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On the Connected Refinements of Topologies of Locally Connected Continua

By

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1. Introduction.

Let I be the closed unit interval and τ_0 the usual topology on I. In [1], S. K. Hildebrand has shown the following theorem: If τ is a finer connected topology for I than τ_0, then any connected subset of I under τ_0 will be a connected subset of I under τ. The purpose of this paper is to generalize this theorem.

2. Definitions.

If σ is a topology on a set X, the resulting space will be denoted by (X, σ) and the family of all connected subsets of (X, σ) by $\boldsymbol{C}(X, \sigma)$. If (X, σ) is connected, then σ is called a connected topology on X. Let A be a subset of X. Then "A is σ-P" means that A has the property P in (X, σ) and $Cl_\sigma A$ denotes the closure of A in (X, σ). The topology $\{U \cup (V \cap A) \mid U, V \in \sigma\}$ on X is said to be the simple extension of σ by A [2]. Other undefined terms can be extracted from Whyburn's Analytic Topology [3].

3. Main results.

THEOREM. *Let (X, σ_0) be a locally connected, connected and compact T_2-space satisfying the second axiom of countability.*

Then in order that for any finer connected topology σ on X than σ_0 we have $\boldsymbol{C}(X, \sigma)=\boldsymbol{C}(X, \sigma_0)$, the following conditions are necessary and sufficient:

(1) *(X, σ_0) does not contain any simple closed curve.*

(2) *Any simple arc in (X, σ_0) contains at most a finite number of branch points of (X, σ_0).*

The proof of the theorem consists of the following four lemmas.

LEMMA 1. *If S is any simple closed curve in (X, σ_0), then S contains an uncountable number of non-cut points of (X, σ_0).*

PROOF. Let $[ab]$ be a simple subarc from a to b of S. Then no points of (X, σ_0) separate (X, σ_0) between a and b, for S is a simple closed curve. Thus Lemma 1 is an immediate consequence of [3, (4.3), p. 51].

LEMMA 2. *If (X, σ_0) contains a simple closed curve, then there exists a finer connected topology σ on X than σ_0 such that $\boldsymbol{C}(X, \sigma) \neq \boldsymbol{C}(X, \sigma_0)$.*

PROOF. Let S be a simple closed curve of (X, σ_0). Then by Lemma 1 S contains a non-cut point p of (X, σ_0). Let $[apb]$ be a simple subarc from a to b of S in which p is an interior point. Let σ be the simple extension of σ_0 by $(X-[pb]) \cup p$, where $[pb]$ is the σ_0-simple subarc of $[apb]$. Then σ satisfies the conditions of Lemma 2.

First, it is obvious from the definition of the simple extensions of σ_0 that σ is finer than σ_0.

Next, we shall show that (X, σ) is connected. Any open subset of $(X-p, \sigma)$ can be written in a form $U \cup [V \cap \{(X-[pb]) \cup p\}]-p$, where U and V are open in (X, σ_0), and hence in a form $(U-p) \cup \{(V-p) \cap (X-[pb])\}$. Therefore any open subset of $(X-p, \sigma)$ is open in $(X-p, \sigma_0)$. On the other hand, $(X-p, \sigma_0)$ is connected since p is a non-cut point of (X, σ_0). Hence $(X-p, \sigma)$ is connected. Since any σ-neighbourhood of p can be written in a form $U \cup [V \cap \{(X-[pb]) \cup p\}]$, where U and V are σ_0-open and either U or V contains p, any σ-neighbourhood of p intersects with $[ap]-p$. Hence p belongs to $Cl_\sigma(X-p)$. Thus it follows from the above two facts that (X, σ) is connected.

Last, the σ_0-simple arc $[apb]$ is not σ-connected because $[apb]=[ap] \cup ([pb]-p)$, $[ap]=[apb] \cap \{(X-[pb]) \cup p\}$ and $[pb]-p=[apb] \cap (X-[ap])$ hold. Therefore we have $\boldsymbol{C}(X, \sigma) \neq \boldsymbol{C}(X, \sigma_0)$.

Thus Lemma 2 is proved.

LEMMA 3. *Assume that (X, σ_0) does not contain any simple closed curve and that some simple subarc of (X, σ_0) contains an infinite number of branch points of (X, σ_0).*

Then there exists a finer connected topology σ on X than σ_0 such that $\boldsymbol{C}(X, \sigma) \neq \boldsymbol{C}(X, \sigma_0)$.

PROOF. Let $[ab]$ be a σ_0-simple arc from a to b which contains an infinite number of branch points of (X, σ_0). Without loss of generality, we may assume that there exists a point p and a sequence $\{b_n\}$ of branch points of (X, σ_0) such that every point of $\{b_n\}$ is in the σ_0-simple subarc $[ap]$ of $[ab]$ and p is a σ_0-limit point of $\{b_n\}$. Let σ be the simple extension of σ_0 by $(X-[ap]) \cup p$. Then σ satisfies the conditions in Lemma 3.

First, it is obvious from the definition of the simple extensions of σ_0 that

σ is finer than σ_0.

Second, to show that (X, σ) is connected, for each n let C_n be a σ_0-component of $X-[ap]$ such that b_n is a σ_0-boundary point of C_n. Then every σ_0-neighbourhood of p contains both a point of each σ_0-component of $X-p$ and all but a finite number of the sets C_n of $\{C_n\}$. The set $(X-[ap])\cup p$ contains both all C_n of $\{C_n\}$ and all σ_0-components not containing $[ap]-p$ of $X-p$. Therefore if V is any σ_0-neighbourhood of p, then the set $V\cap\{(X-[ap])\cup p\}$ intersects with each σ_0-component of $X-p$. Moreover, as in the proof of Lemma 2, any σ-neighbourhood of p can be written in a form $U\cup[V\cap\{(X-[ap])\cup p\}]$ where either U or V is a σ_0-open set containing p. Hence p belongs to the σ-closure of every σ_0-component of $X-p$. On the other hand, it is shown that every σ_0-component of $X-p$ is σ-connected, which is shown in the same way as in Lemma 2 that $X-p$ is σ-connected. Thus (X, σ) is connected.

Last, the σ_0-simple arc $[ap]$ is not σ-connected because $[ap]=([ap]-p)\cup p$, $[ap]-p=[ap]\cap(X-p)$ and $p=[ap]\cap\{(X-[ap])\cup p\}$ hold. Therefore we have $\boldsymbol{C}(X, \sigma)\neq\boldsymbol{C}(X, \sigma_0)$.

Thus Lemma 3 is proved.

Lemma 4. *Assume that (X, σ_0) does not contain any simple closed curve and that any simple arc in (X, σ_0) contains at most a finite number of branch points of (X, σ_0).*

Then for any finer connected topology σ on X than σ_0 we have $\boldsymbol{C}(X, \sigma)=\boldsymbol{C}(X, \sigma_0)$.

Proof. The lemma is proved through the following three steps.

First, if $[ab]$ is any σ_0-simple arc whose interior contains no branch points of (X, σ_0), then $[ab]$ is σ-connected. To prove this, suppose, on the contrary, that $[ab]$ is not σ-connected. Let $[ab]=A\cup B$ be a σ-separation of $[ab]$. Now either $a \in A$ or $a \in B$ and either $b \in A$ or $b \in B$. Assume $a \in A$ and $b \in B$. The proof is similar if any of the other three possible cases is considered. The σ_0-boundary of every σ_0-component of $X-[ab]$ consists of exactly one of a and b, since (X, σ_0) contains no simple closed curve and the interior of $[ab]$ contains no branch points of (X, σ_0). Now let A^* be the sum of A together with all the σ_0-components of $X-[ab]$ whose boundary is a and B^* the sum of B together with all the σ_0-components of $X-[ab]$ whose boundary is b. Then $X=A^*\cup B^*$ is a σ-separation of (X, σ) since (X, σ_0) is locally connected and σ is finer than σ_0, which contradicts the fact that (X, σ) is connected. Hence $[ab]$ is σ-connected.

Second, if $[ab]$ is any σ_0-simple arc whose interior contains a finite number of branch points of (X, σ_0), then $[ab]$ is σ-connected. To show this, denote the branch points belonging to the interior of $[ab]$ by $b_1, b_2, \ldots, b_n$ in the order from a to b. Then each of the σ_0-simple subarcs $[ab_1]$, $[b_1b_2]$, ..., $[b_nb]$ of $[ab]$ contains no branch points of (X, σ_0) in its interior. Therefore, by the result of the preceding paragraph, each of $[ab_1]$, $[b_1b_2]$, ..., $[b_nb]$ is σ-connected. Thus $[ab]$ is σ-connected since $[ab]$ can be written in a form $[ab_1]\cup[b_1b_2]\cup\ldots\cup[b_nb]$.

Last, let C be any σ_0-connected subset of (X, σ_0). Then, by [3, (11.21), p. 82], C is σ_0-arcwise connected and hence every two points x and y of C can be joined in C by a σ_0-simple arc $[xy]$. By the results of preceding paragraphs $[xy]$ is σ-connected. Therefore C is σ-connected. Thus $\boldsymbol{C}(X, \sigma)=\boldsymbol{C}(X, \sigma_0)$.

Lemmas 2 and 3 show that the conditions of the theorem are necessary and Lemma 4 shows that the same are sufficient. Thus Theorem is proved.

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References

[1] S. K. Hildebrand, *A connected topology for the unit interval*, Fund. Math., **61** (1967), 133-140.
[2] N. Levine, *Simple extension of topologies*, Amer. Math. Monthly **71** (1964), 22-25.
[3] G. T. Whyburn, *Analytic Topology*, Amer. Math. Soc. Colloq. Publ., **28**, Providence, 1942.