} s_2 + \cdots + Q_{1\varepsilon_1} s_{\varepsilon_1} \in Z_2(E) \\ x_2^{(2)} + (Q_{12} - R_{12}) s_1 + P_2 s_2 + \cdots + Q_{2\varepsilon_1} s_{\varepsilon_1} \in Z_2(E) \\ \cdots \\ x_2^{(\varepsilon_1)} + (Q_{1\varepsilon_1} - R_{1\varepsilon_1}) s_1 + \cdots + (Q_{\varepsilon_1 - 1, \varepsilon_1} - R_{\varepsilon_1 - 1, \varepsilon_1}) s_{\varepsilon_1 - 1} + P_{\varepsilon_1} s_{\varepsilon_1} \in Z_2(E) \end{cases}$$

and we denote these cycles by $y_2^{(1)}, \ldots, y_2^{(\varepsilon_1)}$. Then,

$$\sum_{i=1}^{h}\alpha_i\pi_i = dx_4+\sum_{k=1}^{\varepsilon_1}x_2^{(k)}s_k-\sum_J\beta_{ij}w_{ij}$$

$$= dx_4+\sum y_2^{(k)}s_k-\{(P_1s_1+Q_{12}s_2+\cdots+Q_{1\varepsilon_1}s_{\varepsilon_1})s_1+\cdots+((Q_{1\varepsilon_1}-R_{1\varepsilon_1})s_1$$

$$+\cdots+(Q_{\varepsilon_1-1,\varepsilon_1}-R_{\varepsilon_1-1,\varepsilon_1})s_{\varepsilon_1-1}+P_{\varepsilon_1}s_{\varepsilon_1})s_{\varepsilon_1}\}-\sum_J\beta_{ij}w_{ij}$$

$$= dx_4+\sum y_2^{(k)}s_k+\sum_{1\le k<h\le\varepsilon_1}R_{kh}s_ks_h-\sum_J\beta_{ij}w_{ij},$$

in view of $s_is_j+s_js_i=0$ $(i\neq j)$ and $s_is_i=0$.

Since

$$\sum_{1\le k<h\le\varepsilon_1}R_{kh}s_ks_h-\sum_J\beta_{ij}w_{ij}=\sum_I R_{pq}s_ps_q+\sum_J R_{ij}s_is_j-\sum_J\beta_{ij}w_{ij}$$

$$=\sum_I R_{pq}s_ps_q+\sum_J R_{ij}(\sum_I r_{pq}^{(ij)}s_ps_q+dw_{ij})-\sum_J\beta_{ij}w_{ij}$$

$$=\sum_I\{R_{pq}+(\sum_J R_{ij}r_{pq}^{(ij)})\}s_ps_q+\sum_J R_{ij}dw_{ij}-\sum_J\beta_{ij}w_{ij}$$

$$=-d(\sum_J R_{ij}w_{ij}),$$

we finally have

$$\sum_{i=1}^{h}\alpha_i\pi_i = dx_4+\sum_{k=1}^{\varepsilon_1}y_2^{(k)}s_k-d(\sum_J R_{ij}w_{ij})\in Z_1(E)Z_2(E)+B_3(E)$$

and consequently $\alpha_i\in\mathfrak{m}$ $(i=1,\dots,h)$, which complete our proof.

Next, we compute ε_4 in some restricted case. Since our computation is quite similar to that of ε_3, the detail of it shall be omitted.

LEMMA 4. *If $n\leq 3$ and $H_1(E)^2=0$, then we have*

$$\varepsilon_4=\varepsilon_1\varepsilon_2-\dim_K H_1(E)H_2(E).$$

PROOF. We have proved that $\varepsilon_4=\dim H_4(X^{(3)})$ (lemma 1) and $\varepsilon_2=\dim H_2(E)$ by our assumption. Let I be a set of integers (p, q), $1\leq p\leq\varepsilon_1$, $1\leq q\leq\varepsilon_2$ such that homology classes of s_pu_q, $(p, q)\in I$, form a base of the vector space $H_1(E)H_2(E)$, and let $J=\{(i, j)\,|\,1\leq i\leq\varepsilon_1,\ 1\leq j\leq\varepsilon_2,\ (i, j)\notin I\}$. Then, for any $(i, j)\in J$, we can find $r_{pq}^{(ij)}\in R$, $(p, q)\in I$, such that

$$z_{ij}=s_iU_j-\sum_I r_{pq}^{(ij)}s_pU_q$$

belongs to $Z_4(X^{(3)})$. With these z_{ij}, $(i, j)\in J$, we can prove $Z_4(X^{(3)})=B_4(X^{(3)})+\sum_J Rz_{ij}$ and, moreover, these z_{ij} are linearly independent cycles modulo $B_4(X^{(3)})$.

§3. An application to the Betti series of local rings of embedding dimension 3.

In this section we restrict the case when the embedding dimension n is 3 and consider the Betti series of R under the additional assumption that the multiplication in $H(E)$ is trivial, i.e., we assume $H_1(E)^2=H_1(E)H_2(E)=0$. Hence

$$Z_1(E)^2\subset B_2(E),\ \ Z_1(E)Z_2(E)\subset B_3(E)=0,\ \ H_3(E)=Z_3(E)\approx 0:\mathfrak{m}.$$

Our assumption also implies that, with the same notations as in §1, $I=\emptyset$ (empty set), $J=\{(i,j)\,|\,1\leq i<j\leq\varepsilon_1\}$ and $r_{pq}^{(ij)}=0$ so that

$$v_{ij}=w_{ij}+s_iS_j\ \ (1\leq i<j\leq\varepsilon_1).$$

Let X be a minimal R-algebra resolution of the residue field K of R [3, 5, 7].

$$X:\ \cdots\to X_i\to X_{i-1}\to\cdots\to X_2\to X_1\to X_0\overset{\varepsilon}{\to}K\to 0,$$

where ε is the augmentation homomorphism. Then, $X_i(i=1,2,\ldots)$ has the following form:

$$X_0=R,\ \ X_1=E_1,\ \ X_2=E_2+\sum_{i=1}^{\varepsilon_1}RS_i,\ \ X_3=E_3+\sum_{j=1}^{\varepsilon_1}E_1S_j+\sum_{i=1}^{\varepsilon_2}RU_i,$$
$$X_4=\sum_{k=1}^{\varepsilon_1}E_2S_k+\sum_{1\leq i<j\leq\varepsilon_1}RS_iS_j+\sum_{i=1}^{\varepsilon_1}RS_i^{(2)}+\sum_{j=1}^{\varepsilon_2}E_1U_j+\sum_{i=1}^{\varepsilon_3}RV_i,\ \cdots$$

where $E=\{E_i\}_{i=0,1,2,3}$ is the Koszul complex of R and $S_i(i=1,\ldots,\varepsilon_1)$, $U_i(i=1,\ldots,\varepsilon_2)$ and V_i $(i=1,\ldots,\varepsilon_3)$ are variables of degree 2, 3 and 4 respectively.

Let $M=dX_4$ and let $\{c_1,\ldots,c_\delta\}$ be a minimal generating system of $0:\mathfrak{m}$ $(\delta=\dim_K(0:\mathfrak{m}))$. Since we can take $c_iT_1T_2T_3(i=1,\ldots,\delta)$ and $v_{ij}(1\leq i<j\leq\varepsilon_1)$ as $v_1,\ldots,v_{\varepsilon_3}$ (theorem 1), M can be written as $M=M_1+M_2+M_3$, where

$$M_1=d(\textstyle\sum E_2S_k+\sum RS_iS_j+\sum RS_i^{(2)})+\sum Rv_{ij}$$
$$M_2=d(\textstyle\sum E_1U_j)$$
$$M_3=(0:\mathfrak{m})\,T_1T_2T_3.$$

Lemma 5. *$M_1\approx\overset{\varepsilon_1}{\bigoplus}N$, where $N=dX_2=B_1(E)+\sum Rs_i$.*

Proof. Let $x\in M_1$. Then

$$x=d(\textstyle\sum x_2^{(k)}S_k+\sum\mu_{ij}S_iS_j+\sum\nu_iS_i^{(2)})+\sum\beta_{ij}v_{ij}\quad(x_2^{(k)}\in E_2,\ \mu_{ij},\ \nu_i,\ \beta_{ij}\in R)$$
$$=(dx_2^{(1)}+\nu_1s_1+\mu_{12}s_2+\cdots+\mu_{1\varepsilon_1}s_{\varepsilon_1})S_1+(dx_2^{(2)}+(\mu_{12}+\beta_{12})s_1+\nu_2s_2$$
$$+\mu_{23}s_3+\cdots+\mu_{2\varepsilon_1}s_{\varepsilon_1})S_2+\cdots+(dx_2^{(\varepsilon_1)}+(\mu_{1\varepsilon_1}+\beta_{1\varepsilon_1})s_1+\cdots+$$
$$(\mu_{\varepsilon_1-1,\varepsilon_1}+\beta_{\varepsilon_1-1,\varepsilon_1})s_{\varepsilon_1-1}+\nu_{\varepsilon_1}s_{\varepsilon_1})S_{\varepsilon_1}+\textstyle\sum x_2^{(k)}s_k+\sum\beta_{ij}w_{ij}.$$

If $\phi: M_1 \to \overset{\varepsilon_1}{\oplus} N$ is an R-homomorphism defined by

$$\phi(x)=(dx_2^{(1)}+\nu_1 s_1+\cdots)S_1+\cdots+(dx_2^{(\varepsilon_1)}+(\mu_{1\varepsilon_1}+\beta_{1\varepsilon_1})s_1+\cdots)S_{\varepsilon_1},$$

i.e., the projection of M_1 on the sum of its first ε_1-factors, then clearly ϕ is surjective.

Now, we shall show that ϕ is injective. Assume $x \in \operatorname{Ker}\phi$. Then,

$$x=\sum_{k=1}^{\varepsilon_1} x_2^{(k)} s_k+\sum_{i<j}\beta_{ij}w_{ij}$$

and

$$\begin{cases} dx_2^{(1)}+\nu_1 s_1+\mu_{12}s_2+\cdots+\mu_{1\varepsilon_1}s_{\varepsilon_1}=0 \\ dx_2^{(2)}+(\mu_{12}+\beta_{12})s_2+\cdots+\mu_{2\varepsilon_1}s_{\varepsilon_1}=0 \\ \cdots \\ dx_2^{(\varepsilon_1)}+(\mu_{1\varepsilon_1}+\beta_{1\varepsilon_1})s_1+\cdots+(\mu_{\varepsilon_1-1,\varepsilon_1}+\beta_{\varepsilon_1-1,\varepsilon_1})s_{\varepsilon_1-1}+\nu_{\varepsilon_1}s_{\varepsilon_1}=0 \end{cases}$$

Hence, ν_i, μ_{ij} and $\beta_{ij} \in \mathfrak{m}$ and

$$\begin{cases} y_2^{(1)}=x_2^{(1)}+P_1 s_1+Q_{12}s_2+\cdots+Q_{1\varepsilon_1}s_{\varepsilon_1} \in Z_2(E) \\ y_2^{(2)}=x_2^{(2)}+(Q_{12}+R_{12})s_1+P_2 s_2+\cdots+Q_{2\varepsilon_1}s_{\varepsilon_1} \in Z_2(E) \\ \cdots \\ y_2^{(\varepsilon_1)}=x_2^{(\varepsilon_1)}+(Q_{1\varepsilon_1}+R_{1\varepsilon_1})s_1+\cdots+(Q_{\varepsilon_1-1,\varepsilon_1}+R_{\varepsilon_1-1,\varepsilon_1})s_{\varepsilon_1-1}+P_{\varepsilon_1}s_{\varepsilon_1} \in Z_2(E), \end{cases}$$

where P_i, Q_{ij} and $R_{ij} \in E_1$ such that $dP_i=\nu_i$, $dQ_{ij}=\mu_{ij}$ and $dR_{ij}=\beta_{ij}$.

Therefore, we have

$$\begin{aligned} x &= \sum y_2^{(i)} s_i-\{(P_1 s_1+Q_{12}s_2+\cdots+Q_{1\varepsilon_1}s_{\varepsilon_1})s_1+\cdots+((Q_{1\varepsilon_1}+R_{1\varepsilon_1})s_1 \\ &\quad +\cdots+(Q_{\varepsilon_1-1,\varepsilon_1}+R_{\varepsilon_1-1,\varepsilon_1})s_{\varepsilon_1-1}+P_{\varepsilon_1}s_{\varepsilon_1})s_{\varepsilon_1}\}+\sum\beta_{ij}w_{ij} \\ &= -\sum R_{ij}s_i s_j+\sum\beta_{ij}w_{ij} \quad (\text{since } \sum y_2^{(i)}s_i \in Z_2(E)Z_1(E)=0) \\ &= d(\sum R_{ij}w_{ij}) \\ &= 0, \end{aligned}$$

since $\sum R_{ij}w_{ij} \in E_4=0$.

LEMMA 6. $M_2 \approx \overset{\varepsilon_2}{\oplus}\mathfrak{m}$ and $M_3 \approx \overset{\delta}{\oplus}K$, where $\delta=\dim_K(0:\mathfrak{m})=\varepsilon_3-\binom{\varepsilon_1}{2}$.

PROOF. Since $M_2=d(\sum_{j=1}^{\varepsilon_2} E_1 U_j)$, $x \in M_2$ can be written as

$$x=\sum_{j=1}^{\varepsilon_2}(dx_1^{(j)})U_j-\sum_{j=1}^{\varepsilon_2}x_1^{(j)}u_j,$$

where $x_1^{(j)}\in E_1$ and $u_j\in Z_2(E)$ (lemma 2). Let ψ: $M_2\to\sum_{j=1}^{\varepsilon_2}\mathfrak{m}U_j$ be an R-homomorphism defined by

$$\psi(x)=\sum_{j=1}^{\varepsilon_2}(dx_1^{(j)})U_j.$$

Obviously, ψ is surjective. If $x\in\operatorname{Ker}\psi$, then $dx_1^{(j)}=0$ $(j=1,\dots,\varepsilon_2)$ so that $x_1^{(j)}\in Z_1(E)$ and, consequently, $x=-\sum x_1^{(j)}u_j\in Z_1(E)Z_2(E)=0$.

For the second assertion of the lemma, we consider the map η: $\overset{\delta}{\oplus}R\to 0:\mathfrak{m}$ defined by

$$\eta(\overset{\delta}{\oplus}r_i)=\sum_{i=1}^{\delta}r_ic_i,$$

where $\{c_1,\dots,c_\delta\}$ ($\delta=\dim(0:\mathfrak{m})=\varepsilon_3-\binom{\varepsilon_1}{2}$, by theorem 1) is a fixed minimal system of generators of $0:\mathfrak{m}$. It is clear that η induces a bijection between $\overset{\delta}{\oplus}K$ and $0:\mathfrak{m}$.

LEMMA 7. $M=M_1\oplus M_2\oplus M_3$ (direct).

PROOF. Let $\theta_i\in M_i$ $(i=1,2,3)$ and suppose $\theta_1+\theta_2+\theta_3=0$. Then, with the same notations of the preceding lemmas, we write θ_i as

$$\theta_1=d\Big(\sum_{k=1}^{\varepsilon_1}x_2^{(k)}S_k+\sum_{i<j}\mu_{ij}S_iS_j+\sum_{i=1}^{\varepsilon_1}\nu_iS_i^{(2)}\Big)+\sum_{i<j}\beta_{ij}v_{ij}$$
$$\theta_2=\sum_{j=1}^{\varepsilon_2}(dx_1^{(j)})U_j-\sum_{j=1}^{\varepsilon_2}x_1^{(j)}u_j$$
$$\theta_3=\sum_{i=1}^{\delta}r_ic_iT_1T_2T_3,\qquad \delta=\varepsilon_3-\binom{\varepsilon_1}{2}.$$

Considering the coefficients of $U_j(j=1,\dots,\varepsilon_2)$, we have $dx_1^{(j)}=0$ so that $\theta_2\in\operatorname{Ker}\psi$ and hence $\theta_2=0$ by lemma 6.

Now we have $\theta_1+\theta_3=0$. Then, each coefficient of $S_i(i=1,\dots,\varepsilon_1)$ is equal to zero and hence $\theta_1\in\operatorname{Ker}\phi$. This implies θ_1 is actually zero by lemma 5 and $\theta_3=0$.

THEOREM 2. *Let $(R,\mathfrak{m})$ be a local ring of embedding dimension 3. Suppose that the multiplication in $H(E)$ is trivial, where E is the Koszul complex of R. Then, the Betti series $\mathscr{B}(R)$ of R has the following form:*

$$\mathscr{B}(R)=\frac{(1+Z)^3}{1-\varepsilon_1Z^2-\varepsilon_2Z^3-\big(\varepsilon_3-\binom{\varepsilon_1}{2}\big)Z^4},$$

where ε_i $(i=1, 2, 3)$ *is the i-th deflection of* R. *Moreover,*

$$\varepsilon_4=\varepsilon_1\varepsilon_2 \quad \textit{and} \quad \varepsilon_3=\varepsilon_2+\binom{\varepsilon_1-1}{2}.$$

PROOF. By preceding lemmas, we have

$$M=dX_4=M_1\oplus M_2\oplus M_3\approx(\overset{\varepsilon_1}{\oplus}N)\oplus(\overset{\varepsilon_2}{\oplus}\mathfrak{m})\oplus(\overset{\delta}{\oplus}K) \quad \left(\delta=\varepsilon_3-\binom{\varepsilon_1}{2}\right).$$

Since the functor Tor is additive,

$$\mathrm{Tor}_p(M, K)=(\overset{\varepsilon_1}{\oplus}\mathrm{Tor}_p(N, K))\oplus(\overset{\varepsilon_2}{\oplus}\mathrm{Tor}_p(\mathfrak{m}, K))\oplus(\overset{\delta}{\oplus}\mathrm{Tor}_p(K, K)),$$

for $p\geq 0$. Hence, for $p\geq 0$,

$$\mathrm{Tor}_{p+4}(K, K)=(\overset{\varepsilon_1}{\oplus}\mathrm{Tor}_{p+2}(K, K))\oplus(\overset{\varepsilon_2}{\oplus}\mathrm{Tor}_{p+1}(K, K))\oplus(\overset{\delta}{\oplus}\mathrm{Tor}_p(K, K)).$$

Therefore, we have

$$B_{p+4}=\varepsilon_1 B_{p+2}+\varepsilon_2 B_{p+1}+\delta B_p \quad (p\geq 0).$$

Now, it is easy to see that this recurrence relation implies the representation of $\mathscr{B}(R)$ for the first part of the theorem.

For the second part, $\varepsilon_4=\varepsilon_1\varepsilon_2$ is an immediate consequence of lemma 4. As for ε_3, the statement is true for regular local rings since in this case $\varepsilon_i=0\,(i=1, 2, \ldots)$. Assume R is not regular, then the Euler-Poincaré characteristic of E is 0, i.e.,

$$\dim_K H_3(E)-\dim_K H_2(E)+\dim_K H_1(E)-\dim_K H_0(E)=0.$$

Combining this with $\dim H_3(E)=\dim(0:\mathfrak{m})$, $\varepsilon_2=\dim H_2(E)/H_1(E)^2=\dim H_2(E)$, $\varepsilon_1=\dim H_1(E)$, $\dim H_0(E)=1$ and $\varepsilon_3=\dim(0:\mathfrak{m})+\binom{\varepsilon_1}{2}$ (theorem 1), we obtain $\varepsilon_3=\varepsilon_2+\binom{\varepsilon_1-1}{2}$.

We remark here that the rational expression of $\mathscr{B}(R)$ obtained by G. Scheja, in the case codh $R\geq n-2$ [6, Satz 9], coincides with that given in theorem 2. But, the following is a simple example of a local ring of Krull dimension 0 which satisfies the assumptions in theorem 2:

$$R=K[[X, Y, Z]]/\mathfrak{a},$$

where K is a field and $\mathfrak{a}$ is defined by $(X^3-Y^3, Y^3-Z^3, XY^2, XZ^2, YZ^2, YX^2, ZX^2, ZY^2)$. Thus, theorem 2 is independent of Scheja's result.

§4. Concluding remarks.

Recently, H. Wiebe showed that if R is a local Gorenstein ring of embedding dimension 3 and is not a complete intersection, then the Betti series of R has the following form:

$$(*) \quad \mathscr{B}(R)=(1+Z)^3/1-\varepsilon_1 Z^2-\varepsilon_1 Z^3+Z^5, \quad \varepsilon_1=\varepsilon_2 \ [10].$$

It is to be mentioned that, in his argument, the multiplicative property of $H(E)$ plays an essential role. Precisely, he proved that $H(E)$ satisfies the relations, $H_1(E)^2=0$, $H_1(E)H_2(E)=H_3(E)$ and $\dim_K H_3(E)=1$ and under these conditions he decided the form of the Betti series mentioned above. If we consider higher deflections in this case, we have $\varepsilon_3=\binom{\varepsilon_1}{2}$ by theorem 1 and $\varepsilon_4=\varepsilon_1\varepsilon_2-1$ by lemma 4. Thus, (*) can be rewritten as

$$(**) \quad \mathscr{B}(R)=(1+Z)^3/1-\varepsilon_1 Z^2-\varepsilon_2 Z^3-(\varepsilon_3-\binom{\varepsilon_1}{2})Z^4-(\varepsilon_4-\varepsilon_1\varepsilon_2)Z^5.$$

On one hand, in the case when $H(E)$ has trivial multiplication, which we treated in theorem 2, we have proved that $\varepsilon_4=\varepsilon_1\varepsilon_2$ so that the Betti series of such ring is also given by (**). If $\mathscr{B}(R)$ has the form (**), we can further calculate ε_5 directly and we find $\varepsilon_5=\varepsilon_1\varepsilon_3-\binom{\varepsilon_1}{3}+\varepsilon_2^2-\binom{\varepsilon_2}{2}$. And, it is easy to check that, if $\varepsilon_i=0$ for $i\geq 2$, the polynomial $1-\varepsilon_1 Z^2-\varepsilon_2 Z^3-(\varepsilon_3-\binom{\varepsilon_1}{2})Z^4-(\varepsilon_4-\varepsilon_1\varepsilon_2)Z^5-\varepsilon' Z^6$, $\varepsilon'=\varepsilon_5-\left\{\varepsilon_1\varepsilon_3-\binom{\varepsilon_1}{3}+\varepsilon_2^2-\binom{\varepsilon_2}{2}\right\}$, is equal to $1-Z^2$, $(1-Z^2)^2$ and $(1-Z^2)^3$ according to $\varepsilon_1=1$, 2 and 3 respectively. Now, we summarize these remarks in the following

THEOREM 3. *If R is of embedding dimension* 3 *and if R is Gorenstein or $H(E)$ has trivial multiplication, then*

$$\mathscr{B}(R)=(1+Z)^3/1-\varepsilon_1 Z^2-\varepsilon_2 Z^3-(\varepsilon_3-\binom{\varepsilon_1}{2})Z^4-(\varepsilon_4-\varepsilon_1\varepsilon_2)Z^5-\varepsilon' Z^6,$$

where $\varepsilon'=\varepsilon_5-\left\{\varepsilon_1\varepsilon_3-\binom{\varepsilon_1}{3}+\varepsilon_2^2-\binom{\varepsilon_2}{2}\right\}$.

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Addendum

Recently, we have informed from T. H. Gulliksen that the same result as Theoren 1 (§2) have been obtained by G. Levin. See T. H. Gulliksen and G. Levin: *Homology of local rings*, Queen's papers in pure and applied mathematics-No. 20, 1969.