J. Math. Tokushima Univ.
Vol. 3. (1969), 1-20

General Projective Connections and Finsler Metric

By

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(Received September 30, 1969)

This paper is the continuation of the paper[1] formerly written by one of the authors (Ichijyô). In the former paper, the general projective connections on the tangent bundle over a C^∞-manifold were discussed. But, in that case, it was necessary to choose canonical parameters independently. In this paper, we first consider a vector bundle having R^{n+1}, the real number space of $(n+1)$-dimensions, as the standard fibre and a subgroup of $GL(n+1; R)$ as the structural group. This vector bundle was introduced by T. Ōtsuki[2] for studying his restricted projective connection and was named a projective vector bundle.

Now, our intention is on the generalization of the former case to the projective vector bundle. In §§1 and 2, we define the projective vector bundle and the general projective connection on it, and discuss some properties of them. Then, a projectively invariant distribution $\mathfrak{p}$ is defined. The integrability condition for $\mathfrak{p}$ is discussed in §3.

§4 is devoted to the study of the holonomy group of the general projective connection, especially the case in which the holonomy group leaves a certain hypercone invariant is studied. In the last section we try to extend some known results on holonomy groups to the case in which the base manifold of the projective vector bundle is assumed to have a Finsler metric. As for the references, we wish to refer the former paper.

§1. Projective vector bundles

Let M be an n-dimensional differentiable manifold of class C^∞. A vector bundle over M which has R^{n+1} as the standard fibre is constructed as follows.

1) The structural group G_0 is formed by all elements of the type

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1) Y. Ichijyô: A note on general projective spaces of paths and tangent bundles I, Jour. of Math., Tokushima Univ. 1(1967) 11-16.

2) T. Ōtsuki: The Geometry of Connections, Kyôritsu-Shuppan (1957) (Japanese).

$$\begin{pmatrix} g^i_j & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ * & 1 \end{pmatrix} \quad \text{where } (g^i_j) \in GL(n; R)^{1)}.$$

Obviously G_0 is a subgroup of $GL(n+1; R)$.

2) Let U and V be two coordinate neighbourhoods of M such that $U \cap V \neq \phi$, and $(x^1, \ldots, x^n)$, $(\bar{x}^1, \ldots, \bar{x}^n)$ be the local coordinate systems valid on U and V respectively. We define the transition functions $g_{UV}: U \cap V \to G_0$ by

$$g_{UV} = \begin{pmatrix} \dfrac{\partial \bar{x}^i}{\partial x^j} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ c\partial_j \log \Delta & 1 \end{pmatrix}, \tag{1.1}$$

where $\partial_j = \dfrac{\partial}{\partial x^j}$, $\Delta = \det\left(\dfrac{\partial \bar{x}^i}{\partial x^j}\right)$, $\partial_j \log \Delta = \dfrac{1}{\Delta}\partial_j \Delta$ and c is a nonvanishing constant.

The vector bundle defined by above 1) and 2) is called a projective vector bundle and we denote it by $P(M, \pi, R^{n+1}, G_0)$ where P is its total space and $\pi: P \to M$ is the projection of the bundle. If we write (x^i, w^λ) and $(\bar{x}^i, \bar{w}^\lambda)$[2] for the canonical coordinate systems valid on $\pi^{-1}(U)$ and $\pi^{-1}(V)$ respectively, determined by the bundle structure, the w^λ and the $\bar{w}^\lambda$ are related by

$$\begin{pmatrix} \bar{w}^1 \\ \vdots \\ \bar{w}^\infty \end{pmatrix} = \begin{pmatrix} \dfrac{\partial \bar{x}^i}{\partial x^j} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ c\partial_j \log \Delta & 1 \end{pmatrix} \begin{pmatrix} w^1 \\ \vdots \\ w^\infty \end{pmatrix}, \tag{1.2}$$

that is,

$$\bar{w}^i = \frac{\partial \bar{x}^i}{\partial x^j} w^j, \quad \bar{w}^\infty = c w^j \partial_j \log \Delta + w^\infty. \tag{1.3}$$

We call a local coordinate system which follows the law of transformation (1.3) the canonical coordinate system of the projective vector bundle P. In this case the Jacobian matrix J of the coordinate transformation on P is given by

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1) Throughout the paper, the Roman indices i, j, k, l etc. run over the range $1, \ldots, n$ and the Greek indices λ, μ, ν etc. over the range $1, \ldots, n$ and ∞.

2) (x^i, w^λ) and $(\bar{x}^i, \bar{w}^\lambda)$ stand for $(x^1, \ldots, x^n, w^1, \ldots, w^\infty)$ and $(\bar{x}^1, \ldots, \bar{x}^n, \bar{w}^1, \ldots, \bar{w}^\infty)$ respectively. Sometimes we also write (x, w) (resp. $(\bar{x}, \bar{w})$) instead of (x^i, w^λ) (resp. $(\bar{x}^i, \bar{w}^\lambda)$).

$$(1.4)\qquad J=\begin{pmatrix} \dfrac{\partial \bar{x}^i}{\partial x^j} & 0 & 0 \\ \dfrac{\partial^2 \bar{x}^i}{\partial x^j \partial x^k} w^k & \dfrac{\partial \bar{x}^i}{\partial x^j} & 0 \\ cw^k \partial_j \partial_k \log \varDelta & c\partial_j \log \varDelta & 1 \end{pmatrix}.$$

Let P_z be the tangent space of P at $z \in P$. If we assign to each $z \in P$ a subspace Φ_z^V of P_z formed by all vectors tangent to the fibre through z, then we have an $(n+1)$-dimensional distribution on P. We call it the vertical distribution and denote it by Φ^V. A distribution complementary to the Φ^V and invariant under the mapping σ^*: $P \to P$ which maps each $z=(x^i, w^\lambda)$ into $\sigma^*(z)=(x^i, \sigma w^\lambda)$ for $\sigma>0$, will be denoted by Φ^h and we call it the horizontal distribution. In other words, the horizontal distribution Φ^h is defined by following two conditions.

1) $P_z=\Phi_z^V \oplus \Phi_z^h$ (direct sum) for all $z \in P$,

2) $d\sigma^* \Phi^h_{(x,w)}=\Phi^h_{(x,\sigma w)}$ for $\sigma>0$.

If we denote by p^V and p^h the projection operators of vector field on P to Φ^V and Φ^h respectively, these operators should have the following forms.

$$(1.5)\qquad p^V=\begin{pmatrix} 0 & 0 \\ \theta_j^\lambda & \delta_\mu^\lambda \end{pmatrix}, \quad p^h=\begin{pmatrix} \delta_j^i & 0 \\ -\theta_j^\lambda & 0 \end{pmatrix}.$$

Since these operators must be (1, 1)-tensors, the conditions $\bar{p}^V J=Jp^V$ and $\bar{p}^h J=Jp^h$ are necessarily satisfied, where $\bar{p}^V$ and $\bar{p}^h$ are the operators obtained by the coordinate transformation from p^V and p^h respectively. So the law of transformation of the components θ_j^λ under the coordinate transformation of M is

$$(1.6)\qquad \begin{cases} \bar{\theta}_l^i \dfrac{\partial \bar{x}^l}{\partial x^j}-\dfrac{\partial \bar{x}^i}{\partial x^l}\theta_j^l=-\dfrac{\partial^2 \bar{x}^i}{\partial x^j \partial x^k} w^k, \\ \bar{\theta}_l^\infty \dfrac{\partial \bar{x}^l}{\partial x^j}-\theta_j^\infty=c\theta_j^l \partial_l \log \varDelta - cw^l \partial_j \partial_l \log \varDelta, \end{cases}$$

and the condition 2) on Φ^h leads us to the

$$(1.7)\qquad \theta_j^\lambda(x, \sigma w)=\sigma\theta_j^\lambda(x, w) \qquad \text{for } \sigma>0.$$

Conversely, if θ_j^λ satisfying (1.6) and (1.7) are given, we can determine p^V and p^h by (1.5) and distributions Φ^V and Φ^h will be determined.

Next, we consider a 1-dimensional distribution Φ^0 spanned by the vector field $\frac{\partial}{\partial w^\infty} \in \Phi^V$. Then the n-dimensional distribution Φ^v complementary to Φ^0 in Φ^V can be considered. In this case the components of the projection operators p^0 and p^v to Φ^0 and Φ^v will be determined as follows.

First, p^0 has to have the form

$$p^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\eta_j & -\zeta_j & 1 \end{pmatrix},$$

and p^v is determined by $p^V = p^0 + p^v$ since $\Phi^V = \Phi^0 \oplus \Phi^v$ (direct sum) by definition. So p^v has the form

$$p^v = \begin{pmatrix} 0 & 0 & 0 \\ \theta^i_j & \delta^i_j & 0 \\ \theta^\infty_j + \eta_j & \zeta_j & 0 \end{pmatrix},$$

by (1.5).

Secondly, the projection operators must be involutive, so $(p^v)^2 = p^v$, and we have

$$-\eta_j = \theta^\infty_j - \zeta_k \theta^k_j.$$

This determines the forms of p^0 and p^v,

$$p^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \theta^\infty_j - \zeta_k \theta^k_j & -\zeta_j & 1 \end{pmatrix}, \quad p^v = \begin{pmatrix} 0 & 0 & 0 \\ \theta^i_j & \delta^i_j & 0 \\ \zeta_k \theta^k_j & \zeta_j & 0 \end{pmatrix}. \tag{1.8}$$

Again the condition that these operators are (1, 1)-tensors gives the law of transformation of the ζ_j.

$$\zeta_j = \bar{\zeta}_k \frac{\partial \bar{x}^k}{\partial x^j} - c\partial_j \log \varDelta. \tag{1.9}$$

Conversely, if ζ_j satisfying (1.9) are given, we can determine, with the θ^λ_j in (1.5), p^0 and p^v by (1.8), so the distributions Φ^0 and Φ^v will be determined.

Now, let us consider two more distributions. One, $\tilde{\Phi}$ is defined by the direct sum $\tilde{\Phi} = \Phi^h \oplus \Phi^v$ and is naturally of $2n$-dimensions. The projection operator $\tilde{p}$ on it is given by

$$\tilde{p}=\begin{pmatrix} \delta^i_j & 0 & 0 \\ 0 & \delta^i_j & 0 \\ \zeta_k\theta^k_j-\theta^\infty_j & \zeta_j & 0 \end{pmatrix}. \tag{1.10}$$

The other, Φ^H is defined by the direct sum $\Phi^H=\Phi^0\oplus\Phi^h$ and is of $(n+1)$-dimensions. The projection operator p^H on it is given by

$$p^H=\begin{pmatrix} \delta^i_j & 0 & 0 \\ -\theta^i_j & 0 & 0 \\ -\zeta_k\theta^k_j & -\zeta_j & 1 \end{pmatrix}. \tag{1.11}$$

We put

$$\begin{cases} X_i=\dfrac{\partial}{\partial x^i}-\theta^k_i\dfrac{\partial}{\partial w^k}-\theta^\infty_i\dfrac{\partial}{\partial w^\infty}, \\ Y_i=\dfrac{\partial}{\partial w^i}+\zeta_i\dfrac{\partial}{\partial w^\infty}, \\ Z=\dfrac{\partial}{\partial w^\infty}. \end{cases} \tag{1.12}$$

The sets of vector fields $\{X_i\}$, $\{Y_i\}$ and $\{Z\}$ form local basis of Φ^h, Φ^v and Φ^0 respectively. The laws of transformation under the coordinate transformation are

$$\bar{X}_i=\frac{\partial x^k}{\partial \bar{x}^i}X_k,\quad \bar{Y}_i=\frac{\partial x^k}{\partial \bar{x}^i}Y_k,\quad \bar{Z}=Z. \tag{1.13}$$

Calculating the brackets between those vector fields, we obtain

$$\begin{cases} [X_i,\ X_j]=R^k_{ij}Y_k+R^0_{ij}Z, \\ [X_i,\ Y_j]=S^k_{ij}Y_k+S^0_{ij}Z, \\ [X_i,\ Z]=\dfrac{\partial\theta^k_i}{\partial w^\infty}Y_k+\left(\dfrac{\partial\theta^\infty_i}{\partial w^\infty}-\zeta_k\dfrac{\partial\theta^k_i}{\partial w^\infty}\right)Z, \\ [Y_i,\ Y_j]=\left(\dfrac{\partial\zeta_j}{\partial w^i}-\dfrac{\partial\zeta_i}{\partial w^j}\right)Z, \\ [Y_i,\ Z]=\dfrac{\partial\zeta_i}{\partial w^\infty}Z, \end{cases} \tag{1.14}$$

where

$$(1.15)\quad \begin{cases} R^k_{ij}=\left(\dfrac{\partial\theta^k_i}{\partial x^j}-\theta^l_j\dfrac{\partial\theta^k_i}{\partial w^l}-\theta^\infty_j\dfrac{\partial\theta^k_i}{\partial w^\infty}\right)-\left(\dfrac{\partial\theta^k_j}{\partial x^i}-\theta^l_i\dfrac{\partial\theta^k_j}{\partial w^l}-\theta^\infty_i\dfrac{\partial\theta^k_j}{\partial w^\infty}\right),\\[2ex] R^0_{ij}=\left(\dfrac{\partial\theta^\infty_i}{\partial x^j}-\theta^l_j\dfrac{\partial\theta^\infty_i}{\partial w^l}-\theta^\infty_j\dfrac{\partial\theta^\infty_i}{\partial w^\infty}\right)-\left(\dfrac{\partial\theta^\infty_j}{\partial x^i}-\theta^l_i\dfrac{\partial\theta^\infty_j}{\partial w^l}-\theta^\infty_i\dfrac{\partial\theta^\infty_j}{\partial w^\infty}\right)-\zeta_l R^l_{ij},\\[2ex] S^k_{ij}=\dfrac{\partial\theta^k_i}{\partial w^j}+\dfrac{\partial\theta^k_i}{\partial w^\infty}\zeta_j,\\[2ex] S^0_{ij}=\dfrac{\partial\theta^\infty_i}{\partial w^j}+\dfrac{\partial\theta^\infty_i}{\partial w^\infty}\zeta_j+\dfrac{\partial\zeta_j}{\partial x^i}-\theta^l_i\dfrac{\partial\zeta_j}{\partial w^l}-\theta^\infty_i\dfrac{\partial\zeta_j}{\partial w^\infty}-\zeta_l S^l_{ij}. \end{cases}$$

The following proposition is the immediate consequence of (1.14) and (1.15).

PROPOSITION 1.1. *Φ^h is integrable, if and only if $R^k_{ij}=R^0_{ij}=0$,*

Φ^v is integrable, if and only if $\dfrac{\partial\zeta_j}{\partial w^i}=\dfrac{\partial\zeta_i}{\partial w^j}$,

$\tilde{\Phi}$ is integrable, if and only if $R^0_{ij}=S^0_{ij}=0$ and $\dfrac{\partial\zeta_i}{\partial w^j}=\dfrac{\partial\zeta_j}{\partial w^i}$,

Φ^H is integrable, if and only if $R^k_{ij}=0$ and $\dfrac{\partial\theta^k_i}{\partial w^\infty}=0$,

Φ^V is always integrable.

From now on, we write y^i and z instead of w^i and w^∞ respectively. Now, let us assume that the θ^i_j and θ^∞_j have the following forms.

$$(1.16)\quad \begin{cases} \theta^i_j=\varphi^i_j(x^k,\ y^k)+B^i_j(x^k,\ y^k)z,\\ \theta^\infty_j=\psi_j(x^k,\ y^k)+A_j(x^k,\ y^k)z, \end{cases}$$

where det $(B^i_j)\neq 0$, and we write $(B^i_j)^{-1}=(C^i_j)$. We should notice that the φ^i_j and ψ_j must be positively homogeneous of degree 1 in the y^k and the A^i_j, B^i_j and C^i_j must be positively homogeneous of degree 0 in the y^k from the condition (1.7) on the θ^λ_j. The laws of transformation of these functions are obtained by substituting the θ^i_j and θ^∞_j in (1.6) by (1.16).

$$(1.17)\quad \begin{cases} \bar{\varphi}^i_l\dfrac{\partial\bar{x}^l}{\partial x^k}-\dfrac{\partial\bar{x}^i}{\partial x^l}\varphi^l_k+\dfrac{\partial^2\bar{x}^i}{\partial x^k\partial x^l}y^l+c\,y^m\partial_m\log\Delta\cdot\bar{B}^i_l\dfrac{\partial\bar{x}^l}{\partial x^k}=0,\\[2ex] \bar{\psi}_l\dfrac{\partial\bar{x}^l}{\partial x^k}+c\,y^m\partial_m\log\Delta\cdot\bar{A}_l\dfrac{\partial\bar{x}^l}{\partial x^k}-\psi_k=c\varphi^l_k\partial_l\log\Delta-c\,y^l\partial_k\partial_l\log\Delta,\\[2ex] \bar{B}^i_l\dfrac{\partial\bar{x}^l}{\partial x^k}=\dfrac{\partial\bar{x}^i}{\partial x^l}B^l_k, \end{cases}$$

$$\bar{A}_l \frac{\partial \bar{x}^l}{\partial x^k} - A_k = cB^l_k \partial_l \log \Delta.$$

PROPOSITION 1.2.

$$\Pi^i_k = \varphi^i_k + B^i_k A_m C^m_l y^l$$

is a non-linear connection, and

$$\Pi_k = \psi_k - A_l C^l_m \varphi^m_k + y^m \partial_k (A_l C^l_m) - \Pi^r_k y^m \dot{\partial}_r (A_l C^l_m)$$

is a quasi-covector, where $\dot{\partial}_r = \dfrac{\partial}{\partial y^r}$.

PROOF. Calculating the laws of transformation, we obtain

$$\bar{\Pi}^i_l \frac{\partial \bar{x}^l}{\partial x^k} - \frac{\partial \bar{x}^i}{\partial x^l} \Pi^l_k = -\frac{\partial^2 \bar{x}^i}{\partial x^k \partial x^l} y^l, \text{ and} \qquad \bar{\Pi}_j \frac{\partial \bar{x}^j}{\partial x^k} = \Pi_k. \qquad \text{q.e.d.}$$

§2. General projective connections

In the previous section, we defined the connection $\theta = \{\theta^\lambda_j\}$ on a projective vector bundle, but it includes more than a projective connection in general. So, in this and following sections, we assume that the θ^λ_j have the following forms in canonical coordinate systems, that is

$$\text{(2.1)} \qquad \begin{cases} \theta^i_j = \varphi^i_j(x^k, y^k) + \delta^i_j z, \\ \theta^\infty_j = \psi_j(x^k, y^k) + A_j(x^k) z, \end{cases}$$

where φ^i_j and ψ_j are positively homogeneous of degree 1 in the y^k and A_j is a function of the x^k only. It is evident from the law of transformation (1.17) that this form of the θ^λ_j is independent of the choice of the canonical coordinate system. Under the assumption (2.1), the Π^i_j and Π_j in Proposition 1.2 are written in the forms

$$\text{(2.2)} \qquad \begin{cases} \Pi^i_j = \varphi^i_j + \delta^i_j A_m y^m, \\ \Pi_j = \psi_j - A_m \varphi^m_j + y^m \partial_j A_m. \end{cases}$$

In such a case, we call θ a general projective connection.

Let us consider a C^1-curve C: $x^i = x^i(t)$ on M. A curve $\tilde{C}$ on P is called the horizontal lift of C if it satisfies following two conditions, that is (1) $\pi(\tilde{C}) = C$ and (2) the tangent vector to $\tilde{C}$ at each point of $\tilde{C}$ belongs to the horizontal distribution Φ^h. The condition (2) can be restated that the tangent vector field along the $\tilde{C}$ is given by $\dfrac{dx^i}{dt}X_i$. And so, a curve $\tilde{C}$: $x^i=x^i(t)$, $y^i=y^i(t)$, $z=z(t)$ on P is the horizontal lift of C: $x^i=x^i(t)$ on M if and only if the condition

$$\frac{dy^i}{dt}=-\theta^i_l\frac{dx^l}{dt}, \qquad \frac{dz}{dt}=-\theta^{\tilde{}}_l\frac{dx^l}{dt}, \tag{2.3}$$

is satisfied.

Let x_1 and x_2 be two distinct points of M and C be a curve joining these points. We define a mapping from $\pi^{-1}(x_1)$, the fibre over x_1, into $\pi^{-1}(x_2)$, the fibre over x_2, as follows. For each point of $\pi^{-1}(x_1)$, there corresponds one and only one horizontal lift of C through the point and this curve on P passes $\pi^{-1}(x_2)$ by one point. In this manner, a point of $\pi^{-1}(x_2)$ correspond to a point of $\pi^{-1}(x_1)$, and this mapping is obviously one to one and onto. We denote it by h_C and call it the transformation associated with the curve C. From the homogeneity of θ^λ_i (1.7), we can see that a straight line through the origin is mapped into a straight line through the origin by h_C. If C_1 is a curve joining x_2 and another point x_3, and if we denote by $C+C_1$ the curve joining x_1 and x_3 which is obtained by combining C and C_1 at x_2, the relation

$$h_{C+C_1}=h_{C_1}\cdot h_C$$

holds good. And the relation

$$h_{C^{-1}}=(h_C)^{-1}$$

also holds good if we denote by C^{-1} the curve obtained by reversing the orientation of C.

Now, if we remove the origin from R^{n+1} and identify all points on a straight line through the origin, we obtain an n-dimensional real projective space P^n. A fibre bundle which has the P^n as the standard fibre, is defined by taking the projective transformation group which leaves fixed the point obtained by identifying the $(n+1)$-th axis of R^{n+1} as the structural group, and the projective transformation determined by (1.1) as the coordinate transformation. We denote this fibre bundle by $\tilde{P}(M, \tilde{\pi}, P^n, \tilde{G}_0)$ where $\tilde{P}$ is its total space, $\tilde{\pi}$ is the projection $\tilde{P}\to M$ of the bundle and $\tilde{G}_0$ is the group.

It is obvious that the transformation associated with the curve C joining two points x_1 and x_2 of M determines a one-to-one mapping between fibres of $\tilde{P}$, $\tilde{\pi}^{-1}(x_1)$ and $\tilde{\pi}^{-1}(x_2)$. We denote this mapping by $\tilde{h}_C$. In general, the mappings h_C and $\tilde{h}_C$ are dependent on the given connection $\theta=\{\theta^\lambda_i\}$, but if for all curves on M the mapping $\tilde{h}_C=\tilde{h}_C(\theta)$ defined by the connection θ and the mapping $h_C=h_C(\hat{\theta})$ defined by another connection $\hat{\theta}$ coincide, we say that two connections are mutually projective and denote by $\theta\overline{\wedge}\hat{\theta}$.

THEOREM 2.1. *In order that two connections θ and $\hat{\theta}$ are mutually projective, it is necessary and sufficient that the coefficients of these connections are related by*

$$\hat{\theta}_j^\lambda = \theta_j^\lambda + \rho_j(x^k) w^\lambda, \tag{2.4}$$

where $\rho_j(x^k)$ is a function of the x^k. Or using (2.1)

$$\begin{cases} \hat{\varphi}_j^i = \varphi_j^i + \rho_j(x^k) y^i, \\ \hat{\psi}_j = \psi_j, \\ \hat{A}_j = A_j + \rho_j(x^k). \end{cases} \tag{2.4$'$}$$

PROOF. Let C: $x^i = x^i(t)$ be a curve on M, and $\tilde{C}(\theta)$ and $\tilde{C}(\hat{\theta})$ be two horizontal lifts of C determined by connections θ and $\hat{\theta}$ respectively. Suppose that the equations of $\tilde{C}(\theta)$ and $\tilde{C}(\hat{\theta})$ are respectively given in the forms of

$$\tilde{C}(\theta):\ x^i = x^i(t), \quad w^\lambda = w^\lambda(t),$$

$$\tilde{C}(\hat{\theta}):\ x^i = x^i(t), \quad \hat{w}^\lambda = \hat{w}^\lambda(t).$$

From (2.3), the differential equations satisfied by w^λ and $\hat{w}^\lambda$ are

$$\frac{dw^\lambda}{dt} + \theta_i^\lambda \frac{dx^i}{dt} = 0 \quad \text{and} \quad \frac{d\hat{w}^\lambda}{dt} + \hat{\theta}_i^\lambda \frac{dx^i}{dt} = 0,$$

respectively. If we notice that $\tilde{h}_C(\theta) = \tilde{h}_C(\hat{\theta})$ means that the image points of a point under the mappings $h_C(\theta)$ and $h_C(\hat{\theta})$ belong to the same straight line through the origin in each fibre, we have

$$\hat{w}^\lambda(t) = \rho(t) w^\lambda(t),$$

providing that $\rho(t)$ is a certain function of t. Substituting the right hand side of above relation for the $\hat{w}^\lambda$ in the differential equations, we have

$$\frac{d\rho}{dt} w^\lambda + \rho \frac{dw^\lambda}{dt} + \hat{\theta}_i^\lambda(x, \rho w) \frac{dx^i}{dt}$$

$$= \frac{d\rho}{dt} w^\lambda - \rho \left\{ \theta_i^\lambda(x, w) - \hat{\theta}_i^\lambda(x, w) \right\} \frac{dx^i}{dt} = 0,$$

that is,

$$\left\{ \hat{\theta}_i^\lambda(x, w) - \theta_i^\lambda(x, w) \right\} \frac{dx^i}{dt} = -\frac{d \log \rho}{dt} w^\lambda.$$

Since this equation should be true for all curves on M, we must have

$$-\frac{d\log\rho}{dt}=\rho_i\frac{dx^i}{dt},$$

where ρ_i is a function of the x^k and w^λ. Moreover, in this circumstance, the direction $\frac{dx^i}{dt}$ may be arbitrary, so we have

$$\hat{\theta}_i^\lambda(x, w)-\theta_i^\lambda(x, w)=\rho_i(x, w)w^\lambda.$$

If we put $\lambda=\infty$, and use (2.1), it becomes clear that ρ_i is a function of the x^k only and does not contain the w^λ.

The converse is obvious. q.e.d.

From this theorem, it is easily verified that the relation $\theta\,\overline{\wedge}\,\hat{\theta}$ is an equivalence relation. So, if two general projective connections θ and $\hat{\theta}$ are in the relation $\theta\,\overline{\wedge}\,\hat{\theta}$, we can say that the two general projective connections are projectively equivalent.

Put $Y=y^i\frac{\partial}{\partial y^i}+z\frac{\partial}{\partial z}$, then Y is a vector field on P. For, from (1.2) we have

$$\frac{\partial}{\partial y^i}=\frac{\partial\bar{x}^m}{\partial x^i}\frac{\partial}{\partial\bar{y}^m}+c\partial_i\log\Delta\cdot\frac{\partial}{\partial\bar{z}},\quad\text{and}\quad\frac{\partial}{\partial z}=\frac{\partial}{\partial\bar{z}},$$

so using (1.3), we have

$$y^i\frac{\partial}{\partial y^i}+z\frac{\partial}{\partial z}=\bar{y}^i\frac{\partial}{\partial\bar{y}^i}+\bar{z}\frac{\partial}{\partial\bar{z}}.$$

This vector field Y on P is independent of connections, and we call it the intrinsic vector field of P.

If we put $\mathfrak{p}(\theta)=\Phi^h\oplus Y$ (direct sum) where Φ^h is the distribution defined by a connection θ and Y is the intrinsic vector field of P, then $\mathfrak{p}(\theta)$ is an $(n+1)$-dimensional distribution on P and we have the following theorem.

Theorem 2.2. *Two connections θ and $\hat{\theta}$ are projectively equivalent, if and only if $\mathfrak{p}(\theta)$ and $\mathfrak{p}(\hat{\theta})$ coincide.*

Proof. Suppose $\theta\,\overline{\wedge}\,\hat{\theta}$, then, since Y is independent of connections, and

$$\begin{aligned}\hat{X}_i&=\frac{\partial}{\partial x^i}-\hat{\theta}_i^m\frac{\partial}{\partial y^m}-\hat{\theta}_i^\infty\frac{\partial}{\partial z}\\&=\frac{\partial}{\partial x^i}-(\theta_i^m+\rho_i y^m)\frac{\partial}{\partial y^m}-(\theta_i^\infty+\rho_i z)\frac{\partial}{\partial z}\\&=X_i-\rho_i Y\in\mathfrak{p}(\theta),\end{aligned}$$

we have $\mathfrak{p}(\hat{\theta}) \subset \mathfrak{p}(\theta)$. In like manner, we have $\mathfrak{p}(\theta) \subset \mathfrak{p}(\hat{\theta})$.

Conversely, suppose $\mathfrak{p}(\theta) = \mathfrak{p}(\hat{\theta})$. Since vector fields $\hat{X}_i$ which form a basis of $\mathfrak{p}(\hat{\theta})$ with Y, belong to $\mathfrak{p}(\theta)$, we can write

$$\begin{aligned}\hat{X}_i &= \frac{\partial}{\partial x^i} - \hat{\theta}_i^m \frac{\partial}{\partial y^m} - \hat{\theta}_i^\infty \frac{\partial}{\partial z} \\ &= p_i^m X_m - \rho_i Y \\ &= p_i^m \left(\frac{\partial}{\partial x^m} - \theta_m^l \frac{\partial}{\partial y^l} - \theta_m^\infty \frac{\partial}{\partial z} \right) - \rho_i y^l \frac{\partial}{\partial y^l} - \rho_i z \frac{\partial}{\partial z}.\end{aligned}$$

Comparing components on both sides and giving care to (2.1), we obtain the following.

$$p_i^m = \delta_i^m, \quad \hat{\theta}_i^m = \theta_i^m + \rho_i y^m, \quad \hat{\theta}_i^\infty = \theta_i^\infty + \rho_i z,$$

where ρ_i is a function of the x^k. q.e.d.

If $\hat{\Pi}_j^i$ is a non-linear connection and $\hat{\Pi}_j$ is a quasi-covector defined by (2.2) using the coefficients of a connection $\hat{\theta}$, then the relations between these and the same quantities from a connection θ which is projectively equivalent with $\hat{\theta}$ are given by

$$(2.5) \qquad \begin{cases} \hat{\Pi}_j^i = \Pi_j^i + \rho_j y^i + \delta_j^i \rho_m y^m, \\ \hat{\Pi}_j = \Pi_j - \rho_m \Pi_j^m - \rho_j \rho_m y^m + (\partial_j \rho_m) y^m. \end{cases}$$

Remark. At the beginning of this section, we assumed that a general projective connection θ had the form of (2.1). If we take a stronger assumption that θ_j^λ is linear in the w^λ, then (2.1) is rewritten in the form

$$(2.6) \qquad \begin{cases} \theta_j^i = \varphi_{jk}^i(x^l) y^k + \delta_j^i z, \\ \theta_j^\infty = \psi_{jk}(x^l) y^k + A_j(x^l) z. \end{cases}$$

In this case, Π_j^i and Π_j of (2.2) take the forms of

$$(2.7) \qquad \Pi_j^i = \Pi_{jk}^i(x^l) y^k, \quad \Pi_j = \Pi_{jk}(x^l) y^k,$$

where

$$(2.8) \qquad \begin{cases} \Pi_{jk}^i(x^l) = \varphi_{jk}^i + \delta_j^i A_k \\ \Pi_{jk}(x^l) = \psi_{jk} - A_m \varphi_{jk}^m + \partial_j A_k. \end{cases}$$

Moreover, it is the direct result of proposition 1.2 that Π_{jk}^i is a linear connection and Π_{jk} is a covariant tensor of degree **2**. The condition (2.5) for projective equivalence becomes

$$\text{(2.9)} \qquad \begin{cases} \hat{\Pi}^{i}_{jk} = \Pi^{i}_{jk} + \delta^{i}_{k}\rho_{j} + \delta^{i}_{j}\rho_{k}, \\ \hat{\Pi}_{jk} = \Pi_{jk} - \rho_{m}\Pi^{m}_{jk} - \rho_{j}\rho_{k} + \partial_{j}\rho_{k}. \end{cases}$$

(2.9) is the condition satisfied by the coefficients of Cartan's projective connection. And so, the linear general projective connection gives Cartan's projective connection, and we call the θ of this case a restricted projective connection.

§3. The integrability condition for $\mathfrak{p}$

We have showed in the previous section that, if a general projective connection θ was given on a projective vector bundle P, the distribution $\mathfrak{p}(\theta) = \Phi^{h} \oplus Y$ was same for all $\hat{\theta}$ which were projectively equivalent to θ. In this section, we will try to seek the integrability conditions for $\mathfrak{p}$.

For this purpose, we first calculate the brackets between X_i, and Y. From (1.12) we can rewrite Y in the form

$$\text{(3.1)} \qquad Y = y^{i} Y_{i} + (z - y^{i}\zeta_{i})Z.$$

So using (1.14) and (1.15), we have

$$\begin{aligned} [X_i, Y] &= [X_i, y^{j} Y_j] + [X_i, (z - y^{j}\zeta_j)Z] \\ &= (y^{j}\dot{\partial}_{j}\varphi^{k}_{i} - \varphi^{k}_{i}) Y_k + \{y^{j}\dot{\partial}_{j}\psi_{i} - \psi_{i} - \zeta_{j}(y^{l}\dot{\partial}_{l}\varphi^{j}_{i} - \varphi^{j}_{i})\} Z. \end{aligned}$$

Both φ^{k}_{i} and ψ_i are positively homogeneous of degree 1 in the y^{k}, so the coefficients of Y_k and Z vanish. And we have

$$\text{(3.2)} \qquad [X_i, Y] = 0.$$

Next we should calculate brackets between X_i and X_j. Since $[X_i, X_j] = R^{k}_{i\,j} Y_k + R^{0}_{i\,j} Z$ by (1.14), $[X_i, X_j]$ does not belong to Φ^{h}. And it can be stated that $\mathfrak{p}$ is integrable, if and only if

$$\text{(3.3)} \qquad \begin{aligned} R^{k}_{i\,j} Y_k + R^{0}_{i\,j} Z &= H_{ij} Y \\ &= H_{ij} y^{k} Y_k + H_{ij}(z - y^{l}\zeta_l) Z, \end{aligned}$$

for suitable H_{ij}, that is,

$$\text{(3.3')} \qquad R^{k}_{i\,j} = H_{ij} y^{k}, \quad R^{0}_{i\,j} = (z - y^{l}\zeta_l) H_{ij}.$$

Putting the first equation into the second, (3.3′) is rewritten in the form

$$\text{(3.4)} \qquad R^{k}_{i\,j} = H_{ij} y^{k}, \quad R^{0}_{i\,j} + \zeta_l R^{l}_{i\,j} = z H_{ij}.$$

We substitute (1.15) by (2.1), then we have for R^k_{ij}

(3.5) $$R^k_{ij}=E^k_{ij}+F^k_{ij}z,$$

where

(3.6) $$\begin{cases} E^k_{ij}=(\partial_j\varphi^k_i-\varphi^m_j\dot{\partial}_m\varphi^k_i-\delta^k_i\psi_j)-(\partial_i\varphi^k_j-\varphi^m_i\dot{\partial}_m\varphi^k_j-\delta^k_j\psi_i), \\ F^k_{ij}=(\dot{\partial}_i\varphi^k_j+\delta^k_jA_i)-(\dot{\partial}_j\varphi^k_i+\delta^k_iA_j). \end{cases}$$

And for R^0_{ij} we have

(3.7) $$R^0_{ij}=P_{ij}+Q_{ij}z,$$

where

(3.8) $$\begin{cases} P_{ij}=(\partial_j\psi_i-\varphi^m_j\dot{\partial}_m\psi_i-A_i\psi_j)-(\partial_i\psi_j-\varphi^m_i\dot{\partial}_m\psi_j-A_j\psi_i)-\zeta_lE^l_{ij}, \\ Q_{ij}=(\partial_jA_i-\dot{\partial}_j\psi_i)-(\partial_iA_j-\dot{\partial}_i\psi_j)-\zeta_lF^l_{ij}. \end{cases}$$

Then the integrability condition (3.4) for $\mathfrak{p}$ is given by

(3.9) $$E^k_{ij}+F^k_{ij}z=H_{ij}y^k,\quad P_{ij}+\zeta_kE^k_{ij}+(Q_{ij}+\zeta_kF^k_{ij})z=H_{ij}z.$$

In the first equation, E^k_{ij} and F^k_{ij} do not contain z, so we can put $H_{ij}=H'_{ij}+H''_{ij}z$, assuming that H'_{ij} and H''_{ij} do not contain z. In the second equation of (3.9), we compare the respective coefficients of z^2, z and z^0 (ζ_k in the left hand side may contain z, but $P_{ij}+\zeta_kE^k_{ij}$ and $Q_{ij}+\zeta_kF^k_{ij}$ do not contain ζ_k by (3.8). So it is reasonable to compare coefficients of z^2, z and z^0 in the second equation of (3.9).).

The second equation of (3.9) is

$$P_{ij}+\zeta_kE^k_{ij}+(Q_{ij}+\zeta_kF^k_{ij})z=H'_{ij}z+H''_{ij}z^2.$$

So, $H''_{ij}=0$, and the first equation of (3.9) becomes

$$E^k_{ij}+F^k_{ij}z=H'_{ij}y^k,$$

which is equivalent to

(3.10) $$E^k_{ij}=H'_{ij}y^k,\quad F^k_{ij}=0.$$

Using this relation and $H''_{ij}=0$, the second equation of (3.9) becomes

$$P_{ij}+\zeta_kE^k_{ij}+Q_{ij}z=H'_{ij}z.$$

Consequently we have

(3.11) $$P_{ij}+\zeta_kE^k_{ij}=0,\quad Q_{ij}y^k=E^k_{ij},$$

and the integrability condition for $\mathfrak{p}$ is expressed as follows.

$$F^k_{ij}=0,\quad P_{ij}+\zeta_k E^k_{ij}=0,\quad E^k_{ij}-Q_{ij}y^k=0. \tag{3.12}$$

Substituting (3.12) by (2.2), (3.6) and (3.8) we obtain the

THEOREM 3.1. *In the projective vector bundle P with a general projective connection θ, the integrability condition for the distribution $\mathfrak{p}(\theta)$ is given by*

$$\begin{cases} \dot{\partial}_i \Pi^k_j - \dot{\partial}_j \Pi^k_i = 0, \\ \partial_i \Pi^k_j - \partial_j \Pi^k_i + \Pi^m_j \dot{\partial}_m \Pi^k_i - \Pi^m_i \dot{\partial}_m \Pi^k_j + \delta^k_i \Pi_j - \delta^k_j \Pi_i + y^k \dot{\partial}_i \Pi_j - y^k \dot{\partial}_j \Pi_i = 0, \\ \partial_i \Pi_j - \partial_j \Pi_i + \Pi^m_j \dot{\partial}_m \Pi_i - \Pi^m_i \dot{\partial}_m \Pi_j = 0. \end{cases} \tag{3.13}$$

REMARK. If we take the assumption that θ is linear, this (3.13) becomes, in terms of (2.7), the condition for a projectively flat space. It is wellknown that, in this case, the space is locally isomorphic to the projective space. However, we have not assumed the linearity in this paper, so we cannot state in general that the space M is locally isomorphic to the projective space even if the condition (3.13) has been satisfied. We call the case "*projectively semi-flat*".

The following theorem is obvious from the remark.

THEOREM 3.2. *A necessary and sufficient condition for a manifold M with a general projective connection θ to be locally isomorphic to a projective space is that the relations* (3.13) *and $\dot{\partial}_j \dot{\partial}_k \theta^\lambda_i = 0$ hold true.*

REMARK. If we put, as in the restricted case,

$$\begin{cases} K^k_{ij} = \partial_i \Pi^k_j - \partial_j \Pi^k_i + \Pi^m_j \dot{\partial}_m \Pi^k_i - \Pi^m_i \dot{\partial}_m \Pi^k_j, \\ K_j = K^m_{mj}, \end{cases} \tag{3.14}$$

we have, from the second equation of (3.13),

$$K_j + n\Pi_j - y^m \dot{\partial}_j \Pi_m = 0.$$

This equation is, of course, a projective one. Solving this equation for Π_j, we have

$$\Pi_j = \frac{-1}{n^2-1}(nK_j + y^m \dot{\partial}_j K_m).$$

We call a general projective connection, in analogy with the restricted case, a *generalized normal projective connection*, if it satisfies the condition

(3.15) $$\dot{\partial}_i \Pi^k_j - \dot{\partial}_j \Pi^k_i = 0, \quad \Pi_j = \frac{-1}{n^2-1}(nK_j + y^m \dot{\partial}_j K_m).$$

§4. The holonomy group of a general projective connection

We defined in §2 a mapping $\tilde{h}_C$ between fibres of $\tilde{P}$. If we take a closed curve, which has the same point x_0 as the initial and the end point, for the curve C, the $\tilde{h}_C$ is a transformation of the fibre $\tilde{\pi}^{-1}(x_0)$ since the θ^λ_j is positively homogeneous of degree 1 in the w^λ. The set $\{\tilde{h}_C\}_{x_0}$ for all closed curve through x_0 forms the holonomy group with reference point x_0, of $\tilde{P}(\theta)$ with a general projective connection determined by a connection θ on P.

Now, we assume that a field of hypercones on fibres of $\tilde{P}(\theta)$ is given and is expressed by the equation

(4.1) $$z^2 - 2G(x^k, y^k)z - H(x^k, y^k) = 0.$$

Since each fibre of $\tilde{P}(\theta)$ is a projective space, $G(x^k, y^k)$ and $H(x^k, y^k)$ are necessarily positively homogeneous of degree 1 and degree 2 in the y^k respectively.

Theorem 4.1. *The laws of transformation of G and H under the coordinate transformation $\bar{x}^i = \bar{x}^i(x^k)$ of M are given by*

(4.2) $$\begin{cases} \bar{G} = G + c\, y^j \partial_j \log \Delta, \\ \bar{H} = H - 2c\, y^j \bar{G} \partial_j \log \Delta + c^2 y^j y^k \partial_j \log \Delta \cdot \partial_k \log \Delta. \end{cases}$$

And $H + G^2$ is invariant under the coordinate transformation, that is, the relation

(4.3) $$\bar{H} + \bar{G}^2 = H + G^2$$

holds good.

Proof. If we rewrite the equation

$$\bar{z}^2 - 2\bar{G}(\bar{x}^k, \bar{y}^k)\bar{z} - \bar{H}(\bar{x}^k, \bar{y}^k) = 0$$

in terms of z by (1.3), we obtain (4.2). (4.3) is directly obtained by substituting $\bar{G}$ in the right hand side of the second equation of (4.2) by the right hand side of the first equation of (4.2). q.e.d.

The quantity $H + G^2$ is invariant under the coordinate transformation, so we put $H + G^2 = L$. Then the equation (4.1) can be written in the form

(4.1′) $$z^2 - 2Gz + (G^2 - L) = 0.$$

THEOREM 4.2. *The field of hypercones* (4.1′) *on fibres of the projective vector bundle P is left invariant by the holonomy group of the general projective connection* (2.1), *if and only if the following condition is satisfied.*

$$(4.4)\quad \begin{cases} G(\dot{\partial}_k G - A_k) - \psi_k - \partial_k G + \varphi^j_k \dot{\partial}_j G + \frac{1}{2}\dot{\partial}_k L = 0, \\ G^2(\dot{\partial}_k G - A_k) - G\psi_k - G\partial_k G + G\varphi^j_k \dot{\partial}_j G + \frac{1}{2}\partial_k L - \frac{1}{2}\varphi^j_k \dot{\partial}_j L - L(\dot{\partial}_k G - A_k) = 0. \end{cases}$$

And this condition is independent of the choice of the projectively equivalent general projective connections on P.

PROOF. We develop (4.1′) along an arbitrary curve $x^i = x^i(t)$ on M. Differentiating both sides of (4.1′) by t and using (2.3) and (2.1), we obtain

$$\Big[z^2(A_k - \dot{\partial}_k G) + z\Big(\psi_k + \partial_k G - \varphi^j_k \dot{\partial}_j G - GA_k + G\dot{\partial}_k G - \frac{1}{2}\dot{\partial}_k L\Big)$$
$$+\Big(G\varphi^j_k \dot{\partial}_j G + \frac{1}{2}\partial_k L - \frac{1}{2}\dot{\partial}_j L \cdot \varphi^j_k - G\psi_k - G\partial_k G\Big)\Big]\frac{dx^k}{dt} = 0.$$

This equation should hold for an arbitrary direction $\frac{dx^k}{dt}$, so the inside of the bracket vanishes. The hypercone defined by putting the inside of the bracket $=0$ coinsides with the hypercone defined by (4.1′), if and only if the coefficients of z^2, z and z^0 are proportional. So we obtain (4.4). The converse is evident from above consideration.

The last part of the theorem is obvious from Theorem 2.1 and homogeneity of quantities. q.e.d.

§5. *F*-hypersurfaces

As we have shown in the previous section, the L in the equation of the hypercone (4.1′) is positively homogeneous of degree 2 in the y^k and is invariant under the coordinate transformation. So we assume, in this section, that the function L of $2n$ variables x^k and y^k defines a Finsler metric on M.

If we denote the components of the Finsler metric tensor by g_{ij} then

$$(5.1)\qquad L = g_{ij} y^i y^j, \quad \frac{1}{2}\dot{\partial}_j L = g_{ij} y^i, \quad \frac{1}{2}\dot{\partial}_i \dot{\partial}_j L = g_{ij}.$$

Moreover, let G^i_j and G^i_{jk} be the coefficients of the Berwald's connection, that is

(5.2) $$G^i_j = \frac{1}{2}\partial_j(\gamma^i_{ml}y^m y^l),\quad G^i_{jk} = \dot\partial_k G^i_j,$$

where γ^i_{ml} is the Christoffel's symbol. The law of transformation for G^i_j is given by

$$\bar G^i_j = G^p_q \frac{\partial \bar x^i}{\partial x^p}\frac{\partial x^q}{\partial \bar x^j} - y^r \frac{\partial^2 \bar x^i}{\partial x^r \partial x^m}\frac{\partial x^m}{\partial \bar x^j},$$

from which we obtain

$$\bar G^h_h = G^p_p - y^r \partial_r \log \Delta.$$

So if we notice the first equation of (4.2), we can put

(5.3) $$G = -cG^h_h,$$

for the G in (4.1′).

We call the field of hypercones (4.1′) with (5.1) and (5.3) for its L and G, an *F-hypersurface*. In other words, an F-hypersurface is a field of hypercones which assigns to each fibre of P a hypercone having the origin as its vertex and having the equation

(5.4) $$z^2 + 2cG^h_h z + (c^2 G^h_h G^l_l - L) = 0.$$

THEOREM 5.1. *Suppose that a Finsler metric L is given on M. The holonomy group of a general projective connection $\hat\theta$ on P leaves an F-hypersurface* (5.4) *invariant, if and only if $\hat\theta$ is projectively equivalent with θ which is defined by the following φ^j_k and ψ_k,*

(5.5) $$\begin{cases} \varphi^j_k = (G^j_{kh} + c\delta^j_k G^m_{mh} + c\delta^j_h G^m_{mk})y^h + T^j_k + y^j A_k, \\ \psi_k = (g_{kh} - cG^l_{lm}G^m_{kh} + c\partial_k G^m_{mh} - c^2 G^l_{lk} G^m_{mh})y^h - cG^m_{mh}T^h_k, \end{cases}$$

where T^i_j is an arbitrary (1, 1)*-tensor satisfying*

(5.6) $$g_{ik}T^i_j y^k = 0,$$

and being positively homogeneous of degree 1 *in the y^k.*

PROOF. Putting (5.1) into the condition (4.4) for the holonomy group of θ to leave invariant the field of hypercones (4.1′), we obtain

(5.7) $$\begin{cases} Gg_{ik}y^i - \frac{1}{2}y^i y^j \partial_k g_{ij} + \varphi^j_k g_{ij} y^i + g_{ij} y^i y^j \dot\partial_k G - g_{ij} y^i y^j A_k = 0, \\ \psi_k = G(\dot\partial_k G - A_k) - \partial_k G + \varphi^j_k \dot\partial_j G + g_{jk} y^j. \end{cases}$$

We put

$$C^j_{kl}=\frac{1}{2}g^{jm}\dot{\partial}_m g_{kl},$$

and

$$\varphi^j_k=y^jA_k-\delta^j_kG-y^j\dot{\partial}_kG+\gamma^j_{kl}y^l-C^j_{km}\gamma^m_{pr}y^py^r+T^j_k. \tag{5.8}$$

Substituting (5.8) into the first equation of (5.7), we obtain (5.6). Writing down the condition for φ^j_k given by (5.8) to satisfy the law of transformation (1.17), we see that the T^j_k is a tensor. On using the famous relation between the connections of Rund and Berwald, we obtain

$$G^j_k=(\gamma^j_{kl}-y^pC^j_{km}\gamma^m_{pl})y^l.$$

So, giving care to (5.2) and (5.3), (5.8) becomes the first equation of (5.5). The second equation of (5.5) follows from the second equation of (5.7). q.e.d.

The first curvature tensor $R_{ijk}{}^h$ of the Berwald's connection G^i_{jk} is given by

$$R_{ijk}{}^h=(\partial_iG^h_{jk}-G^m_i\dot{\partial}_mG^h_{jk})-(\partial_jG^h_{ik}-G^m_j\dot{\partial}_mG^h_{ik})+G^h_{im}G^m_{jk}-G^h_{jm}G^m_{ik}. \tag{5.9}$$

It can be stated that a Finsler space which satisfies

$$y^kR_{ijk}{}^h=y^k(g_{ik}\delta^h_j-g_{jk}\delta^h_i), \tag{5.10}$$

is a generalization of a certain kind of Riemannian spaces of constant curvature. So, in this case, we call such a Finsler space a generalized space of constant curvature. Then, as a generalization of the restricted case, we obtain the

Theorem 5.2. *Let M be a Finsler space with a metric function L, P a projective vector bundle over M and θ a general projective connection on P. If the distribution $\mathfrak{p}(\theta)$ (defined in §2) is integrable and the holonomy group of θ leaves a F-hypersurface invariant, then M is a generalized space of constant curvature.*

Proof. In consequence of Theorem 5.1, φ^j_k and ψ_k of (5.5) must satisfy the integrability condition for $\mathfrak{p}(\theta)$ (3.13). So, if we rewrite (3.13) using (2.2) and (5.5), we obtain

$$\left\{\begin{aligned}
&\dot{\partial}_jT^k_i-\dot{\partial}_iT^k_j=0,\\
&(\partial_iG^k_{jh}-\partial_jG^k_{ih}+G^m_{jh}G^k_{im}-G^m_{ih}G^k_{jm}+G^m_{jh}\dot{\partial}_mT^k_i-G^m_{ih}\dot{\partial}_mT^k_j)y^h\\
&+\partial_iT^k_j-\partial_jT^k_i+T^m_jG^k_{im}-T^m_iG^k_{jm}+T^m_j\dot{\partial}_mT^k_i-T^m_i\dot{\partial}_mT^k_j\\
&\qquad\qquad+(g_{jh}\delta^k_i-g_{ih}\delta^k_j)y^h=0,\\
&(\partial_jg_{ih}-\partial_ig_{jh}+g_{jm}G^m_{ih}-g_{im}G^m_{jh})y^h+g_{jm}T^m_i-g_{im}T^m_j=0.
\end{aligned}\right. \tag{5.11}$$

The well known formulae on Cartan's $\overset{*}{\Gamma}{}^m_{jh}$

$$\overset{*}{\Gamma}{}^m_{jh} y^h = G^m_j, \quad \overset{*}{\Gamma}{}^m_{jk} = \overset{*}{\Gamma}{}^m_{kj} \qquad \text{and}$$

$$\partial_j g_{ih} - G^m_j \dot{\partial}_m g_{ih} - \overset{*}{\Gamma}{}^m_{ji} g_{mh} - \overset{*}{\Gamma}{}^m_{jh} g_{im} = 0,$$

and the relation

$$y^m \dot{\partial}_k g_{im} = 0$$

lead us to

$$y^h(\partial_j g_{ih} - \partial_i g_{jh} + g_{jm} G^m_{ih} - g_{im} G^m_{jh}) = 0.$$

Then the third equation of (5.11) is reduced to

$$g_{jm} T^m_i - g_{im} T^m_j = 0.$$

Using this and (5.6) we obtain

$$g_{jm} T^m_k y^k = g_{km} T^m_j y^k = 0,$$

which shows

$$T^m_k y^k = 0.$$

Differentiating both sides by y^j and using the first equation of (5.11) we obtain

$$0 = y^k \dot{\partial}_j T^m_k + T^m_j = y^k \dot{\partial}_k T^m_j + T^m_j = 2 T^m_j,$$

that is

$$T^m_j = 0. \tag{5.12}$$

And, finally, if we give care to the relation $\dot{\partial}_m G^k_{jh} y^h = 0$, the second equation of (5.11) becomes (5.10). q.e.d.

In this theorem we assumed that $\mathfrak{p}(\theta)$ was integrable. Now let us take a weaker assumption that the connection is a generalized normal projective connection. Or equivalently the first equation of (3.13) and the equation obtained by contracting k and i in the second equation of (3.13) are satisfied. So, in this case, the first equation of (5.11) and the equation obtained by contracting k and i in the second equation of (5.11) must be satisfied, thus we obtain

$$
(5.13)\quad \begin{cases} \dot{\partial}_j T^k_i - \dot{\partial}_i T^k_j = 0, \\ (\partial_r G^r_{jh} - \partial_j G^r_{rh} + G^m_{jh} G^r_{rm} - G^m_{rh} G^r_{jm} + G^m_{jh} \dot{\partial}_m T^r_r - G^m_{rh} \dot{\partial}_m T^r_j) y^h \\ + \partial_r T^r_j - \partial_j T^r_r + T^m_j G^r_{rm} - T^m_r G^r_{jm} + T^m_j \dot{\partial}_m T^r_r - T^m_r \dot{\partial}_m T^r_j + (n-1) g_{jh} y^h = 0. \end{cases}
$$

If we put

$$
(5.14)\quad \Gamma^k_{jh} = G^k_{jh} + T^k_{jh}, \quad \text{where } T^k_{jh} = \dot{\partial}_h T^k_j,
$$

then, from $G^k_{jh} = G^k_{hj}$ and the first equation of (5.13), we have

$$
(5.15)\quad \Gamma^k_{jh} = \Gamma^k_{hj}.
$$

And if we put

$$
(5.16)\quad K_{ijk}{}^h = (\partial_i \Gamma^h_{jk} - G^m_i \dot{\partial}_m \Gamma^h_{jk}) - (\partial_j \Gamma^h_{ik} - G^m_j \dot{\partial}_m \Gamma^h_{ik}) + \Gamma^h_{im} \Gamma^m_{jk} - \Gamma^h_{jm} \Gamma^m_{ik},
$$

and

$$
(5.17)\quad K_{jh} = K_{rjh}{}^r,
$$

then, from the second equation of (5.13), we obtain

$$
(5.18)\quad K_{jh} y^h = -(n-1) g_{jh} y^h.
$$

The following theorem is thus obtained.

THEOREM 5.3. *If the holonomy group of a generalized normal projective connection on a projective vector bundle P over a Finsler space M leaves an F-hypersurface invariant, then the M satisfies* (5.18).

If we take a Riemannian space for M, and the connection is a restricted one, then this theorem is the wellknown result of Yano-Sasaki and Ōtsuki. In this case, (5.18) becomes $R_{jh} = -(n-1) g_{jh}$ and M is an Einstein space.

From above consideration, a Finsler space which satisfies (5.18) may be called a *Finsler-Einstein* space.

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