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Arithmetic Genera of Some Local Rings

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Introduction

Let A be a zero dimensional ideal of a local ring $\mathfrak{O}$. The constant term of the Hilbert-Samuel's function of A is called the arithmetic genus of A and is denoted by $p_a(A)$. In this note, we investigate this less studied invariant of A in connection with the properties of complete ideals. We show that if $\mathfrak{O}$ is analytically unramified and of (Krull) dimension 1, $p_a(A)$ is the same for any zero dimensional normal ideal A and that the similar result holds for an analytically irreducible normal local domain of dimension 2 which satisfies the condition N [2].

It is to be mentioned that these results are considered, in a sense, as an arithmetic analog of those obtained by Muhly and Zariski [3]. The possibility of the study of the arithmetic genus in this direction was suggested to me by H. T. Muhly, to whom I would express my sincere gratitude. The theorem 1 in §1 is essentially due to him.

All rings (resp. local rings) considered here are assumed commutative (resp. commutative Noetherian) with identity. By a local ring $(\mathfrak{O}, \mathfrak{m})$ we mean that $\mathfrak{O}$ is a local ring and $\mathfrak{m}$ is its maximal ideal.

§1. Arithmetic genus of one dimensional local ring

Let $(\mathfrak{O}, \mathfrak{m})$ be an analytically unramified local ring and let $\bar{\mathfrak{O}}$ be its integral closure in its total quotient ring. Then, $\bar{\mathfrak{O}}$ is a finite $\mathfrak{O}$-module, say, $\bar{\mathfrak{O}}=\mathfrak{O}\omega_1+\cdots+\mathfrak{O}\omega_s$ and there is a non zero divisor c in $\mathfrak{O}$ such that $c\,\bar{\mathfrak{O}}\subset\mathfrak{O}$. Denote by $\mathfrak{c}$ the conductor of $\bar{\mathfrak{O}}$ with respect to $\mathfrak{O}$, then $c\in\mathfrak{c}$ so that $\mathfrak{c}$ is an $\mathfrak{m}$-primary ideal if $\mathfrak{O}$ is one dimensional.

LEMMA 1. *Let $A_1, \cdots, A_r$ be ideals in $\mathfrak{O}$, then there is an integer k such that $(A_1^{n_1}\cdots A_r^{n_r})_a=(A_1^{n_1-k}\cdots A_r^{n_r-k})(A_1^k\cdots A_r^k)_a$ for $n_i\geq k$ $(i=1, \cdots, r)$, where B_a is the integral closure of B. In particular, for an ideal A of $\mathfrak{O}$, $(A^k)_a$ is normal for some integer k. We call this ideal a derived normal ideal of A.*

PROOF. The proof of this lemma is quite similar to the one given in theorem 1 in [8], so that we shall state it briefly in the case when $r=2$. Let $A=(a_1, \cdots, a_r)$ and $B=(b_1, \cdots, b_s)$ and let $\mathfrak{O}(A, B)=\mathfrak{O}[a_1t, \cdots, a_rt, t^{-1}, b_1u, \cdots, b_su, u^{-1}]$ where t and u are indeterminates. Then, $\mathfrak{O}(A, B)$ is a bigraded subring of $\mathfrak{O}[t, t^{-1}, u, u^{-1}]$. If $\mathfrak{O}^*(A, B)$ is the integral closure of $\mathfrak{O}(A, B)$ in $\mathfrak{O}[t, t^{-1}, u, u^{-1}]$, then $\mathfrak{O}^*(A, B)$ is again bigraded. Since $\mathfrak{O}$ is analytically unramified, there is an integer k such that $(A^nB^m)_a \subset A^{n-k}B^{m-k}$ for all $n, m \geq k$. Hence, by the argument similar to lemma 1 and 2 in [8], we obtain our lemma.

LEMMA 2. *$A\bar{\mathfrak{O}} \cap \mathfrak{O} \subset A_a$. If A is a principal ideal generated by a non zero divisor, we have $A\bar{\mathfrak{O}} \cap \mathfrak{O} = A_a$.*

PROOF. If $x \in A\bar{\mathfrak{O}} \cap \mathfrak{O}$, then we can write $x\omega_i$ as $x\omega_i = \sum_{j=1}^{s} a_{ij}\omega_j$ with $a_{ij} \in A$. Therefore $\sum_{j=1}^{s}(a_{ij}-\delta_{ij}x)\omega_j=0$ $(i=1, \cdots, s)$ and whence $\det\|a_{ij}-\delta_{ij}x\|=0$. Consequently $x \in A_a$. Suppose $A=(x)$, x is a non zero divisor of $\mathfrak{O}$. If $y \in (x)_a$, then y satisfies the equation of the form, $y^\rho + c_1xy^{\rho-1} + \cdots + c_\rho x^\rho = 0$, where $c_i \in \mathfrak{O}$. Dividing by x^ρ, we get $(y/x)^\rho + c_1(y/x)^{\rho-1} + \cdots + c_\rho = 0$ and hence $y/x \in \bar{\mathfrak{O}}$.

LEMMA 3. *Let α and β be non zero divisors in $\mathfrak{O}$ such that $\alpha^{m_0} \in \mathfrak{c}$, then we have $(\alpha^m\beta^n)_a = (\alpha^{m-m_0}\beta^n)(\alpha^{m_0})_a$ for $n \geq 0$ and $m \geq m_0$.*

PROOF. We have $(\alpha^m\beta^n)_a = (\alpha^m\beta^n)\bar{\mathfrak{O}} \cap \mathfrak{O} = \alpha^m\beta^n\bar{\mathfrak{O}} = \alpha^{m-m_0}\beta^n\alpha^{m_0}\bar{\mathfrak{O}}$ (since $\alpha^{m_0}\bar{\mathfrak{O}} \subset \mathfrak{O}) = (\alpha^{m-m_0}\beta^n)(\alpha^{m_0}\bar{\mathfrak{O}} \cap \mathfrak{O}) = (\alpha^{m-m_0}\beta^n)(\alpha^{m_0})_a$.

LEMMA 4. *If $\mathfrak{O}$ is one dimensional and if A is a normal $\mathfrak{m}$-primary ideal, then for any $\mathfrak{m}$-primary ideal B, there exist positive integers p, q and non-negative integer m_0 such that $A'^mB'^n$ is complete for $n \geq 0$ and $m \geq m_0$ where $A' = A^p$ and $B' = B^q$.*

PROOF. For suitable powers $A^p = A'$ and $B^q = B'$ of A and B, A' and B' have minimal reductions of order 1, i.e., $A' = A'_a = (\alpha)_a$ and $B'_a = (\beta)_a$ for some non zero divisors α and β [4, 5]. If m_0 is an integer such that $\alpha^{m_0} \in \mathfrak{c}$, then, from the relation, $(A'^mB'^n)_a = (\alpha^m\beta^n)_a = (\alpha^{m-m_0}\beta^n)(\alpha^{m_0})_a \subset A'^{m-m_0}B'^nA'^{m_0} = A'^mB'^n$ (lemma 3), our assertion follows.

LEMMA 5. *Let A and B be $\mathfrak{m}$-primary ideals in an one dimensional local ring $\mathfrak{O}$ and assume that A and B have minimal reductions of order 1 and that A is normal, then we have $A^mB^{n-1}/A^mB^n \approx A^{m_0}/A^{m_0}B$ for $n > 0$ if $m \geq m_0$ where m_0 is some integer depending on A and B. Consequently, $l(A^mB^n) - l(A^mB^{n-1})$ is constant for $n > 0$ if $m \geq m_0$, where $l(A)$ denotes the length of an $\mathfrak{m}$-primary ideal of A.*

PROOF. By lemma 4, we have $A^m B^n = (\alpha^{m-m_0}\beta^n)(\alpha^{m_0})_a = \alpha^{m-m_0}\beta^n A^{m_0}$. Hence $A^m B^{n-1}/A^m B^n = \alpha^{m-m_0}\beta^{n-1}A^{m_0}/\alpha^{m-m_0}\beta^n A^{m_0}$. Since α and β are non zero divisors in $\mathfrak{O}$, we see immediately that $A^m B^{n-1}/A^m B^n \approx A^{m_0}/\beta A^{m_0} = A^{m_0}/B A^{m_0}$.

For fixed $\mathfrak{m}$-primary ideals A and B, we denote by $r(m, n)$ the length of $(A^m B^n)_a$ and by $s(m, n)$ the length of $A^m B^n$. By a theorem, due to Bhattacharya, $s(m, n)$ becomes polynomial in m and n if $m \geq M$ and $n \geq N$ for some integers M and N [1]. We denote this polynomial by $\rho(m, n)$. In the one dimensional local ring, we can write $\rho(m, n) = \tilde{\alpha}m + \tilde{\beta}n + \tilde{\gamma}$.

LEMMA 6. *In the situation of lemma* 5, *we have* $r(m, n) = \rho(m, n)$ *if* $m \geq \text{Max}\{m_0, M\}$ *and* $n \geq 0$.

PROOF. Both $r(m, n)$ and $\rho(m, n)$ are integer valued functions and coincide if m and n are sufficiently large (lemma 4). Since $r(m, n) - r(m, n-1)$ is constant, say b, if $m \geq m_0$ (lemma 5) and $\rho(m, n) - \rho(m, n-1) = \tilde{\beta}$. Hence $\tilde{\beta} = b$ and $r(m, n) = \rho(m, n)$ for $m \geq \text{Max}\{m_0, M\}$ and $n \geq 0$.

THEOREM 1. *If A and B are normal $\mathfrak{m}$-primary ideals in an analytically unramified local ring of dimension* 1, *then $p_a(A) = p_a(B)$ where $p_a(C)$ is the arithmetic genus of C.*

PROOF. Since $p_a(A) = p_a(A^p)$ for any positive integer p, we can assume that both A and B have minimal reductions of order 1. By lemma 6, there exist integers m^* and n^* such that

$$r(m, n) = \rho(m, n) \text{ for } m \geq m^* \text{ and } n \geq 0, \text{ or for } m \geq 0 \text{ and } n \geq n^*.$$

Hence, $r(m, 0) = \rho(m, 0) = \tilde{\alpha}m + \tilde{\gamma} = l(A^m) = e(A)m + p_a(A)$ and $r(0, n) = \rho(0, n) = \tilde{\beta}n + \tilde{\gamma} = l(B^n) = e(B)n + p_a(B)$ where $e(C)$ is the multiplicity of C. Consequently, we have $p_a(A) = \tilde{\gamma} = p_a(B)$.

We denote this common arithmetic genus of normal $\mathfrak{m}$-primary ideals by $p_a(\mathfrak{O})$ and call it the arithmetic genus of the local ring $\mathfrak{O}$.

THEOREM 2. *For any $\mathfrak{m}$-primary ideal A of $\mathfrak{O}$, we have $p_a(A) \geq p_a(\mathfrak{O})$.*

PROOF. Let $A_g = (A^g)_a$ be a derived normal ideal of A (lemma 1). Then, by theorem 1, $p_a(A_g) = p_a(\mathfrak{O})$. Hence our assertion follows from the relations: $e(A^g)n + p_a(A^g) = l(A^{gn}) \geq l((A_g)^n) = e(A_g)n + p_a(\mathfrak{O})$, $e(A^g) = e(A_g)$ and $p_a(A) = p_a(A^g)$, q.e.d.

§2. Arithmetic genus of two dimensional local ring which satisfies the condition N

First, we recall the following definitions which will be needed in this section [2]. Let $(\mathfrak{O}, \mathfrak{m})$ be a normal local domain and let v be a discrete rank 1 valuation of the quotient field F of $\mathfrak{O}$ which dominates $\mathfrak{O}$. We say that v is a divisor of second kind if the residue field R_v/M_v of v is a finitely generated extension of transcendence degree $r-1$ over the residue field $\mathfrak{O}/\mathfrak{m}$ of $\mathfrak{O}$ where r=dimension $\mathfrak{O}$. Let $A_0=\mathfrak{O}$ and $A_i=\{x \in \mathfrak{O} \mid v(x)>v(A_{i-1})\}(i>0)$, then the sequence $A_0=\mathfrak{O}\supset A_1\supset A_2\supset \ldots$ of valuation ideals has the property that $A_iA_j\subset A_{i+j}$. If $B_0=\mathfrak{O}\supset B_1\supset \ldots$ is any subsequence of $A_0\supset A_1\supset \ldots$ such that $v(B_{i+j})=v(B_i)+v(B_j)$, then the direct sum $G(B)=\Sigma B_i/B_{i+1}$ becomes a graded ring. The divisor v is said to be Noetherian if there exists a subsequence $B_0\supset B_1\supset \ldots$ such that $G(B)$ is Noetherian. If every divisor of second kind is Noetherian, we say that $\mathfrak{O}$ satifies the condition N. If $\bar{v}_A$ is the homogeneous pseudo valuation associated with powers of an $\mathfrak{m}$-primary ideal A, $\bar{v}_A$ can be represented as a subvaluation, $\bar{v}_A=\min\left\{\frac{v_1}{e_1}, \cdots, \frac{v_s}{e_s}\right\}$, $e_i=v_i(A)$, and v_i is a divisor of 2nd kind if $\mathfrak{O}$ is analytically irreducible. The valuations $v_1, \cdots, v_s$ are uniquely determined by A and are called the Rees valuations associated with A. We denote the set $\{v_1, \cdots, v_s\}$ by $S(A)$. An ideal W in $\mathfrak{O}$ is called asymptotically irreducible if all powers W^n of W are the valuation ideals of some divisor of 2nd kind relative to $\mathfrak{O}$ [2, §2].

LEMMA 7. *Let $(\mathfrak{O}, \mathfrak{m})$ be an analytically irreducible 2 dimensional normal local domain which satisfies the condition N and let A and B be $\mathfrak{m}$-primary ideals. Suppose $S(A)\subset S(B)$ and B is normal, then for some powers $A'=A^p$ and $B'=B^q$ of A and B, $A'^mB'^n$ is complete for $m\geq 0$ and $n\geq 1$ where $S(A)$ and $S(B)$ are the sets of Rees valuations associated with A and B respectively.*

PROOF. Let $S(A)=\{v_1, \cdots, v_s\}$ and $S(B)=\{v_1, \cdots, v_s, v_{s+1}, \cdots, v_t\}$. Since $\mathfrak{O}$ satisfies the condition N, each v_i defines an asymptotically irreducible ideal $W_i (i=1, \cdots, t)$ and there exist sets of positive integers $p, \alpha_1, \cdots, \alpha_s$ and $q, \beta_1, \cdots, \beta_t$ such that

$$(A^p)_a=(W_1^{\alpha_1}\cdots W_s^{\alpha_s})_a \text{ and } B^q=(B^q)_a=(W_1^{\beta_1}\cdots W_s^{\beta_s}\cdots W_t^{\beta_t})_a$$

[2, theorem 4.3]. Moreover, these sets of integers are defined up to proportionality, so that we can assume $\alpha_i\leq\beta_i$ $(i=1, \cdots, s)$. Now, let

$$C=W_1^{\beta_1-\alpha_1}\cdots W_s^{\beta_s-\alpha_s} W_{s+1}^{\beta_{s+1}}\cdots W_t^{\beta_t}.$$

Then, $B^q=(B^q)_a=(A^pC)_a$. Hence, if n is sufficiently large, say $n\geq n_0$, we have

$$(A'^m B'^n)_a = (A'^m (A'C)^n)_a = (A'^{m+n} C^n)_a = (A'^{m+n-n_0} C^{n-n_0})(A'^{n_0} C^{n_0})_a$$
$$= A'^m (A'C)^{n-n_0} (A'C)_a^{n_0} \subset A'^m B'^{n-n_0} (B'^{n_0})_a = A'^m B'^n,$$

in view of lemma 1, where $A' = A^p$ and $B' = B^q$.

Lemma 8. *With the same hypothesis as in lemma 7, for suitable powers A^* and B^* of A and B, $\rho^*(m, n) = r^*(m, n) = s^*(m, n)$ for $m \geq 0$ and $n \geq 1$, where $\rho^*(m, n), \ldots$ denote the corresponding functions relative to A^* and B^*.*

Proof. Since $(A'^m B'^n)_a = (A'^m (A'C)^n)_a = (A'^{m+n-n_0} C^{n-n_0})(A'C)_a^{n_0}$ for $n \geq n_0$, the length $l(A'^m B'^n)_a$ becomes polynomial in m and n if $n - n_0$ is sufficiently large, say $n - n_0 \geq n_1$. Hence we can take $A^* = A'^{n_0+n_1}$ and $B^* = B'^{n_0+n_1}$.

Theorem 3. *If A and B are normal $\mathfrak{m}$-primary ideals in an analytically irreducible two dimensional normal local domain which satisfies the condition N and if $S(A) = S(B)$, then $p_a(A) = p_a(B)$ where $S(C)$ is the set of Rees valuations associated with C.*

Proof. Since $S(A) = S(A^p)$ and $p_a(A) = p_a(A^p)$ for any positive integer p, we can replace A and B by their powers. Hence, by lemma 8, we can assume A and B satisfy the following relation:

$$\rho(m, n) = r(m, n) = s(m, n) \text{ for } m \geq 0,\ n \geq 1, \text{ or for } m \geq 1,\ n \geq 0.$$

If $P_B(n)$ is the Hilbert-Samuel's function of B, then $P_B(n) = \rho(0, n)$. Hence $p_a(B) = P_B(0) = \rho(0, 0)$. Similarly, we have $p_a(A) = \rho(0, 0)$, and consequently $p_a(A) = p_a(B)$.

We know that every two dimensional regular local ring satisfies the condition N [2, coroll. 5. 4] and if a simple ideal W corresponds to the divisor v of 2nd kind [9], then W is the asymptotically irreducible ideal of minimal value for v. From these remarks, we get the following

Corollary. *Let $A = \mathfrak{P}_1^{\alpha_1} \ldots \mathfrak{P}_s^{\alpha_s}$ and $B = \mathfrak{P}_1'^{\beta_1} \ldots \mathfrak{P}_t'^{\beta_t}$ be the factorizations of the complete ideals A and B in a two dimensional regular local ring into the products of simple ideals. If $s = t$ and $\{\mathfrak{P}_1, \cdots, \mathfrak{P}_s\} = \{\mathfrak{P}_1', \cdots, \mathfrak{P}_t'\}$, then $p_a(A) = p_a(B)$.*

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