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On the Mean Values of Integral Functions and Their Derivatives Defined by Dirichlet Series

By

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1. Consider the Dirichlet Series $f(s)=\sum_{n=1}^{\infty}a_n e^{s\lambda_n}$, where $\lambda_{n+1}>\lambda_n$, $\lambda_1\geqslant 0$, $\lim_{n\to\infty}\lambda_n=\infty$, $s=\sigma+it$ and

$$\lim_{n\to\infty}\sup\frac{\log n}{\log\lambda_n}=E<\infty. \tag{1.1}$$

Let σ_c and σ_a be the abscissa of convergence and abscissa of absolute convergence, respectively, of $f(s)$. Let $\sigma_c=\infty$ then σ_a will also be infinite, since according to a known result ([1], p. 4) a Dirichlet Series which satisfies (1.1) has its abscissa of convergence equal to its abscissa of absolute convergence, and so, $f(s)$ is an integral function.

The Mean Value of $f(s)$ is

$$I_2(\sigma)=I_2(\sigma,f)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(\sigma+it)|^2dt, \tag{1.2}$$

and extending this definition to $f^{(p)}(s)$, the p^{th}-derivative of $f(s)$,

$$I_2(\sigma,f^{(p)})=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f^{(p)}(\sigma+it)|^2dt. \tag{1.3}$$

Let $\mu(\sigma)=\max_{n\geqslant 1}\{|a_n|e^{\sigma\lambda_n}\}$; $M(\sigma)=\underset{-\infty<t<\infty}{\text{l.u.b.}}|f(\sigma+it)|$ be respectively the maximum term and the maximum modulus of an integral function.

It is known ([2], p. 67) that

$$\log\mu(\sigma)=0(1)+\int_{\sigma_0}^{\sigma}\lambda_{\nu(t)}\,dt, \tag{1.4}$$

where $\nu(\sigma)$ is the rank of the maximum term.

Further, we know ([3], p. 265, Theorem 5) that

$$\log M(\sigma)\sim\log\mu(\sigma), \tag{1.5}$$

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provided (1.1) holds and $f(s)$ is of finite order.

It is also known* ([4], p. 523) that for functions of finite non-zero linear order ρ and lower order λ,

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log I_2(\sigma)}{\sigma} = \begin{matrix}\rho\\ \lambda\end{matrix} \tag{1.6}$$

and

$$\log\{I_2(\sigma)\}^{1/2} \sim \log M(\sigma). \tag{1.7}$$

Through out this paper we shall assume that the function $f(s)$ is of finite non-zero linear order and satisfies (1.1). In this paper we have obtained a few properties of $I_2(\sigma)$ and its derivative, also of $I_2(\sigma, f^{(p)})$.

2. THEOREM 1. *Let $f(s)$ be an integral function of linear order ρ and lower order λ, then*

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left\{\frac{\log I_2'(\sigma)/I_2(\sigma)}{\sigma}\right\} = \begin{matrix}\rho\\ \lambda\end{matrix}. \tag{2.1}$$

PROOF. We know ([4], p. 521) that $\log I_2(\sigma)$ is an increasing convex function of σ. Therefore, $\log I_2(\sigma)$ is differentiable almost everywhere with an increasing derivative; the set of points where the left hand derivative is less than the right-hand derivative is of measure zero. This enables us to express $\log I_2(\sigma)$ in the following form:

$$\log I_2(\sigma) = \log I_2(\sigma_0) + \int_{\sigma_0}^{\sigma} \frac{I_2'(x)}{I_2(x)}\, dx,$$

for an arbitrary σ_0.

We have,

$$\log I_2(\sigma) \leqslant \log I_2(\sigma_0) + \frac{I_2'(\sigma)}{I_2(\sigma)}(\sigma-\sigma_0)$$

or

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left\{\frac{\log\log I_2(\sigma)}{\sigma}\right\} \leqslant \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left[\frac{1}{\sigma}\left\{\log\left(\frac{I_2'(\sigma)}{I_2(\sigma)}\right)\right\}\right].$$

Again, for an arbitrary fixed $k > 0$

* Results (1. 6) and (1. 7) has been proved under the condition $\lim_{n\to\infty} \sup \frac{\log n}{\lambda_n} = D = 0$, though the results also hold for $E < \infty$.

$$\log I_2(\sigma+k)=\log I_2(\sigma)+\int_\sigma^{\sigma+k}\frac{I_2'(x)}{I_2(x)}\,dx\geqslant k\,\frac{I_2'(\sigma)}{I_2(\sigma)},$$

and therefore,

$$\lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\left\{\frac{\log\log I_2(\sigma+k)}{\sigma}\right\}\geqslant\lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\left[\frac{1}{\sigma}\left\{\log\left(\frac{I_2'(\sigma)}{I_2(\sigma)}\right)\right\}\right].$$

Thus

$$\lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\left\{\frac{\log\log I_2(\sigma)}{\sigma}\right\}=\lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\left[\frac{1}{\sigma}\left\{\log\left(\frac{I_2'(\sigma)}{I_2(\sigma)}\right)\right\}\right].$$

Further, from (1. 7) we have

$$\lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\left\{\frac{\log\log I_2(\sigma)}{\sigma}\right\}=\lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\left\{\frac{\log\log M(\sigma)}{\sigma}\right\}.$$

Hence

$$\lim_{\sigma\leftarrow\infty}\begin{matrix}\sup\\ \inf\end{matrix}\left\{\frac{\log I_2'(\sigma)/I_2(\sigma)}{\sigma}\right\}=\lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\left\{\frac{\log\log M(\sigma)}{\sigma}\right\}=\begin{matrix}\rho\\ \lambda\end{matrix}.$$

THEOREM 2. *Let $f(s)$ be an integral function of linear order ρ and lower order λ, then*

$$\lim_{\sigma\to\infty}\sup\left\{\frac{\log I_2(\sigma)}{\sigma\lambda_{\nu(\sigma)}}\right\}\leqslant 2(1-\lambda/\rho) \tag{2.2}$$

and

$$\lim_{\sigma\to\infty}\sup\left\{\frac{\log I_2(\sigma)}{\lambda_{\nu(\sigma)}\log\lambda_{\nu(\sigma)}}\right\}\leqslant 2(1/\lambda-1/\rho). \tag{2.3}$$

PROOF. It is known* ([5], p. 84) that, for $0<\rho<\infty$

$$\lim_{\sigma\to\infty}\sup\left\{\frac{\log\mu(\sigma)}{\sigma\lambda_{\nu(\sigma)}}\right\}\leqslant 1-\lambda/\rho \tag{2.4}$$

and

$$\lim_{\sigma\to\infty}\sup\left\{\frac{\log\mu(\sigma)}{\lambda_{\nu(\sigma)}\log\lambda_{\nu(\sigma)}}\right\}\leqslant 1/\lambda-1/\rho. \tag{2.5}$$

Using (1.5) and (1.7), we have

* Results (2.4) and (2.5) has been proved under the condition $\lim_{n\to\infty}\sup\frac{\log n}{\lambda_n}=D=0$, though the result also hold for $E<\infty$.

$$\limsup_{\sigma\to\infty}\left[\frac{\log\{I_2(\sigma)\}^{1/2}}{\sigma\lambda_{\nu(\sigma)}}\right]=\limsup_{\sigma\to\infty}\left\{\frac{\log M(\sigma)}{\sigma\lambda_{\nu(\sigma)}}\right\}$$

$$=\limsup_{\sigma\to\infty}\left\{\frac{\log\mu(\sigma)}{\sigma\lambda_{\nu(\sigma)}}\right\}$$

$$\leqslant 1-\lambda/\rho.$$

Hence

$$\limsup_{\sigma\to\infty}\left\{\frac{\log I_2(\sigma)}{\sigma\lambda_{\nu(\sigma)}}\right\}\leqslant 2(1-\lambda/\rho).$$

Proceeding as above and using (2.5), we have

$$\limsup_{\sigma\to\infty}\left\{\frac{\log I_2(\sigma)}{\lambda_{\nu(\sigma)}\log\lambda_{\nu(\sigma)}}\right\}\leqslant 2(1/\lambda-1/\rho).$$

THEOREM 3. *Let $f(s)$ be an integral function of finite linear order ρ and lower order λ, then*

$$\limsup_{\sigma\to\infty}\left\{\frac{\log I_2(\sigma)}{\sigma\lambda_{\nu(\sigma)}}\right\}\leqslant 2. \tag{2.6}$$

PROOF. Using (1.5) and (1.7), we have

$$\limsup_{\sigma\to\infty}\left\{\frac{\log I_2(\sigma)}{\sigma\lambda_{\nu(\sigma)}}\right\}=2\limsup_{\sigma\to\infty}\left\{\frac{\log\mu(\sigma)}{\sigma\lambda_{\nu(\sigma)}}\right\}.$$

From (1.4), we get

$$\limsup_{\sigma\to\infty}\left\{\frac{\log\mu(\sigma)}{\sigma\lambda_{\nu(\sigma)}}\right\}\leqslant 1.$$

Hence

$$\limsup_{\sigma\to\infty}\left\{\frac{\log I_2(\sigma)}{\sigma\lambda_{\nu(\sigma)}}\right\}\leqslant 2.$$

THEOREM 4. *Let $f(s)$ be an integral function of finite linear order ρ and lower order λ, then for $\sigma>\sigma_0$, $\varepsilon>0$*

$$I_2'(\sigma)>\frac{I_2(\sigma)\log I_2(\sigma)}{(1+\varepsilon)\sigma}, \tag{2.7}$$

where $\varepsilon=\varepsilon(\sigma)\to 0$ as $\sigma\to\infty$.

PROOF. For the left hand derivative of $\log I_2(\sigma)$, we have

$$\frac{I_2'(\sigma)}{I_2(\sigma)} \geqslant \frac{\log I_2(\sigma)-\log I_2(\sigma_1)}{\sigma-\sigma_1}$$

$$> \frac{\log I_2(\sigma)}{(1+\varepsilon)\sigma}$$

where $\sigma_1<\sigma$ and $\varepsilon=\varepsilon(\sigma)\to 0$ as $\sigma\to\infty$. Hence,

$$I_2'(\sigma) > \frac{I_2(\sigma)\log I_2(\sigma)}{(1+\varepsilon)\sigma},$$

where $\varepsilon=\varepsilon(\sigma)\to 0$ as $\sigma\to\infty$.

Corollary. *Let $f(s)$ be an integral function of finite order ρ and if $I_2'(\sigma)$ is the derivative of $I_2(\sigma)$. Then for $\sigma_0<\sigma_1<\sigma_2$*

$$\frac{I_2'(\sigma_1)}{I_2(\sigma_1)} \leqslant \frac{\log I_2(\sigma_2)-\log I_2(\sigma_1)}{\sigma_2-\sigma_1} \leqslant \frac{I_2'(\sigma_2)}{I_2(\sigma_2)}. \tag{2.8}$$

Proof. From (1.4), we have

$$\log I_2(\sigma_2)=\log I_2(\sigma_1)+\int_{\sigma_1}^{\sigma_2}\frac{I_2'(x)}{I_2(x)}\,dx \tag{2.9}$$

$$\leqslant \log I_2(\sigma_1)+\frac{I_2'(\sigma_2)}{I_2(\sigma_2)}(\sigma_2-\sigma_1)$$

and

$$\log I_2(\sigma_2)=\log I_2(\sigma_1)+\int_{\sigma_1}^{\sigma_2}\frac{I_2'(x)}{I_2(x)}\,dx \tag{2.10}$$

$$\geqslant \log I_2(\sigma_1)+\frac{I_2'(\sigma_1)}{I_2(\sigma_1)}(\sigma_2-\sigma_1).$$

Combining (2.9) and (2.10) we get the result.

3. Theorem 5. *Let $f(s)$ be an integral function of linear order ρ and lower order λ, then for $Re(s)=\sigma$ and $\lambda\geqslant\delta>0$*

$$I_2(\sigma, f)< I_2(\sigma, f^{(1)})<\cdots< I_2(\sigma, f^{(p)}), \tag{3.1}$$

where p an integer.

Proof. It is known ([4], p. 522)

$$I_2(\sigma, f^{(1)}) \geqslant \frac{1}{2^2}\left(\frac{\log I_2(\sigma, f)}{\sigma}\right)^2 I_2(\sigma, f). \tag{3.2}$$

Taking limits on both the sides and using (1.6), we get

$$\lim_{\sigma\to\infty} \inf\left[\frac{1}{\sigma}\log\left\{\frac{I_2(\sigma, f^{(1)})}{I_2(\sigma, f)}\right\}^{1/2}\right] \geqslant \lambda.$$

Again, for $\epsilon > 0$ and σ sufficiently large

$$\left\{\frac{I_2(\sigma, f^{(1)})}{I_2(\sigma, f)}\right\} > e^{2\sigma(\lambda-\varepsilon)}.$$

If $\lambda \geqslant \delta > 0$,

$$I_2(\sigma, f) < I_2(\sigma, f^{(1)}),$$

and the result follows for subsequent derivatives.

THEOREM 6. *Let $f(s)$ be an integral function, then for $\sigma > 0$ and $\lambda \geqslant \delta > 0$*

$$I_2(\sigma, f^{(p)}) \geqslant \frac{1}{2^{2p}}\left\{\frac{\log I_2(\sigma, f)}{\sigma}\right\}^{2p} I_2(\sigma, f), \tag{3.3}$$

where p is an integer.

PROOF. Writing (3.2) for p^{th}-derivative, we have

$$I_2(\sigma, f^{(p)}) \geqslant \frac{1}{2^2}\left\{\frac{\log I_2(\sigma, f^{(p-1)})}{\sigma}\right\}^2 I_2(\sigma, f^{(p-1)})$$

$$\geqslant \frac{1}{(2^2)^2}\left\{\frac{\log I_2(\sigma, f^{(p-1)})}{\sigma}\right\}^2\left\{\frac{\log I_2(\sigma, f^{(p-2)})}{\sigma}\right\}^2 I_2(\sigma, f^{(p-2)})$$

$$\geqslant \frac{1}{2^{2p}}\left\{\frac{\log I_2(\sigma, f^{(p-1)})}{\sigma}\right\}^2\left\{\frac{\log I_2(\sigma, f^{(p-2)})}{\sigma}\right\}^2 \cdots \left\{\frac{\log I_2(\sigma, f)}{\sigma}\right\}^2 I_2(\sigma, f).$$

Using (3.1), we get

$$I_2(\sigma, f^{(p)}) \geqslant \frac{1}{2^{2p}}\left\{\frac{\log I_2(\sigma, f)}{\sigma}\right\}^{2p} I_2(\sigma, f).$$

COROLLARY. *Let $f(s)$ be an integral function of linear order ρ and lower order λ, then*

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix}\left[\frac{1}{\sigma}\log\left\{\frac{I_2(\sigma, f^{(p)})}{I_2(\sigma, f)}\right\}^{1/2p}\right] \geqslant \begin{matrix}\rho\\ \lambda\end{matrix}, \tag{3.4}$$

where p is an integer.

This follow immediately from (3.3).

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