J. Math. Tokushima Univ.
Vol. 2. (1968), 1-5

On the Connectivity Structures of Spaces

By

Tadashi TANAKA

(Received September 30, 1968)

1. Introduction

If σ is a topology on a set X, the resulting space will be denoted by (X, σ). The family of all connected subsets of (X, σ) is called the connectivity structure of (X, σ) and denoted by $\boldsymbol{C}(X, \sigma)$. A function $f: (X, \sigma)\to(Y, \tau)$ is a connected function if for every connected subset C of (X, σ), $f(C)$ is connected. Also f is a connectivity function if the graph function $g: (X, \sigma)\to(X, \sigma)\times(Y, \tau)$, defined by $g(p)=p\times f(p)$, is a connected function [1], [2]. In [3], S. K. Hildebrand and D. E. Sanderson have shown that $f: (X, \sigma)\to(Y, \tau)$ is a connectivity function if and only if $\boldsymbol{C}(X, \sigma)=\boldsymbol{C}(X, \sigma\cup f^{-1}(\tau))$ where $\sigma\cup f^{-1}(\tau)$ is the topology on X generated by σ and $f^{-1}(\tau)$.

In this paper we investigate what conditions, if we have such a space (X, σ_2) that is finer than (X, σ_1), will lead the relation $\boldsymbol{C}(X, \sigma_1)=\boldsymbol{C}(X, \sigma_2)$. Some results concerned with this problem may be found [4] and [5].

2. Definitions and preliminary results

Let (X, σ_1) be a T_1-space and denote it by $\{U(p)\}$, the family of all neighbourhoods of a point p in (X, σ_1). Let $\boldsymbol{F}$ be a family of subsets of X having the following properties:

($\mathrm{F_I}$) The empty set ϕ belongs to $\boldsymbol{F}$.

($\mathrm{F_{II}}$) If F_α and F_β belong to $\boldsymbol{F}$, then the sum $F_\alpha\cup F_\beta$ belongs to $\boldsymbol{F}$.

Let $\{V(p)\}$ be the family of all subsets $V(p)$ of X, $V(p)$ of which takes the form $(U(p)-F)\cup p$ where $U(p)\in\{U(p)\}$ and $F\in\boldsymbol{F}$. Then the following proposition holds.

PROPOSITION. *If to each point p of X there corresponds the family $\{V(p)\}$ of subsets of X, then there is a unique T_1-space (denoted by (X, σ_2)) such that, for each point p of X, $\{V(p)\}$ is a base of neighbourhoods of p in (X, σ_2). And moreover (X, σ_2) is finer than (X, σ_1).*

PROOF. To prove this, we shall show that $\{V(p)\}$ corresponded to each point p of X satisfies the three conditions on the bases of neighbourhoods of a point.

First, it is obvious from the definition of $\{V(p)\}$ that any set belonging to $\{V(p)\}$ contains p.

Second, if $V_\alpha(p)$ and $V_\beta(p)$ belong to $\{V(p)\}$, then $V_\alpha(p)\cap V_\beta(p)$ belongs to $\{V(p)\}$. For, since $V_\alpha(p)\equiv(U_\alpha(p)-F_\alpha)\cup p$ and $V_\beta(p)\equiv(U_\beta(p)-F_\beta)\cup p$, where $U_\alpha(p)$ and $U_\beta(p)$ belong to $\{U(p)\}$ and F_α and F_β belong to $\boldsymbol{F}$, it follows that

$$\begin{aligned}V_\alpha(p)\cap V_\beta(p)&=\{(U_\alpha(p)-F_\alpha)\cap(U_\beta(p)-F_\beta)\}\cup p\\&=\{(U_\alpha(p)\cap F_\alpha^C)\cap(U_\beta(p)\cap F_\beta^C)\}\cup p\\&=\{(U_\alpha(p)\cap U_\beta(p))\cap(F_\alpha^C\cap F_\beta^C)\}\cup p\\&=\{(U_\alpha(p)\cap U_\beta(p))\cap(F_\alpha\cup F_\beta)^C\}\cup p\\&=\{(U_\alpha(p)\cap U_\beta(p))-(F_\alpha\cup F_\beta)\}\cup p.\end{aligned}$$

Therefore $V_\alpha(p)\cap V_\beta(p)$ belongs to $\{V(p)\}$.

Third, if $V_\alpha(p)$ belongs to $\{V(p)\}$ and q is any point of $V_\alpha(p)$, then there exists $V_\beta(q)$ belonging to $\{V(q)\}$ such that $V_\beta(q)\subset V_\alpha(p)$. For, let $V_\alpha(p)\equiv(U_\alpha(p)-F_\alpha)\cup p$ exactly as before and let $U_\beta(q)$ be a set belonging to $\{U(q)\}$ such that $U_\beta(q)\subset U_\alpha(p)$. Then it follows that

$$(U_\beta(q)-F_\alpha)\cup q\subset U_\alpha(p)-F_\alpha\subset(U_\alpha(p)-F_\alpha)\cup p=V_\alpha(p).$$

Hence $(U_\beta(q)-F_\alpha)\cup q$ is a set satisfying the required condition, which belongs to $\{V(q)\}$.

Finally, Since (X,σ_1) is a T_1-space and $\boldsymbol{F}$ contains the empty set, it is obvious that (X,σ_2) is a T_1-space and is finer than (X,σ_1).

Thus our proposition is proved.

The space (X,σ_2) defined above is said to be the refined space of (X,σ_1) by $\boldsymbol{F}$.

Let A be any set of X. Then "A is σ_i-P" means that A has the property P in (X,σ_i), and $Cl_{\sigma_i}A$ denotes the closure of A in (X,σ_i) where $i=1, 2$.

3. Connectivity structures of (X, σ_i)

THEOREM 1. *Let (X,σ_2) be the refined space of (X,σ_1) by $\boldsymbol{F}$ and C any nondegenerate subset of X. In order that C be σ_2-connected, it is necessary and sufficient that* (1) *C be σ_1-connected and* (2) *if F_α is any set belonging to $\boldsymbol{F}$ then $Cl_{\sigma_1}(C-F_\alpha)\supset C$.*

PROOF. The condition is necessary. For let C be σ_2-connected. Then C

is σ_1-connected since (X, σ_2) is finer than (X, σ_1). To show (2), suppose, on the contrary, that there exist a point p of C and a set F_α belonging to $\boldsymbol{F}$ such that p is not in $Cl_{\sigma_1}(C-F_\alpha)$. Then there exists a set $U_\alpha(p)$ belonging to $\{U(p)\}$ such that $U_\alpha(p)$ is disjoint from $C-F_\alpha$. Let set $V_\alpha(p)\equiv(U_\alpha(p)-F_\alpha)\cup p$, where F_α and $U_\alpha(p)$ are the sets defined above. Then it follows that

$$\begin{aligned} C\cap V_\alpha(p) &= C\cap\{(U_\alpha(p)-F_\alpha)\cup p\} \\ &= \{C\cap(U_\alpha(p)-F_\alpha)\}\cup\{C\cap p\} \\ &= \{C\cap U_\alpha(p)\cap F_\alpha^C\}\cup p \\ &= \{U_\alpha(p)\cap(C-F_\alpha)\}\cup p = p. \end{aligned}$$

Hence in the case in which C is a subspace of (X, σ_2), p is both open and closed in C. Therefore the nondegenerate set C is not σ_2-connected, contrary to the supposition, and thus (2) is proved.

The condition is sufficient. For suppose, on the contrary, that C satisfies the conditions (1) and (2) in this theorem, and that C is not σ_2-connected. Let $C=A\cup B$ be a σ_2-separation of C. Then by (1), we assume, there exists a point a of A such that a is in $Cl_{\sigma_1}B$ without losing generality. Let $V_\alpha(a)\equiv(U_\alpha(a)-F_\alpha)\cup a$ be a set belonging to $\{V(a)\}$ that is disjoint from B. Then there exists a point of b of B such that b is in $U_\alpha(a)$. Let $V_\beta(b)\equiv(U_\beta(b)-F_\beta)\cup b$ be a set belonging to $\{V(b)\}$ such that $U_\beta(b)\subset U_\alpha(a)$ and $V_\beta(b)$ is disjoint from A. Then it follows that

$$\begin{aligned} B\cap U_\beta(b)\subset B\cap U_\alpha(a)\subset B\cap\{(U_\alpha(a)-F_\alpha)\cup F_\alpha\} \\ =\{B\cap(U_\alpha(a)-F_\alpha)\}\cup(B\cap F_\alpha)=\phi\cup(B\cap F_\alpha)\subset F_\alpha \end{aligned}$$

and

$$\begin{aligned} A\cap U_\beta(b)\subset A\cap\{(U_\beta(b)-F_\beta)\cup F_\beta\} \\ =\{A\cap(U_\beta(b)-F_\beta)\}\cup(A\cap F_\beta)=\phi\cup(A\cap F_\beta)\subset F_\beta. \end{aligned}$$

Hence

$$C\cap U_\beta(b)=(A\cap U_\beta(b))\cup(B\cap U_\beta(b))\subset F_\alpha\cup F_\beta.$$

Therefore the point b of B is not in $Cl_{\sigma_1}(C-(F_\alpha\cup F_\beta))$. This contradicts the condition (2) since $F_\alpha\cup F_\beta$ belongs to $\boldsymbol{F}$.

Thus the sufficiency is proved.

LEMMA 1. *If F_α is any set belonging to $\boldsymbol{F}$, then F_α is either empty or σ_2-totally disconnected.*

PROOF. Let F_α be any set belonging to $\boldsymbol{F}$, p be any point of F_α, and $U_\alpha(p)$ be any set belonging to $\{U(p)\}$. Let us define $V_\alpha(p)\equiv(U_\alpha(p)-F_\alpha)\cup p$. Then we have

$$F_\alpha\cap V_\alpha(p)=F_\alpha\cap\{(U_\alpha(p)-F_\alpha)\cup p\}=p.$$

Therefore, in the case in which F_α is a subspace of (X, σ_2), p is open in F_α and hence p is a component of F_α.

Thus F_α is σ_2-totally disconnected.

LEMMA 2. *If we have $\boldsymbol{C}(X, \sigma_1)=\boldsymbol{C}(X, \sigma_2)$, then any set belonging to $\boldsymbol{F}$ is either empty or σ_1-totally disconnected.*

PROOF. Suppose, on the contrary, that there exists a set F_α belonging to $\boldsymbol{F}$ which is neither empty nor σ_1-totally disconnected. Let C be a nondegenerate σ_1-connected subset of F_α. Then, by the hypothesis $\boldsymbol{C}(X, \sigma_1)=\boldsymbol{C}(X, \sigma_2)$, C is σ_2-connected. On the other hand, by Lemma 1 F_α is σ_2-totally disconnected and so is C.

This contradiction proves Lemma 2.

LEMMA 3. *Assume that (X, σ_1) satisfies the condition as follows:*

If C is any nondegenerate σ_1-connected subset of X, p is any point of C, and $U_\alpha(p)$ is any set belonging to $\{U(p)\}$, then $C\cap U_\alpha(p)$ is not σ_1-totally disconnected.

Then if each set belonging to $\boldsymbol{F}$ is either empty or σ_1-totally disconnected, we have $\boldsymbol{C}(X, \sigma_1)=\boldsymbol{C}(X, \sigma_2)$.

PROOF. Let C be any nondegenerate σ_1-connected subset of X. To prove this, by Theorem 1 it is only need to show that for any set F_α belonging to $\boldsymbol{F}$ we have $Cl_{\sigma_1}(C-F_\alpha)\supset C$. Suppose, on the contrary, that there exist a point p of C and a set $U_\alpha(p)$ belonging to $\{U(p)\}$ such that $U_\alpha(p)\cap(C-F_\alpha)$ is empty. Then $U_\alpha(p)\cap C\subset F_\alpha$. Hence F_α contains a nondegenerate σ_1-connected set since $U_\alpha(p)\cap C$ contains the same. This is impossible because F_α is σ_1-totally disconnected. Therefore we have $Cl_{\sigma_1}(C-F_\alpha)\supset C$.

Thus Lemma 3 is proved.

Combining Lemma 2 with Lemma 3 we have the following theorem.

THEOREM 2. *Assume that (X, σ_1) satisfies the condition as follows:*

(*) *If C is any nondegenerate σ_1-connected subset of X, p is any point of C, and $U_\alpha(p)$ is any set belonging to $\{U(p)\}$, then $C\cap U_\alpha(p)$ is not σ_1-totally disconnected.*

Then in order that we have $\boldsymbol{C}(X, \sigma_1)=\boldsymbol{C}(X, \sigma_2)$ it is both necessary and sufficient that

(**) *Each set belonging to* $\boldsymbol{F}$ *is either empty or* σ_1*-totally disconnected.*

On the condition (*) in Theorem 2 used only for the proof of sufficency, we have the following results.

COROLLARY. *In order to have* $\boldsymbol{C}(X, \sigma_1)=\boldsymbol{C}(X, \sigma_2)$ *for every refined space* (X, σ_2) *of* (X, σ_1) *by any family* $\boldsymbol{F}$ *of subsets of* X *satisfying the conditions* $(\mathrm{F}_{\mathrm{I}})$, $(\mathrm{F}_{\mathrm{II}})$ *and* (**), *the condition* (*) *is both necessary and sufficient.*

PROOF. By Theorem 2 the condition (*) is sufficient. To prove the necessity suppose, on the contrary, that the condition (*) is not satsified. Then there exist a nondegenerate σ_1-connected set C, a point p of C and a set $U_\alpha(p)$ belonging to $\{U(p)\}$ such that $C\cap U_\alpha(p)$ is σ_1-totally disconnected. Then the family of subsets of X consisting of the empty set and $C\cap U_\alpha(p)$ satisfies the conditions $(\mathrm{F}_{\mathrm{I}})$, $(\mathrm{F}_{\mathrm{II}})$ and (**). Let (X, σ_2) be the refined space of (X, σ_1) by the family above. Then C is not σ_2-connected since if $V_\alpha(p)\equiv\{U_\alpha(p)-(C\cap U_\alpha(p))\}\cup p$ we have $C\cap V_\alpha(p)=p$. This contradicts our hypothesis. Thus the corollary is proved.

REMARK. In [6], Mazurkiewicz has shown the existence of nondegenerate connected set in a plane containing none of bounded nondegenerate connected subsets. Accordingly, no spaces containing a subspace homeomorphic to the plane satisfy the condition (*).

On the other hand any nondegenerate continuum which is locally connected and contains no simple closed curve (called the dendrite) satisfies the condition (*), because every connected subset of any dendrite is arcwise connected [7].

Faculty of Engineering
Tokushima University

References

[1] O. H. Hamilton, *Fixed points for certain non-continuous transformations*, Proc. Amer. Math. Soc., vol. **8** (1957), pp. 750-756.

[2] J. Stallings, *Fixed point theorems for connectivity maps*, Fund. Math., vol. **47** (1959), pp. 249-263.

[3] S. K. Hildebrand and D. E. Sanderson, *Connectivity functions and retracts*, Fund. Math., vol. **57** (1965), pp. 237-245.

[4] S. K. Hildebrand, *A connected topology for the unit interval*, Fund. Math., vol. **61** (1967), pp. 133-140.

[5] Carlos J. R. Borges, *On extensions of topologies*, Canadian J. Math., vol. **19** (1968) pp. 474-487.

[6] S. Mazurkiewicz, *Sur l'existence d'un ensemble plan connexe ne contenant aucun sous ensemble connexe, borné*, Fund. Math., vol. **2**, (1921), pp. 96-103.

[7] G. T. Whyburn, *Analytic Topology*. Amer. Math. Soc. Colloq. publ., vol. **28**, Providence, 1942.