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On the Maximum Term and Rank of an Entire Dirichlet Series

By

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Consider the Dirichlet series,

$$f(s)=\sum_{n=1}^{\infty} a_n \cdot e^{s\lambda_n}$$

where $\lambda_{n+1}>\lambda_n$, $\lambda_1\geqq 0$, $\lim_{n\to\infty}\lambda_n=\infty$, $s=\sigma+it$ and $\lim_{n\to\infty}\sup\frac{\log n}{\lambda_n}=0$. Let σ_c and σ_a be the abscissa of convergence and the abscissa of absolute convergence of $f(s)$ respectively. If $\sigma_c=\sigma_a=\infty$ then $f(s)$ is an entire function. It is known [1],

$$\log^{\mu}(\sigma)=A+\int_{\sigma_0}^{\sigma}\lambda_{\nu(x)}\,dx$$

where $\nu(\sigma)$ gives the rank of the maximum term.

THEOREM 1. *Let,*

$$\lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\lambda_{\nu(\sigma)}}{\log\mu(\sigma)}=\frac{S}{T},\ (0<T\leqq S<\infty)$$

then,

i) $$T^2/S\leqq\lambda\leqq\rho\leqq S^2/T$$

ii) $$\lim_{\sigma\to\infty}\inf\frac{\lambda_{\nu(\sigma)}}{\sigma^p\cdot e^{\frac{\sigma}{n}\cdot\lambda(\sigma/n)}}\geqq\frac{T^{p+2}}{\lambda}\cdot m$$

and

$$\lim_{\sigma\to\infty}\sup\frac{\lambda_{\nu(\sigma)}}{\sigma^p\cdot\log\mu(\sigma/n)}\geqq\frac{T^{p+2}}{\lambda}\cdot m \quad \textit{where } m \textit{ and } n \textit{ are constants.}$$

iii) $$\lim_{\sigma\to\infty}\inf\frac{\lambda_{\nu(\sigma)}}{\sigma^p\cdot\log\mu(\sigma)}\leqq\frac{S^{p+2}}{\rho}.$$

THEOREM 2. *Let $f_1(s)$ and $f_2(s)$ be two entire functions defined by Diri-*

chlet series of orders ρ_1, ρ_2 and lower orders λ_1, λ_2 respectively. Let,

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \log\left\{\frac{\mu(\sigma, f_2)}{\mu(\sigma, f_1)}\right\} = \frac{A}{B},$$

and

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \lambda_{\nu(\sigma,f_2)} - \lambda_{\nu(\sigma,f_1)} = \frac{C}{D},$$

then,

i) $$\limsup_{\sigma\to\infty} e^{\sigma(\rho_1-\lambda_1+\varepsilon)} \leqq A/D.$$

ii) If, $\rho_1 \geqq \rho_2$, $\dfrac{\log\mu(\sigma, f_1)}{\lambda_{\nu(\sigma,f_1)}} \sim \dfrac{1}{\rho_1}$ and $\dfrac{\log\mu(\sigma, f_2)}{\lambda_{\nu(\sigma,f_2)}} \sim \dfrac{1}{\rho_2}$

then,

$$1/\rho_2 \leqq \sqrt{AB/CD}$$

THEOREM 3.

i) $$\lim_{\sigma\to\infty} \frac{\{\log\mu(k_1\sigma)\}^{p_1}}{\{\lambda_{\nu(k_2\sigma)}\}^{p_2}} = \begin{matrix}\infty\\ 0\end{matrix}, \text{ if } \begin{matrix}k_1p_1\lambda > k_2p_2\rho.\\ k_1p_1\rho < k_2p_2\lambda.\end{matrix}$$

ii) $$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\{\log\mu(k_1\sigma)\}^{p_1}}{\{\lambda_{\nu(k_2\sigma)}\}^{p_2}} = \begin{matrix}\infty\\ 0\end{matrix}, \text{ if } \begin{matrix}k_1p_1 > k_2p_2.\\ k_2p_2 > k_1p_1.\end{matrix}$$

where k_1, k_2, p_1 and p_2 are constants.

REMARK. Linear regular growth is not essential.

THEOREM 4.

i) $$e^{\sigma p}\cdot\frac{\mu(\sigma, f^n)}{\mu(\sigma, f)} \to \begin{matrix}0\\ \infty\end{matrix}, \ if \ \begin{matrix}n\rho+p<0 \text{ ....(A)}\\ n\lambda+p>0 \text{ ....(B)}\end{matrix}$$

as $\sigma\to\infty$

REMARK. *$\lambda>0$ for (B) and $\rho>0$ for (A) are not essential as R. P. Srivastav has proved [2].*

ii) $$e^{\sigma p}\sum_{n=1}^{k}\frac{\mu(\sigma, f^n)}{\mu(\sigma, f^{n-1})} \to \begin{matrix}0\\ \infty\end{matrix}, \text{ if } \begin{matrix}\rho+p<0.\\ \lambda+p>0.\end{matrix}$$

as $\sigma\to\infty$

where p is a constant.

THEOREM 5.

$$\prod_{i=1}^{n}\{\mu(\sigma_i)\}^{k_i}\geqq\mu\{\sum_{i=1}^{n}\sigma_i k_i\}$$

where $\mu(\sigma)$=*maximum term of* $f(s)$, $k_i=m_i/M$, $M=\sum_{i=1}^{n}m_i$ *and* k_i*'s are rational positive numbers.*

PROOF of THEOREM 1. i)

$$\log\mu(\sigma)=\mathrm{A}+\int_{\sigma_0}^{\sigma}\lambda_{\nu(x)}\,dx.$$

But

$$\lambda_{\nu(\sigma)}>(T-\varepsilon)\cdot\log\mu(\sigma),\ \text{for } \sigma\geqq\sigma_0.$$

So,

$$\lambda_{\nu(\sigma)}>(T-\varepsilon)\int_{\sigma_0}^{\sigma}\lambda_{\nu(x)}\,dx>(T-\varepsilon)\int_{\sigma_0}^{\sigma}(T-\varepsilon)\cdot\log\mu(x)\,dx$$

$$\lambda_{\nu(\sigma)}>(T-\varepsilon)^2\int_{\sigma_0}^{\sigma}\log\mu(x)\,dx.$$

Now let $\lambda(\sigma)$ be lower proximate order with respect to $\log\mu(\sigma)$, [3]. So,

$$\lambda_{\nu(\sigma)}>(T-\varepsilon)^2\int_{\sigma_0}^{\sigma}e^{x\cdot\lambda(x)}\,dx>(T-\varepsilon)^2\cdot\frac{e^{\sigma\cdot\lambda(\sigma)}}{\lambda}.$$

Hence,

$$\liminf_{\sigma\to\infty}\frac{\lambda_{\nu(\sigma)}}{e^{\sigma\cdot\lambda(\sigma)}}\geqq\frac{T^2}{\lambda}.$$

and so,

$$\limsup_{\sigma\to\infty}\frac{\lambda_{\nu(\sigma)}}{\log\mu(\sigma)}=S\geqq\frac{T^2}{\lambda},\ \text{i.e,}\ \frac{T^2}{S}\leqq\lambda.$$

Similarly other part follows.

ii) $$\log\mu(\sigma)=A+\int_{\sigma_0}^{\sigma}\lambda_{\nu(x)}dx>\int_{\sigma/k_1}^{\sigma}\lambda_{\nu(x)}dx\quad(k_1>1)$$

$$>\lambda_{\nu(\sigma/k_1)}\cdot\sigma\left(1-\frac{1}{k_1}\right).$$

But,

$$\lambda_{\nu(\sigma)} > (T-\varepsilon)\log\mu(\sigma).$$

So,

$$\lambda_{\nu(\sigma)} > (T-\varepsilon)\lambda_{\nu(\sigma/k_1)}\cdot\sigma\left(1-\frac{1}{k_1}\right)$$

$$> (T-\varepsilon)^2\lambda_{\nu(\sigma/k_1k_2)}\frac{\sigma^2}{k_1}\left(1-\frac{1}{k_1}\right)\left(1-\frac{1}{k_2}\right),\ (k_2>1)$$

$$\vdots$$

$$> (T-\varepsilon)^p\lambda_{\nu(\sigma/k_1k_2\cdots k_p)}\left(\frac{\sigma^p}{k_1k_2\cdots k_{p-1}}\right)\left(1-\frac{1}{k_1}\right)\cdot\left(1-\frac{1}{k_2}\right)\cdots\left(1-\frac{1}{k_p}\right),\ (k_p>1).$$

Put $k_1k_2\cdots k_p=n$ and $\dfrac{\left(1-\frac{1}{k_1}\right)\left(1-\frac{1}{k_2}\right)\cdots\left(1-\frac{1}{k_p}\right)}{k_1k_2\cdots k_{p-1}}=m$. So,

$$\lambda_{\nu(\sigma)} > (T-\varepsilon)^p\cdot\sigma^p\cdot\lambda_{\nu(\sigma/n)}\cdot m$$

$$> (T-\varepsilon)^{p+1}\cdot\sigma^p\cdot\log\mu(\sigma/n)\cdot m.$$

$$\lambda_{\nu(\sigma)} > (T-\varepsilon)^{p+1}\cdot\sigma^p\cdot m\cdot\int_{\sigma_0}^{\sigma/n}\lambda_{\nu(x)}\,dx$$

$$> (T-\varepsilon)^{p+2}\cdot\sigma^p\cdot m\cdot\left\{\int_{\sigma_0}^{\sigma/n}\log\mu(x)\,dx \geqq \frac{e^{x\cdot\lambda(x)}}{\lambda}\right]_{x=\sigma_0}^{x=\sigma/n}\right\}$$

where $\lambda(\sigma)$ is a lower proximate order with respect to $\log\mu(\sigma)$. So,

$$\lambda_{\nu(\sigma)} > (T-\varepsilon)^{p+2}\cdot\sigma^p\cdot m\cdot\left\{\frac{e^{(\sigma/n)\cdot\lambda(\sigma/n)}}{\lambda}-K\right\},\ (K \text{ is a constant}).$$

But,

$$e^{\sigma\cdot\lambda(\sigma)}=\log\mu(\sigma) \text{ for a sequence of values of } \sigma\to\infty.$$

So,

$$\liminf_{\sigma\to\infty}\frac{\lambda_{\nu(\sigma)}}{e^{(\sigma/n)\cdot\lambda(\sigma/n)}\cdot\sigma^p} \geqslant \frac{T^{p+2}}{\lambda}\cdot m$$

and

$$\limsup_{\sigma\to\infty}\frac{\lambda_{\nu(\sigma)}}{\sigma^p\cdot\log\mu(\sigma/n)} \geqslant \frac{T^{p+2}}{\lambda}\cdot m.$$

iii) We omit the proof.

PROOF of THEOREM 2. i)

$$\log\mu(\sigma, f_2)-\log\mu(\sigma, f_1)<(A+\varepsilon), \text{ for } \sigma\geqq\sigma_0$$

$$\lambda_{\nu(\sigma,f_2)}-\lambda_{\nu(\sigma,f_1)}>(D-\varepsilon), \qquad \text{for } \sigma\geqq\sigma_0$$

So,

$$\limsup_{\sigma\to\infty}\frac{\log\mu(\sigma, f_2)-\log\mu(\sigma, f_1)}{\lambda_{\nu(\sigma,f_2)}-\lambda_{\nu(\sigma,f_1)}}\leqq\frac{A}{D}.$$

But,

$$\frac{e^{\sigma(\lambda_2-\varepsilon_2')}-e^{\sigma(\rho_1+\varepsilon_1)}}{e^{\sigma(\rho_2+\varepsilon_2)}-e^{\sigma(\lambda_1-\varepsilon_1')}}<\frac{\log\mu(\sigma, f_2)-\log\mu(\sigma, f_1)}{\lambda_{\nu(\sigma,f_2)}-\lambda_{\nu(\sigma,f_1)}}, \text{ for } \sigma\geqq\sigma_0$$

So,

$$\limsup_{\sigma\to\infty}\frac{e^{\sigma(\lambda_2-\varepsilon_2')}-e^{\sigma(\rho_1+\varepsilon_1)}}{e^{\sigma(\rho_2+\varepsilon_2)}-e^{\sigma(\lambda_1-\varepsilon_1')}}\leqq\frac{A}{D} \quad \cdots\cdots\cdots\cdots \text{(I)}$$

But,

$$\rho_1+\rho_2\geqq\lambda_1+\lambda_2$$

So,

$$e^{\rho_1+\rho_2+\varepsilon_1+\varepsilon_2}>e^{\lambda_1+\lambda_2-\varepsilon_1-\varepsilon_2}$$

i.e,

$$\frac{e^{\sigma(\rho_1+\varepsilon_1)}}{e^{\sigma(\lambda_1-\varepsilon_1')}}>\frac{e^{\sigma(\lambda_2-\varepsilon_2')}}{e^{\sigma(\rho_2+\varepsilon_2)}}$$

So,

$$\frac{e^{\sigma(\rho_1+\varepsilon_1)}-e^{\sigma(\lambda_2-\varepsilon_2')}}{e^{\sigma(\lambda_1-\varepsilon_1')}-e^{\sigma(\rho_2+\varepsilon_2)}}>\frac{e^{\sigma(\rho_1+\varepsilon_1)}}{e^{\sigma(\lambda_1-\varepsilon_1')}} \quad \cdots\cdots\cdots\cdots\cdots \text{(II)}$$

Hence by (I) and (II),

$$\limsup_{\sigma\to\infty} e^{\sigma(\rho_1-\lambda_1+\varepsilon)}\leqq\frac{A}{D}.$$

ii) We omit the proof.

PROOF of THEOREM 3. i)

$$\log\mu(\sigma)<e^{(\rho+\varepsilon)\sigma}, \text{ for } \sigma\geqq\sigma_0$$

$$\log\mu(\sigma)>e^{(\lambda-\varepsilon)\sigma}, \text{ for } \sigma\geqq\sigma_0$$

So,

$$\frac{\{\log \mu(\sigma k_1)\}^{p_1}}{\{\lambda_{\nu(\sigma k_2)}\}^{p_2}} < e^{\sigma(k_1 p_1 \rho - k_2 p_2 \lambda + \varepsilon')}$$

Hence,

$$\lim_{\sigma\to\infty} \frac{\{\log \mu(\sigma k_1)\}^{p_1}}{\{\lambda_{\nu(\sigma k_2)}\}^{p_2}} = 0, \text{ if } k_1 p_1 \rho < k_2 p_2 \lambda.$$

Similarly,

$$\lim_{\sigma\to\infty} \frac{\{\log \mu(\sigma k_1)\}^{p_1}}{\{\lambda_{\nu(\sigma k_2)}\}^{p_2}} = \infty, \text{ if } k_1 p_1 \lambda > k_2 p_2 \rho.$$

ii)

$\log \mu(\sigma) < e^{\sigma(\lambda+\varepsilon)}$, for a sequence of values of $\sigma \to \infty$.

$\lambda_{\nu(\sigma)} > e^{\sigma(\lambda-\varepsilon)}$, for $\sigma \geqq \sigma_0$.

Hence,

$$\frac{\{\log \mu(\sigma k_1)\}^{p_1}}{\{\lambda_{\nu(\sigma k_2)}\}^{p_2}} < e^{\sigma\{\lambda(k_1 p_1 - k_2 p_2) + \varepsilon'\}}$$

So,

$$\lim_{\sigma\to\infty} \inf \frac{\{\log \mu(\sigma k_1)\}^{p_1}}{\{\lambda_{\nu(\sigma k_2)}\}^{p_2}} = 0, \text{ if } k_1 p_1 < p_2 k_2.$$

Similarly,

$$\lim_{\sigma\to\infty} \sup \frac{\{\log \mu(\sigma k_1)\}^{p_1}}{\{\lambda_{\nu(\sigma k_2)}\}^{p_2}} = \infty, \text{ if } k_1 p_1 > k_2 p_2.$$

PROOF OF THEOREM 4. i) We know,

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \left\{ \dfrac{\mu(\sigma, f^n)}{\mu(\sigma, f)} \right\}^{1/n}}{\sigma} = \begin{matrix}\rho\\ \lambda\end{matrix},$$

So,

$$e^{n\cdot\sigma(\lambda-\varepsilon)} < \frac{\mu(\sigma, f^n)}{\mu(\sigma, f)} < e^{n\cdot\sigma(\rho+\varepsilon)}, \text{ for } \sigma \geqq \sigma_0.$$

Hence,

$$e^{\sigma(n\lambda+p-\varepsilon)} < e^{\sigma p} \frac{\mu(\sigma, f^n)}{\mu(\sigma, f)} < e^{\sigma(n\rho+p+\varepsilon)}.$$

Thus,

$$e^{\sigma p}\cdot\frac{\mu(\sigma, f^n)}{\mu(\sigma, f)}\to\begin{matrix}0\\ \infty\end{matrix},\ \begin{matrix}\text{if } n\rho+p<0\\ \text{if } n\lambda+p>0\end{matrix},\ \text{as } \sigma\to\infty.$$

ii) Put $n=1$ in theorem 4, i), then we get,

$$e^{\sigma p}\cdot\frac{\mu(\sigma, f^1)}{\mu(\sigma, f)}\to\begin{matrix}0\\ \infty\end{matrix},\ \text{if}\ \begin{matrix}\rho+p<0\\ \lambda+p>0\end{matrix},\ \text{as } \sigma\to\infty.$$

Thus,

$$e^{\sigma p}\cdot\frac{\mu(\sigma, f^n)}{\mu(\sigma, f^{n-1})}\to\begin{matrix}0\\ \infty\end{matrix},\ \text{if}\ \begin{matrix}\rho+p<0\\ \lambda+p>0\end{matrix},\ \text{as } \sigma\to\infty.$$

Put $n=1, 2, \cdots, k$ and add, as each term $\to\begin{matrix}0\\ \infty\end{matrix}$, the sum of finite number of terms $\to\begin{matrix}0\\ \infty\end{matrix}$.

Thus we get,

$$e^{\sigma p}\sum_{n=1}^{k}\frac{\mu(\sigma, f^n)}{\mu(\sigma, f^{n-1})}\to\begin{matrix}0\\ \infty\end{matrix},\ \text{if}\ \begin{matrix}\rho+p<0\\ \lambda+p>0\end{matrix},\ \text{as } \sigma\to\infty.$$

PROOF OF THEOREM 5. It is well known that $\log\mu(\sigma)$ is a convex increasing function of σ. We draw the graph of $\log\mu(\sigma)$ and take n points $p_1, p_2, \cdots, p_n$ on it, with coordinates,

$$p_i=(\sigma_i, \log\mu(\sigma_i)).$$

The coordinates of Q which divides the chord p_1p_2 in the ratio $m_1: m_2$ are,

$$\frac{m_1\cdot\sigma_1+m_2\cdot\sigma_2}{m_1+m_2},\ \frac{m_1\cdot\log\mu(\sigma_1)+m_2\cdot\log\mu(\sigma_2)}{m_1+m_2}.$$

Next join Q_1 and p_3. The coordinates of Q_2 dividing the line Q_1p_3 in the ratio $m_3: m_1+m_2$ are,

$$\frac{\sum_{i=1}^{3}m_i\cdot\sigma_i}{\sum_{i=1}^{3}m_i},\ \frac{\sum_{i=1}^{3}m_i\cdot\log\mu(\sigma_i)}{\sum_{i=1}^{3}m_i}.$$

Lastly, we get the coordinates of Q_{n-1},

$$\frac{\sum_{i=1}^{n}m_i\cdot\sigma_i}{\sum_{i=1}^{n}m_i},\ \frac{\sum_{i=1}^{n}m_i\cdot\log\mu(\sigma_i)}{\sum_{i=1}^{n}m_i}.$$

Now using the convexity property we get,

$$\frac{\log \prod_{i=1}^{n} \{\mu(\sigma_i)\}^{m_i}}{\sum_{i=1}^{n} m_i} \geqslant \log \mu \left\{ \frac{\sum_{i=1}^{n} m_i \cdot \sigma_i}{\sum_{i=1}^{n} m_i} \right\}$$

i.e,

$$\left\{ \prod_{i=1}^{n} \{\mu(\sigma_i)\}^{m_i} \right\}^{\frac{1}{\sum_{i=1}^{n} m_i}} \geqslant \mu \left\{ \frac{\sum_{i=1}^{n} m_i \sigma_i}{\sum_{i=1}^{n} m_i} \right\}$$

So,

$$\prod_{i=1}^{n} \{\mu(\sigma_i)\}^{k_i} \geqq \mu \left\{ \sum_{i=1}^{n} \sigma_i \cdot k_i \right\}$$

COROLLARY. If $m_1 = m_2 = \cdots = m_n$, then, $k_i = 1/n$, $i = 1, 2, 3, \cdots, n$. Then we have,

$$\prod_{i=1}^{n} \{\mu(\sigma_i)\}^{1/n} \geqq \mu \left\{ \sum_{i=1}^{n} \sigma_i / n \right\}$$

As, Arith mean $\geqq$ Geom mean $\geqq$ Harmonic mean and $\log \mu(\sigma)$ is an increasing function of σ, we get,

$$\prod_{i=1}^{n} \{\mu(\sigma_i)\}^{1/n} \geqq \mu\left(\frac{\sigma_1 + \sigma_2 + \cdots + \sigma_n}{n} \right)$$

$$\geqq \mu\{(\sigma_1 \cdot \sigma_2 \cdots \sigma_n)^{1/n}\}$$

$$\geqslant \mu\left(\frac{n}{\frac{1}{\sigma_1} + \frac{1}{\sigma_2} + \cdots + \frac{1}{\sigma_n}} \right)$$

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